Differential Geometry II

Holomorphic Curves

Klaus Mohnke

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Almost Complex Structures

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Theorem 103: (i) Let (M, ω) be a symplectic manifold. The space of compatible almost complex structures

$$\mathcal{J}(M, \omega) := \{J \mid J \text{ almost complex structure compatible with } \omega\}$$

is a non-empty contractible space.
(ii) Assume that on a open subset U there is an almost copmplex structure tamed by ω such that $M \setminus U$ is compact, then there is an almost complex structure on M which is tamed by ω.

Almost Complex Structures

Proof 2: The closedness $d\omega = 0$ plays no role in the proof.

1) (V,ω) ... symplectic vector space

g ... scalar product

$\exists!\ A \in Gl(V)$: $g(v,w) = \omega(v, Aw)\ \forall\ v,w \in V$

claim : $A^* = -A$

pf: $g(Av,w) = \omega(Av, Aw) = -\omega(Aw, Av) = g(-Aw, v)$

$= g(v, (-A)w)$ $\square$

$\Rightarrow (-A^2) \in Gl(V)$ symmetric & pos. definit

$\Rightarrow \exists \sqrt{-A^2} \in Gl(V)$ symmetric & pos. def.

$J \quad \sqrt{-A^2}$

claim: (a) $\bar{g}$ defined via $\bar{g}(v,w) = \omega(v, Jw)$ is a scalar product

(b) $J^2 = -id_V$.

Almost Complex Structures

Pf:

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For example: An oriented closed 4–manifold admits an almost complex structure (inducing this orientation) if and only if there is an integer class $c \in H^2_{DR}(M)$ such that $\int u^* c$ for $u : \Sigma \to M$ is even if and only if $u^* TM$ admits a spin structure (Σ a closed oriented surface) and

$$\int_M c^2 = 2\chi(M) + 3\sigma(M).$$

S^4 and $2k\mathbb{C}P^2 \sharp 2\ell\overline{\mathbb{C}P^2}$ do not admit almost complex structures for any orientation.

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Definition 104: The **Chern classes** of a symplectic manifold (M, ω) are the Chern classes $c_k(TM, J)$ of an almost complex structure J which is compatible to ω.

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Remark: Since the space of such structures is connected, via the Chern-Weil construction we see that $c_k(M, \omega)$ is well-defined. i.e. does not depend on J.

Pseudoholomorphic Curves

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Definition 106: Let (M, J) be an almost complex manifold. A J–**holomorphic** (or **pseudoholomorphic**) curve is a Riemann surface (Σ, j) together with a map $u : \Sigma \to M$ such that for all $z \in \Sigma$

$$d_z u \circ j_p z = J_{u(z)} \circ d_z u.$$

Pseudoholomorphic Curves

In complex coordinates, $z = x + \mathrm{i}y$, this takes the form

$$\frac{\partial u}{\partial x}(z) + J(u(z))\frac{\partial u}{\partial y}(z) = 0.$$

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Proposition 107: Let h be a Hermitian metric on M, ω its Kähler form, $u : (\Sigma, j) \to (M, J)$ a J–holomorphic curve. Then $u^*\omega$ is compatible with the orientation of (Σ, j) wherever $d_z u \neq 0$.

Proof:

Exact Lagrangian Submanifolds

Let (M, ω) be a symplectic manifold with an exact symplectic form, i.e. $\omega = d\alpha$ for a one form $\alpha \in \Omega^1(M)$ (e.g. $(\mathbb{C}^n, \omega_{st})$. If $L \subset M$ is a Lagrangian submanifold, then $\alpha|_{TL}$ defines a closed one form on L

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Theorem 109 (Gromov): There exists no closed, exact Lagrangian submanifold in $(\mathbb{C}^n, \omega_{st})$.

Remark: This can be considered as a generalization of Jordan's Curve Theorem.

Exact Lagrangian Submanifolds

Proof:

Non-Squeezing

Theorem 110 (Gromov): Let $Z^{2n}(R) := B^2(R) \times \mathbb{R}^{2n-2} \subset \mathbb{R}^{2n}$ and $B^{2n}(r) \subset \mathbb{R}^{2n}$ be equipped with the standard symplectic structure. Assume there is an embedding

$$\varphi : B^{2n}(r) \subset Z^{2n}(R)$$

which is a symplectomorphism onto its image: $\varphi^*\omega_{st} = \omega_{st}$. Then $r \leq R$.

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Corollary: The group of symplectomorphisms, $\mathrm{Symp}_c(\mathbb{R}^{2n})$, of $\mathbb{R}^{2n}$ with compact support is not dense in C^0-topology in the group of volume preserving diffeomorphisms with compact support. Let $\varphi_0 : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ be a volume preserving diffeomorphism such that $\varphi_0(B^{2n}(2R)) \subset Z^{2n}(R)$. Then there is no sequence $(\varphi_k)_k \subset \mathrm{Symp}_c(\mathbb{R}^{2n})$ such that $\varphi_k \to \varphi_0$ uniformly (in C^0).

Non-Squeezing

Monotonicity

For the proof of Theorem 110 we will need the following

Proposition 111: Let $u : (\Sigma, j) \to \mathbb{C}^n$ be a holomorphic curve (in Algebra: "complex curve"), $u(p) = 0$ for $p \in \Sigma$ and $r > 0$ such that $u^{-1}(B^{2n}(r)) \subset \Sigma$ is compact. Then

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$$\text{area}(u(\Sigma)) \geq \pi r^2.$$

Proof of Non-Squeezing

Embedd $\iota : (Z^{2n}(R), \omega_{st}) \hookrightarrow (S^2(R) \times \mathbb{R}^{2n-2}, \omega) =: (M, \omega)$ with $\omega = \pi_1^* \omega_R + \pi_2^* \omega_{st}$, where $\omega_R \in S^2$ area form with

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$$\int_S^2 \omega_R = \pi R^2.$$

Sufficient to show that $\varphi : (B^{2n}(r), \omega_{st}) \hookrightarrow (M, \omega)$ implies $r \leq R$.