Differential Geometry II

Manifolds with Boundary

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Manifolds with Boundary

Definition 19: An n–dimensional smooth **manifold with boundary** is a metric space M with an open covering $(U_\iota)_{\iota\in I}$ together with homeomorphisms

$$\varphi_\iota : V_\iota \longrightarrow U_\iota,$$

on open subsets $V_\iota \subset \mathbb{H}^n : \{x \in \mathbb{R}^n \mid x_n \geq 0\}$ of the upper half space with the induced topology from $\mathbb{R}^n$ such that every transition map

$$(\varphi_\kappa)^{-1} \circ \varphi_\iota : \varphi_\iota^{-1}(U_\iota \cap U_\kappa) \to \varphi_\kappa^{-1}(U_\iota \cap U_\kappa)$$

is a diffeomorphism between open subsets of $\mathbb{H}^n$. The family of triples $\{(U_\iota, \varphi_\iota, V_\iota)\}_{\iota\in I}$ is called an **atlas** of the manifold, the elements are called **charts**, U_ι, coordinate neighborhood, φ_ι a **parametrization** its inverse φ_ι^{-1} the **coordinate map**, its components **(local) coordinates** of M.

Manifolds with Boundary

Remarks: (1) Manifolds as defined last semester (i.e. all $V_\iota \subset \mathbb{R}^n$ are open subsets) are also manifolds with boundary.
(2) $[0, 1)$ is an open subset of $\mathbb{H}^1 = [0, \infty)$!

Definition 19 (cont'd): The **boundary**, ∂M, of M is the set of all points which map to $\mathbb{R}^{n-1} \times \{0\} \subset \mathbb{R}^n$ under the coordinate maps

$$\partial M := \{\varphi_\iota(x) \mid \iota \in I, x \in V_\iota \cap \left(\underline{\mathbb{R}^{n-1} \times \{0\}}\right)\}$$

Points in the complement $M \setminus \partial M$ will be referred to as **interior points** of M.

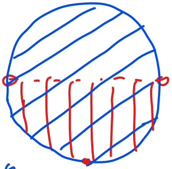
Examples:

(0) $\mathbb{R}^n, \mathbb{H}^n$ are smooth manifolds with boundary:
$\partial\mathbb{R}^n = \emptyset, \partial\mathbb{H}^n = \mathbb{R}^{n-1} \times \{0\}$.

$\mathbb{R}^n \underset{\text{dif.}}{\simeq} \mathring{\mathbb{H}}^n$

(1) The closed ball $B^n(r) := \{x \in \mathbb{R}^n \mid \|x\| \leq r\}$ is a smooth manifold with boundary: $\partial B^n = S^{n-1}(r) = \{x \in \mathbb{R}^n \mid \|x\| = r\}$.

$n=2$



B^n

$(0,-1)$

$U_1 = \{(x_1,x_2) \in B^2(1) \mid x_2 < 0\} \xleftarrow{\varphi_1} V_1$

$\varphi_1(y_1,y_2) = (y_1, y_2 - \sqrt{1-y_1^2})$

$U_2 = \{(x_1,x_2) \in B^2(1) \mid x_1 > 0\}$

$U_3 = \{(x_1,x_2) \in B^2(1) \mid x_2 > 0\}$

$U_4 = \{(x_1,x_2) \in B^2(1) \mid x_1 < 0\}$

$V_0 \xrightarrow{\varphi_0} U_0 = \{(x_1,x_2) \mid \|x\| < 1\}$,

$V_1 = \{(y_1,y_2) \in \mathbb{H} \mid \|y\| < 1\} = V_2 = V_3 = V_4$

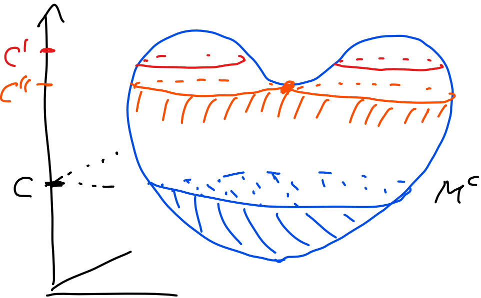
$V_0 = \{(y_1,y_2) \mid y_1^2 + (y_2-1)^2 < 1\} \subset \mathbb{H}$ open

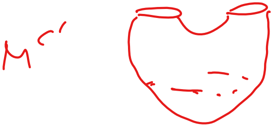
Manifolds with Boundary

Examples: (2) Let M be a manifold without boundary, $f : M \to \mathbb{R}$ a smooth function, $c \in \mathbb{R}$ a regular value, i.e. for all $p \in M$, $f(p) = c$ we have $d_p f \neq 0$. Then the sublevel set

$$M^c := \{p \in M \mid f(p) \leq c\}$$

is a smooth manifold with boundary: $\partial M^c = \{p \in M \mid f(p) = c\}$. (Exercise)





Differentiable maps between Manifolds with Boundary

Definition 20: Let M, N be smooth manifolds with boundary of possibly different dimension. Let $\{(U_\iota, \varphi_\iota, V_\iota)\}_{\iota \in I}$ and $\{(\hat{U}_\kappa, \hat{\varphi}_\kappa, \hat{V}_\kappa)\}_{\kappa \in \hat{I}}$ be the corresponding collection of coverings of M and N, together with the homeomorphisms to open subsets of $\mathbb{H}^m$ and $\mathbb{H}^n$, respectively.

(1) A continuous map $F : M \to N$ is **differentiable** if for any $p \in M$, any $\iota \in I$ and $\kappa \in \hat{I}$ the composition $\hat{\varphi}_\kappa^{-1} \circ F \circ \varphi_\iota : \varphi_\iota^{-1}(F^{-1}(\hat{U}_\kappa)) \to V_\kappa$ is smooth as a map between open subsets of $\mathbb{H}^m$ and $\mathbb{H}^n$. F is a diffeomeorphism if it is bijective and together with its inverse F^{-1} differentiable. In particular, the coordinate maps $\varphi_\iota^{-1} : U_\iota \to \cancel{\mathbb{R}^n}\, V_\iota \subset \mathbb{H}^n$ are diffeomorphisms.

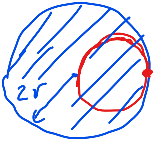
(2) A chart (U, φ, V) of M is given by a diffeomorphism $\varphi : V \to U$ between open subsets $V \subset \mathbb{H}^m$ and $U \subset M$.

Submanifolds

Definition 21: Let M be a manifold with boundary of dimension m. A subset $N \subset M$ is a submanifold of M of dimesnion n if for each $p \in N$ there is a chart (U, φ, V) such that $\varphi|_{V \cap \{0_{m-n}\} \times \mathbb{R}^{n-1} \times [a,\infty)}$ is a homeomorphism onto $U \cap N$ for some $a \geq 0$. Such charts are sometimes called ironing charts ("Bügelkarten" in German).

Remark: (1) A submanifold is a smooth manifold with boundary. (Exercise)
(2) Note that $N \cap \partial M \subset \partial N$: the interior points of N are interior points of M.

Example: (1) $B^n \times \{0_{n-m}\} \subset \mathbb{R}^m$ is a submanifold.
(2) $B^2((r,0),r) \subset B^2(0,2r)$ is not a submanifold.



Boundaries

Lemma 21: Let M be an m–dimensional smooth manifold with boundary.
(1) A point $p \in \cancel{\partial} M$ lies in the boundary of M if and only if there is a chart (U, φ, V) such that $x \in \varphi(V \cap \mathbb{R}^{n-1} \times \{0\})$ if and only if that condition holds for any chart containing p.
(2) The boundary ∂M is a ~~sub~~manifold ~~of M~~ of dimension $(m-1)$, a closed subset of M and has empty boundary: $\partial(\partial M) = \emptyset$.

Proof of Lemma 21:

(1) The existence of such a chart for $p \in \partial M$ is provided by the definition (namely for one of the charts from the atlas).
Now assume that there exists a chart (U, φ, V), $p \in U \subset M$ such that $p = \varphi(x_1, \ldots, x_{n-1}, 0)$. Let $(\hat{U}, \hat{\varphi}, \hat{V})$ be any other chart around p. The intersection $U \cap \hat{U}$ is also open and contains p. By taking its image under both coordinate maps, we may assume w.l.o.g. that $U = \hat{U}$. We need to show that for the transition function $F := \hat{\varphi}^{-1} \circ \varphi$, $F(x_1, \ldots, x_{n-1}, 0) \in \mathbb{R}^{n-1} \times \{0\}$. The n–th component of the inverse $G := F^{-1}$, $G^n : \hat{V} \to \mathbb{R}$ attains its absolute minimim, namely 0, at $y := F(x_1, \ldots x_{n-1}, 0)$. Now if $y_n > 0$ it would be an interior point of $\mathbb{H}^n$ and therefore $d_y G^n = 0$. !
That would contradict, that $d_y G$ is invertible, since G was a diffeomorphism.
Finally, if the condition is satisfied for all charts, then in particular for charts from the atlas and we are back at the definition. $\square$
(2) Exercise.

Tangent Vectors

As for manifolds, we define tangent vectors of manifolds with boundary and the differential of smooth funtions between them.

Definition 22: Let M be an m–dimensional manifold with boundary ∂M, $p \in M$.
(1) Let I, J be intervals containing 0. Two differentiable curves $\gamma :: I \to M$, $\delta : J \to M$ with $\gamma(0) = \delta(0) = p$ are called äquivalent if for a chart (U, φ, V) around p we have

$\varphi^{-1} \circ \gamma : I \to \mathbb{H}^n$

$\varphi^{-1} \circ \delta : J \to \mathbb{H}^n$

$$(\varphi^{-1} \circ \gamma)'(0) = (\varphi^{-1} \circ \delta)'(0).$$

A tangent vector of M at p is an equivalence class of such curves. The set of tangent vectors, T_pM, the so-called **tangent space** of M at p.

Claim: The tangent space, T_pM, is a real vector space. (Exercise: see "Differential Geometry I") use $T_p\mathbb{H}^n \simeq \mathbb{R}^n \ \forall p \in \mathbb{H}^n$.

Tangent Vectors

(2) Let $p \in \partial M$ be a boundary point, $v \in T_pM$ a tangent vector. v is called **tangent to** ∂M if there is a differentiable curve $\gamma : I \to \partial M$ representing v. v is **pointing inward**/**pointing outward** if v is not tangent to ∂M and represented by a curve $\gamma : [0, a) \to M/\gamma : (-a, 0] \to M$ for some $a > 0$.

(3) Let $F : M \to N$ be a differentiable map between manifolds with boundary. Its differential at $p \in M$, $d_pF : T_pM \to T_{F(p)}N$ is defined via

$$d_pF([\gamma]) := [F \circ \gamma],$$

i.e. the image of $[\gamma] \in T_pM$ represented by a differentiable curve $\gamma : I \to M$ is given by the equivalence class $[F \circ \gamma]$.

Claim: The differential is a linear map between vector spaces. (Exercise)

Vector fields,
Differential forms together with wedge-product, inner product, pull-back and exterior derivative are given analogously on manifolds with boundary.

Orientation

Definition 23: An **orientation** of an m–dimensional manifold with boundary, M, is a choice of orientation for the tangent spaces at all interior points such that for each $p \in M \setminus \partial M$ there exists a chart (U, φ, V), $p \in U$ such that the coordinate vector fields $\{\frac{\partial}{\partial x_k}\}_{k=1}^n$ form an oriented basis of $T_q M$ at each $q \in U$. If M admits an orientation it is called **orientable**, if an orientation is chosen M is called **oriented**.

V_ι, U_ι are connected

Remark: (1) If M is oriented and $\{U_\iota, \varphi_\iota, V_\iota\}_{\iota \in I}$ an atlas, we can modify it by replacing $(U_\iota, \varphi_\iota, V_\iota)$ by $(U_\iota, \varphi_\iota \circ \sigma, \sigma(V_\iota))$ with $\sigma : \mathbb{H}^n \to \mathbb{H}^n$ given by $\sigma(x_1, x_2, ..., x_m) = (-x_1, x_2, ..., x_m)$ if $m \geq 2$ to obtain an atlas, whose charts satisfy the condition of Defintion 23, called an **oriented atlas**.
(2) 1–dimensional manifolds are always orientable. If such a manifold is oriented one can distinguish between boundary points for which charts induce oriented bases and those whose charts induces bases with the opposite orientation.

Orientation

Lemma 24: Let M be an oriented manifold with boundary of dimenson $m \geq 2$. Then the tangent spaces at all boundary points can be oriented so that there exists a chart around each which is oriented in the sense of Definition 23. In particular, the boundary can be oriented so that for any $p \in \partial M$ an oriented basis of $T_p(\partial M)$ extended by an inward pointing tangent vector gives an oriented basis of $T_p M$.