

Differential Geometry II

Differential Forms

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Differential Forms on \mathbb{R}^n

Definition 8: Let $U \subset \mathbb{R}^n$ be an open subset. A **(differential) k -form** on U is a differentiable map $\alpha : U \rightarrow \Lambda^k((\mathbb{R}^n)^*)$, i.e. the components α_I

$$\alpha_p = \sum_{I=\{1 \leq i_1 < i_2 < \dots < i_k\}} \alpha_I(p) \underbrace{e^{i_1} \wedge e^{i_2} \wedge \dots \wedge e^{i_k}}$$

where $\{e^1, \dots, e^n\}$ denotes the basis dual to the standard basis are smooth functions (or C^m -functions etc.). We typically write α_p instead of $\alpha(p)$. *The $\alpha_I : U \rightarrow \mathbb{R}$ are differentiable.*
The space of such forms will be denoted by $\Omega^k(U)$.

Differential Forms on \mathbb{R}^n

Examples: (1) Let $X : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a smooth vector field. Then $\langle \cdot, X(\cdot), \cdot \rangle$ defines a differential one form.

(2) Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be a smooth function. Then its differential df defines also a one form. We have

$$d_p f = \sum_{i=1}^n \frac{\partial f}{\partial x_i} e^i. \quad x^i(x_1, \dots, x_n) = x_i$$

In particular, for the i -th coordinate function x^i we see that $dx^i = e^i$. Therefore and since the e^i are closely related to the choice of coordinates, we will from now on write dx^i instead of e^i :

$$d_p f = \sum_{i=1}^n \frac{\partial f}{\partial x_i} dx^i.$$

Wedge-Product and Inner product

(1) Given a differential k -form $\alpha \in \Omega^k(U)$ and a differential ℓ -form $\beta \in \Omega^\ell(U)$, the wedge-product $\alpha \wedge \beta \in \Omega^{k+\ell}(U)$ is defined by pointwise applying the wedge-product:

$$(\alpha \wedge \beta)_p := \alpha_p \wedge \beta_p. \quad \text{last line}$$

Check that the Result is a differential form, i.e. smooth.

(2) Given a smooth vector field X on U and a differential form $\alpha \in \Omega^k(U)$, then the inner product is also defined pointwise:

$$(X \lrcorner \alpha)_p := X_p \lrcorner \alpha_p \quad \text{last line}$$

giving rise to a differential $(k - 1)$ -form (Check that!)

Pull-Back

(3) For a smooth map $F : U \rightarrow V$, where $V \subset \mathbb{R}^m$ open and a differential k -form $\alpha \in \Omega^k(V)$, its pull-back, $F^*\alpha$ is defined via

$$p \in U \quad (F^*\alpha)_p := (d_p F)^* \alpha_{F(p)} \in \wedge^k ((\mathbb{R}^m)^*)$$

Once again, $F^*\alpha \in \Omega^k(U)$, i.e. smooth (Check it!).

Examples: (1) $g \in \Omega^0(V) = C^\infty(V)$: $F^*g = g \circ F$. ←

$$(2) F^*(dx^j) = \sum_{i=1}^n \frac{\partial F^j}{\partial x_i} dx^i = dF^j / F = (F^1, F^2, \dots, F^m)$$

$$(3) m = n : F^*(dx^1 \wedge dx^2 \wedge \dots \wedge dx^n) = \underbrace{\det(dF)}_{\text{Jacobian}} dx^1 \wedge dx^2 \wedge \dots \wedge dx^n.$$

Proposition 9: The pull-back is a homomorphism of graded algebras over \mathbb{R} .

$$\Omega^*(U) = \bigoplus_{k=0}^n \Omega^k(U) \quad \left\{ \begin{array}{l} f \in C^\infty(V), \alpha \in \Omega^k(V) \\ f \cdot \alpha \in \Omega^k(V) \\ F^*(f \cdot \alpha) = F^*(f) \cdot F^*(\alpha) \end{array} \right.$$

Proof: Exercise.

Notice that $\Omega^*(U), \Omega^*(V)$ are algebras over $C^\infty(U), C^\infty(V)$, respectively, but even if $U = V$ the pull-back is not an algebra homomorphism w.r.t. that structure.

The Exterior Derivative

Intuition: 1) $f \in \Omega^0(U) = C^\infty(U)$
then $df \in \Omega^1(U)$ is the differential
2) $\alpha \in \Omega^k, \beta \in \Omega^l$
 $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^k \alpha \wedge d\beta$

Definition 10: For

$$\alpha = \sum_{\substack{I=\{1 \leq i_1 < i_2 < \dots < i_k\} \\ \subseteq n}} \alpha_I dx^{i_1} \wedge dx^{i_2} \wedge \dots \wedge dx^{i_k} \in \underline{\Omega^k(U)}$$

its **exterior derivative**, $d\alpha \in \Omega^{k+1}(U)$ is defined via

$$d\alpha = \sum_{I=\{1 \leq i_1 < i_2 < \dots < i_k\}} \sum_{j=1}^n \frac{\partial \alpha_I}{\partial x^j} dx^j \wedge dx^{i_1} \wedge dx^{i_2} \wedge \dots \wedge dx^{i_k}.$$

Notice that the summand vanishes for those I and j for which $j \in I$.

Examples: (1) If $\alpha \in \Omega^n(U)$, then $d\alpha = 0$ by the remark at the end.

(2) For $\alpha = x_1 dx_2 \in \Omega^1(\mathbb{R}^2)$: $d\alpha = dx_1 \wedge dx_2$.

The Exterior Derivative

(3) Let $f \in C^\infty(U)$. Then

$$d(\underline{df}) = d\left(\sum_{i=1}^n \frac{\partial f}{\partial x_i} dx^i\right) = \sum_{i=1}^n \sum_{j=1}^n \frac{\partial^2 f}{\partial x_j \partial x_i} dx^j \wedge dx^i.$$

Now, $dx^i \wedge dx^j = -dx^j \wedge dx^i$ hence all summands with $i = j$ vanish and the sum reduces to

$$= \sum_{1 \leq i < j \leq n} \left(\frac{\partial^2 f}{\partial x_i \partial x_j} - \frac{\partial^2 f}{\partial x_j \partial x_i} \right) dx^i \wedge dx^j = 0$$

due to Schwarz' Lemma. \square

The Exterior Derivative

Theorem 11: $(\Omega^*(U), \wedge, d)$ is a **differential graded algebra** over \mathbb{R} , meaning that $(\Omega^*(U), \wedge)$ is an algebra over \mathbb{R} , d increases the degree by 1, is \mathbb{R} -linear and (1) d satisfies Leibniz' rule:

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^k \alpha \wedge d\beta$$

for $\alpha \in \Omega^k(U), \beta \in \Omega^\ell(U)$

(2) $d^2 = d \circ d = 0.$

(3) Moreover, if $F : U \rightarrow V$ is a smooth map as before the pull-back defines a morphism of differential graded algebras, i.e. in particular for $\alpha \in \Omega^k(V)$ we have

$$d(F^* \alpha) = F^*(d\alpha).$$

Proof of Theorem 11 (1):

Proof of Theorem 11 (2):

Proof of Theorem 11 (3):

Poincaré's Lemma

Definition 12: A k -form $\alpha \in \Omega^k(U)$ is called **closed** or **cocycle**, if $d\alpha = 0$, it is called **exact** or **coboundary** if there exists $\beta \in \Omega^{k-1}(U)$ such that $d\beta = \alpha$. The set of boundaries, $B^k(U)$, and of cycles, $Z^k(U)$, are linear subspaces of the set of k -Forms, i.e. $B^k(U) \subset Z^k(U) \subset \Omega^k(U)$. The quotient

$$H_{DR}^k(U) := Z^k(U)/B^k(U).$$

is called **de Rham cohomology** of U .

Examples: (1) $H_{DR}^0(U) \cong \mathbb{R}$ if U is connected and freely generated.
(2) $U \subset \mathbb{R}^n$: $H_{DR}^n(U) = \{0\}$.

Exercise.

Poincaré's Lemma

Definition 13: A subset $U \subset \mathbb{R}^n$ is called **starshaped** if there exists a point $p_0 \in U$ such that for all $p \in U$ all points of the segment between p_0 and p are contained in U .

Theorem 14: Let $U \subset \mathbb{R}^n$ be an open, starshaped set. Then for every $k > 0$, every closed k -Form $\alpha \in \Omega^k(U)$ is exact, i.e. there exists a $(k - 1)$ -form $\beta \in \Omega^{k-1}(U)$ such that $d\beta = \alpha$. In particular,

$$H_{DR}^k(U) = \{0\}$$

for every $k > 0$.

Chain Homotopies

Poincaré's Lemma is a special case of a much more general statement. Let $f, g : U \rightarrow V$ be two differentiable maps between open subsets $U \subset \mathbb{R}^n$, $V \subset \mathbb{R}^m$, $\Phi : U \times [0, 1] \rightarrow V$ differentiable such that $\Phi(x, 0) = f(x)$ and $\Phi(x, 1) = g(x)$ for all $x \in U$.

Example: $U \subset \mathbb{R}^n$ open and starshaped w.r.t. p_0 . $f := id_U$, $g := c_{p_0} \equiv p_0$ and $\Phi(x, t) = p_0 + (1 - t)(x - p_0)$.

Theorem 15: There exists linear maps $P_k : \Omega^k(V) \rightarrow \Omega^{k-1}(U)$ such that for all k

$$P \circ d + d \circ P = f^* - g^* : \Omega^k(V) \rightarrow \Omega^k(U).$$

Proof of Poincaré's Lemma

Theorem 15 applied to example and noting $c_{p_0}^* \alpha = 0$ for any k -form α with $k > 0$ yields:

$\alpha \in \Omega^k(U)$ closed, i.e. $d\alpha = 0$. Then

$$id^* \alpha - c_{p_0}^* \alpha = \alpha = P(d\alpha) + d(P(\alpha)) = d(P(\alpha))$$

so, $\beta := P(\alpha) \in \Omega^{k-1}(U)$ gives the desired $(k-1)$ -form.

Proof of Theorem 15

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