Differential Geometry II

Symplectic Manifolds

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Bonnet's Theorem

Theorem 90: Assume that (M, g) is a complete connected Riemannian manifold for which the sectional curvature satisfies

$$K \geq \frac{1}{R^2}$$

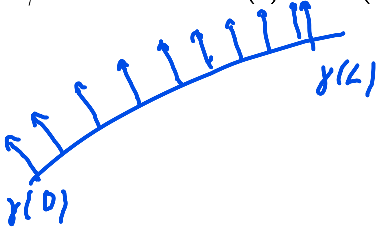
Then the diameter of M is bounded by

$$\mathrm{diam}(M, g) \leq \pi R.$$

In particular, M is compact and its fundamental group is finite.

Proof of Bonnet's Theorem

Consider a geodesic $\gamma : [0, L] \to M$. Let X be vector field along γ, $X \perp \dot{\gamma}$ and $\nabla_{\dot{\gamma}} X = 0$. Define $Y(t) := \sin(\pi t/L)$. Then



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\begin{aligned}
I(Y, Y) &= -\int_0^L g(Y, \nabla_{\dot{\gamma}}^2 Y + R(Y, \dot{\gamma})\dot{\gamma}) dt \\
&= \int_0^L (\sin(\pi/L))^2 (\pi^2/L^2 - g(R(X, \dot{\gamma})\dot{\gamma}, X)) dt
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$\|X\|_g = 1$

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Let $L \geq \pi R$. Then $K \geq 1/R^2$ implies hat $I(Y,Y) \leq 0$.

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Assume γ has no conjugate points. Then the unique Jacobi field J with $Z(0) = Y(0) = 0$ and $Z(L) = Y(L) = 0$ (i.e. $J \equiv 0$) would satisfy

$$0 = I(J, J) < I(Y, Y) \leq 0$$

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Hence γ has conjugate points and is not minimal. $\square$

Proof of Bonnet's Theorem

Symplectic Manifolds

Definition 92: Let M be a smooth manifold (without boundary, but not necessarily compact). A **symplectic structure** of **symplectic form** on M is a closed, non-degenerate 2-form $\omega \in \Omega^2(M)$, i.e. $d\omega = 0$ and for all $p \in M$

$$X \in T_pM \mapsto X \lrcorner \omega \in T_p^*M$$

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Lemma 93 : From the existence of a symplectic structure follows:

(1) The non-degeneracy implies that dim $M = 2n$ is even.

$V_0 = T_pM \ni v_1 \neq 0$ % $\omega_p(v_1, v_1) = 0$ $\qquad \omega_0 = \omega_p$

$(*) \Rightarrow \exists\, v_2 \notin \mathrm{span}(v_1) : \omega(v_1, v_2) = 1.$

$\dim \operatorname{Ker} \omega(v_1, \cdot) = \dim \operatorname{Ker} \omega(v_2, \cdot) = \dim M - 1$

& $\operatorname{Ker} \omega(v_1, \cdot) \neq \operatorname{Ker} \omega(v_2, \cdot) \Rightarrow \dim(\operatorname{Ker} \omega(v_1, \cdot) \cap \operatorname{Ker}(v_2, \cdot)) = \dim M - 2$ $= V_1$

$\omega_1 := \omega|_{V_1 \times V_1}$ is non-deg. $\quad v \in V_1 : \omega(v, v_1) = \omega(v, v_2) = 0$

$(*) \Rightarrow \exists\, v' \in V_1 : \omega(v, v') \neq 0 \overset{\text{Induction}}{\Longrightarrow} (v_1, v_2, \dots, v_{2n-1}, v_{2n})$ with [illegible])

w.r.t. $\mathcal{B}$: $\omega = \begin{pmatrix} 0 & 1 & & & 0 \\ -1 & 0 & & & \\ & & 0 & 1 & \\ & & -1 & 0 & \\ 0 & & & \ddots & \\ & & & & 0 \;\; 1 \\ & & & & -1 \;\; 0 \end{pmatrix}$

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(2) It is equivalent to

$$\omega^n = \omega \wedge \ldots \wedge \omega \neq 0$$

is a volume form. In particular, M has to be oriented.

$\mathcal{B}^* = (v_j^*) \subset T_p^*M$ is dual to $\mathcal{B}$

ω w.r.t. $\mathcal{B}$: $\omega = v_1^* \wedge v_2^* + v_3^* \wedge v_4^* + \ldots + v_{2n-1}^* \wedge v_{2n}^*$

$\omega^n = n!\, v_1^* \wedge v_2^* \wedge v_3^* \wedge \ldots \wedge v_{2n-1}^* \wedge v_{2n}^*$

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(3) If M is a closed manifold, then

$$b_2(M) := \dim H^2_{DR}(M) \geq 1.$$

Symplectic Manifolds

(3) $\omega^n \neq 0 \Rightarrow \int_M \omega^n >$

Examples

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With standard coordinates $(x_1, ..., x_n)$ on $\mathbb{R}^n$ and adapted coordinates $(p_1, ..., p_n, q_1, ..., q_n)$ where

$$(p_1, ..., q_n) \mapsto \theta_{(p_1,...,q_n)} := (q_1, ..., q_n, \sum_{k=1}^{n} p^k dq^k)$$

parametrizes $T^*\mathbb{R}^n$ so that with the projection $\pi : T^*\mathbb{R}^n \to \mathbb{R}^n$

$$\pi(\theta(p_1, ..., q_n)) = (q_1, ..., q_n).$$

and for the cordinate vector fields

$$\theta_{(p_1,...,q_n)}\Big(\frac{\partial}{\partial q_k}\Big) = p^k(p_1, ..., q_n) = p_k.$$

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The identification is given by $(p_1, p_2, ..., p_n, q_1, q_2, ..., q_n) \mapsto (x_1, x_3, ..., x_{2n-1}, x_2, x_4, ..., x_{2n})$ and the symplectic form

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the symplectic form can be described as

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Notice

$$\omega_{st} = \Re\mathrm{e}(\langle ., \rangle)$$

for the standard Hermitian product on $\mathbb{C}^n \cong T_z\mathbb{C}^n$, i.e. the **Kähler form**.

Examples

(2) Let $A \in \Omega^1(S^{2n+1}; \mathrm{i}\mathbb{R})$ be the connection one form on the total space of the Hopf bundle $S^{2n+1} \to \mathbb{C}P^n$ where the connection is given by the horizontal spaces $T_z^h S^{2n+1}$ which are the orthogonal complements to the orbits of the S^1-action

$$(z_1, ..., z_{n+1}) \cdot g = (z_1 g, ..., z_{n+1} g).$$

The curvature $F_A \in \Omega^2(\mathbb{C}P^n; \mathrm{i}\mathbb{R})$ defines the symplectic form

$$\omega_{FS} := -\mathrm{i} F_A,$$

called **Fubini Study form**.
(3) Let F be an oriented surface. Any **area form** on F defines a symplectic structure on F.

Cotangent Bundles

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Proof:

Cotangent Bundles

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(3) Let $g : M \to N$ be a diffeomorphism ($\dim M = \dim N$). Then $\varphi : T^*N \to T^*M$ given by

$$\alpha \in T_p^* N \mapsto (d_{g^{-1}(p)} g)^* \alpha \in T^*_{g^{-1}(p)} M$$

is a symplectomorphism.

Hamiltonian Dynamics

Definition 96: A **Hamiltonian system** is a tripel (M, ω, H) where (M, ω) is a symplectic manifold, and $H : M \times \mathbb{R} \to \mathbb{R}$ is a smooth function. If $H : M \to \mathbb{R}$ i.e. independent on teh second $\mathbb{R}$-component, the system is called **autonomous**. H defines a $\mathbb{R}$-dependent vector field X_H on M via

$$\omega_p(X_H(p,t), Y) = -d_{p,t}H(Y)$$

for all $p \in M$ and $Y \in T_pM$, which is called **Hamiltonian vector field**, or with $H_t = H(.,t) : M \to \mathbb{R}$ in short

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H is sometimes called Hamiltonian function – although it is still just an ordinary function.

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Let $U : \Omega \times \mathbb{R} \subset \mathbb{R}^3 \to \mathbb{R}$ be the potential of a force field, i.e. the force acting on a point mass with mass m is given by

$$F = -\nabla U.$$

where derivatives are taken in space direction only. The point mass will move along curves $x : I \to \mathbb{R}^3$ which satisfy Newton's equations of motion:

$$m\ddot{x}(t) = F(x(t), t) = -\nabla U(x(t), t)$$

Consider $(T^*\mathbb{R}^3, d\theta, H)$ with $H : T^*\mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}$ given by

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The first summand is the **kinetic energy** the second the **potential energy**, H the **total energy** of the system.

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and with $q(t) = x(t), p(t) = m\dot{x}(t)$ we obtain Newton's equation again.

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(2) **Conservation of the symplectic structure:** Consider a general Hamiltonian system (M, ω, H). Let $\Phi : U \times (t_0 - \epsilon, t_0 + \epsilon) \to M$ be a smooth map for an open subset $U \subset M$ satisfying $\Phi(t_0, x) = x$ for all $x \in U$ and

$$\frac{\partial \Phi}{\partial t}(x, t) = X_H(\Phi(x, t), t)$$

also called the **flow of** X_H. We abbreviate $\Phi_t(X) := \Phi(x, t)$. Then $\Phi_t : U \to M$ is an embedding and

$$\Phi_t^* \omega = \omega.$$

Conservation Laws

(3) **Transformation under symplectomorphisms:** Let $\varphi : (M_1, \omega_1) \to (M_2, \omega_2)$ be a symplectomorphism. Let $H : M_2 \times \mathbb{R} \to \mathbb{R}$ be a smooth function on M_2. Then for the Hamiltonian vector fields of H and $H \circ \varphi$

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Proof:

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Isotropic, Coisotropic and Lagrangian Immersions

For a symplectic vector space (V, ω) and a subspace $U \subset V$ we define

$$\mathrm{Ann}_\omega(U) := \{v \in V \mid \omega(v, u) = 0 \quad \forall u \in U\}.$$

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Notice: If ι is isotropic, then $\dim N \leq \frac{1}{2} \dim M$, coisotropic, then $\dim N \geq \frac{1}{2} \dim M$. Hence, if ι is Lagrangian, then $\dim N = \frac{1}{2} \dim M$.

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(4) Let $\varphi : (M_1, \omega_1) \to (M_2, \omega_2)$ be a symplectomorphism ($\dim M_1 = \dim M_2$). Then the graph

$$\Gamma_\varphi := \{(x, \varphi(x)) \mid x \in M_1\} \subset M_1 \times M_2$$

is a Lagrangian submanifold where the symplectic structure on $M_1 \times M_2$ is given by

$$\omega := \pi_1^*\omega_1 - \pi_2^*\omega_1.$$