Differential Geometry II

Differential Forms

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Differential Forms on Manifolds

Definition 16: Let M be a differentiable manifold of dimension n. A differential k–form is a family $\alpha = \{\alpha_p \in \Lambda^k(T_p^* M)\}_{p \in M}$ which satisfies the following condition w.r.t. any parametrization $\varphi : V \subset \mathbb{R}^n \to U \subset M$: Let $\{\frac{\partial}{\partial x_i}\}_i$ be the coordinate vector fields on ~~a~~ the coordinate neighbourhood, $\mathcal{U}$ $\{dx^j_p\}_j$ the dual basis w.r.t. $\{\frac{\partial}{\partial x_i}(p)\}_i \subset T_pM$. Then the uniquely determined functions $\alpha_I : U \to \mathbb{R}$ in

$$\alpha|_U = \sum_{I=\{1 \leq i_1 < i_2 < \ldots < i_k \leq n\}} \alpha_I dx^{i_1} \wedge dx^{i_2} \wedge \ldots \wedge dx^{i_k}.$$

are smooth.
The set of differential k–forms on M is denoted by $\Omega^k(M)$.

Differential Forms on Manifolds

$d_x\varphi : T_xV = \mathbb{R}^n \xrightarrow{\cong} T_{\varphi(x)}M$

$d_p(\varphi^{-1}) : T_pM \xrightarrow{\cong} T_{\varphi^{-1}(p)}V = \mathbb{R}^n$

Remark: If the coefficients α_I are smooth around p with respect to one coordinate, then they are smooth w.r.t. any other.
chart

Proof: Notice that $dx_p^i = d_p(\varphi^{-1})^* e^i$ where $\{e^i\}$ is the standard dual basis of $(\mathbb{R}^n)^*$, correspondigly for exterior forms. Varying p we denote the result by

$$\varphi^*\alpha = \sum_{I=\{1\le i_1<i_2<\ldots<i_k\le n\}} \alpha_I dx^{i_1} \wedge dx^{i_2} \wedge \ldots \wedge dx^{i_k}.$$

where we wrote dx^i instead of $\{e^i\}$ as explained before and understand that α_I is the composition of the original $\alpha_I : U \to \mathbb{R}$ with φ.

Differential Forms on Manifolds

Let $\varphi : V \subset \mathbb{R}^n \to U \subset M$ and $\tilde{\varphi} : \tilde{V} \subset \mathbb{R}^n \to U \subset M$ be two parametrizations of some open neighbourhood U. Denote by $F : \tilde{V} \to V$ the composition $F := \varphi^{-1} \circ \tilde{\varphi}$. transition map

Then

$$\boxed{\tilde{\varphi}^* \alpha = F^*(\varphi^* \alpha)}$$

Now, pull-backs via smooth maps preserve smoothness.

of k-forms on open subsets of $\mathbb{R}^n$

$\square$

Wedge–Product, Inner Product, Pull-Back

(1) Given differential forms $\alpha \in \Omega^k(M), \beta \in \Omega^\ell(M)$ we define the wedge–product $\alpha \wedge \beta \in \Omega^{k+\ell}(M)$ via $(\alpha \wedge \beta)_p := \alpha_p \wedge \beta_p$.

(2) Given a differential form $\alpha \in \Omega^k(M)$ and a smooth vector field X on M we define the inner product $(X \lrcorner \alpha) \in \Omega^{k-1}(M)$. via

$$(X \lrcorner \alpha)_p := X_p \lrcorner \alpha_p$$

(2) Let $F : M \to N$ be a smooth map between manifolds, $\alpha \in \Omega^k(N)$ a differential k–form. Then the pull-back $F^*\alpha \in \Omega^k(M)$ is defined to be $(F^*\alpha)_p = (d_pF)^*\alpha_{\cancel{p}F(p)}$ $\quad d_pF : T_pM \to T_{F(p)}N$

(Check the condition of Definition 16 in all three cases).

Exterior Derivative

Definition 17: Let $\alpha \in \Omega^k(M)$. The exterior derivative $d\alpha \in \Omega^{k+1}(M)$ is the differential $(k+1)$–form such that $(d\alpha)_p$ is defined as follows: in a coordinate chart (U, φ, V) around p

$$d\alpha = (\varphi^{-1})^*(d(\varphi^*\alpha)) \qquad \varphi : V \to U$$

Lemma 18: The definition of $(d\alpha)_p$ does not depend on the choice of coordinates around p.

Proof of Lemma 18

Let $\varphi : V \subset \mathbb{R}^n \to U \subset M$ and $\tilde{\varphi} : \tilde{V} \subset \mathbb{R}^n \to U \subset M$ be two parametrizations of some open neighbourhood U. Denote by $F : \tilde{V} \to V$ the composition $F := \varphi^{-1} \circ \tilde{\varphi}$. Recall from previous proof, that with respect to corresponding coordinates

$$\tilde{\varphi}^* \alpha = F^*(\varphi^* \alpha) \quad \in \Omega^k(\tilde{V})$$

hence using $d \circ F^* = F^* \circ d$ on differential forms on $\mathbb{R}^n$

$$\begin{aligned}
\rightarrow \quad (\tilde{\varphi}^{-1})^*(d(\tilde{\varphi}^* \alpha)) &= (\tilde{\varphi}^{-1})^*(d(F^*(\varphi^* \alpha))) \\
&= (\tilde{\varphi}^{-1})^* F^*(d(\varphi^* \alpha)) \\
&= (F \circ \tilde{\varphi}^{-1})^*(d(\varphi^* \alpha)) \\
&= (\varphi^{-1})^*(d(\varphi^* \alpha)). \quad \square
\end{aligned}$$

Rmk: One can define the exterior derivative using vector fields :
due to Cartan : $d\alpha(X_1, \ldots, X_{k+1}) = X_1(\alpha(X_2, \ldots, X_{k+1})) - X_2(\alpha(X_1, X_3, \ldots)) \pm \alpha([X_1, X_2], X_3, \ldots, X_{k+1}) \pm \cdots$

De-Rham Cohomology of Manifolds : $d \circ d = 0$!

We define the sets of k–**cocycles**

$$Z^k(M) := \{\alpha \in \Omega^k(M) \mid d\alpha = 0\},$$

to consist of **closed** k–forms, **coboundaries**,

$$B^k(M) := \{d\beta \mid \beta \in \Omega^{k-1}(M)\},$$

to consist of **exact** k–forms and the k**-th de Rham cohomology** of M to be the quotient

$$H^k_{DR}(M) := Z^k(M)/B^k(M).$$

There is an analogous version of Theorem 15.