Differential Geometry II

Integration of Differential Forms

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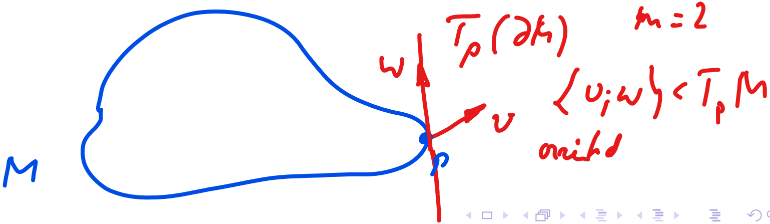
Orientation: Correction

Lemma 24: Let M be an oriented manifold with boundary of dimension $m \geq 2$. Then the tangent spaces at all boundary points can be oriented so that there exists a chart around each which is oriented in the sense of Definition 23.

In particular, the boundary can be oriented so that for any $p \in \partial M$ an oriented basis of $T_p(\partial M)$ extended by an **outward** pointing tangent vector put in the **first position** gives an oriented basis of T_pM.



Measurable subsets

Definition 25: A metric space is called **separable** if it contains a countable dense subset.

From now on we assume that the manifolds we consider are separable metric spaces without always mentioning it.

Definition 26: Let M be an n–dimensional separable manifold with boundary.
(1) A subset $A \subset M$ is **(Lebesgue) measurable** if for every chart (U, φ, V) of M, $\varphi^{-1}(A) \subset V$ is a Lebesgue measurable subset of $\mathbb{R}^n$.
(2) A subset $N \subset M$ is a **zero set** if for every chart (U, φ, V) of M, $\varphi^{-1}(A) \subset V$ is a zero set of $\mathbb{R}^n$.

Signed Measures

Remark: The separability of M implies, that there is a countable base of its topology. Then the measurable/null sets $A \subset M$ are exactly countable unions of Lebesgue measurable/null sets of coordinate neighbourhoods (identified with open subsets of $\mathbb{R}^n$). Therefore, these locally defined sets generate the σ–algebras.

Recall

Definition 27: A **finite signed measure** μ on a manifold M assignes to each measurable set a real number and is σ–additive: i.e. for each countable family $\{A_k\}_{k\in\mathbb{N}}$ of pairwise disjoint measurable subsets

$$\mu(\cup_{k=1}^{\infty} A_k) = \sum_{k=1}^{\infty} \mu(A_k).$$

Integrals of Differential Forms

Proposition 28: Let $\alpha \in \Omega^n(M)$ be a differential n–forms on the n–dimensional, oriented manifold with boundary M with compact support: the closure in M

$$\overline{\{p \in M \mid \alpha_p \neq 0\}} \subset M$$

is compact.
There exists a unique signed measure μ on M which satisfies the following: Let (U, φ, V) be an *oriented* chart and let $f : U \to \mathbb{R}$ be given by

$$\alpha = f dx^1 \wedge dx^2 \ldots \wedge dx^n.$$

Let $A \subset U$ be measurable. Then

$$\mu(A) := \int_{\varphi^{-1}(A)} (f \circ \varphi) d\lambda^n$$

defines a unique signed measure on M. By λ^n we denote the Lebesgue measure on $\mathbb{R}^n$.

Proof of Proposition 28:

(1) First notice that $f \circ \varphi$ is continuous with compact support, hence the integral is defined and finite.

(2) We need to show that for a measurable set $A \subset U$ for a coordinate chart (U, φ, V) the right hand side in the definition remains unchanged if we use different **oriented** coordinates $(U, \hat{\varphi}, \hat{V})$. Now

$F = \varphi^{-1} \circ \hat{\varphi}$

$$\alpha|_U = f dx^1 \wedge \ldots \wedge dx^n = (f \circ F) \det(dF) d\hat{x}^1 \wedge \ldots \wedge d\hat{x}^n.$$

Therefore since $\det(dF) > 0$ by the transformation rule for integrals (involving the factor $|\det(dF)|$!) we see that $\mu(A)$ is independent of the choice of oriented coordinates.

Proof of Proposition 28:

(3) Now it suffices to show, that for a coordinate chart (U, φ, V) and a measurable set $A \subset U$ with

$$A = \cup_{k=1}^{\infty} B_k$$

disjoint

for measurable sets $B_k \subset U_k$ for coordinate charts (U_k, φ_k, V_k) we have

$$\mu(A) = \sum_{k=1}^{\infty} \mu(B_k).$$

But this is true since the Lebesgue integral

$$\int_A f d\lambda^n = \sum_{k=1}^{\infty} \int_{B_k} f d\lambda^n$$

is σ–additive. $\square$

Definition 29: Let M be an oriented manifold with boundary, $\dim M = n$, $\alpha \in \Omega^n(M)$ a differential n-form with compact support, μ the signed measure defined by it, $A \subset M$ be a measurable set. Then

$$\int_A \alpha := \mu(A).$$

In particular,

$$\int_M \alpha = \mu(M).$$

Remark: Notice, that change of orientation changes the overall sign of the signed measure μ: if $-M$ denotes the same manifold with the opposite orientation, then

$$\int_{-M} \alpha = -\int_M \alpha.$$

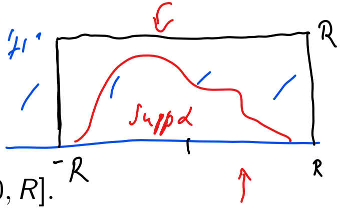
Stokes' Theorem

Theorem 30: Let M be an oriented n-dimensional manifold with boundary, $\alpha \in \Omega^{n-1}(M)$ a differential $(n-1)$–~~manifold~~ form with compact support. Then

$$\int_M d\alpha = \int_{\partial M} \alpha,$$

where α on the right hand side denotes the pull-back of α under the inclusion $\partial M \hookrightarrow M$ and ∂M is equipped with the induced orientation of Lemma 24.

Proof of Stokes' Theorem



(1) $M = \mathbb{H}^n$, $R > 0$ such that

$$\operatorname{supp}(\alpha) \subset [-R, R]^{n-1} \times [0, R].$$

Let

$$\alpha = \sum_{k=1}^{n} \alpha_k dx^1 \wedge \ldots \wedge \widehat{dx^k} \wedge \ldots \wedge dx^n.$$

where $\widehat{dx^k}$ means, that dx_k is left out. We have

$$d\alpha = \Big(\sum_{k=1}^{n} (-1)^{k-1} \frac{\partial \alpha_k}{\partial x_k} \Big) dx^1 \wedge \ldots \wedge dx^n$$

$\iota_{\partial \mathbb{H}^n} : \mathbb{R}^{n-1} \to \mathbb{H}^n$

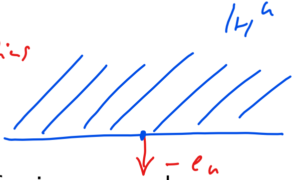
and

$(x_1, \ldots, x_{n-1}) \mapsto (x_1, \ldots, x_{n-1}, 0)$

$$\iota^*_{\partial \mathbb{H}^n} \alpha = \alpha_n dx^1 \wedge \ldots \wedge dx^{n-1}.$$

$\Rightarrow \iota^*_{\partial \mathbb{H}} dx_n = 0$

Proof of Stokes' Theorem



Notice that

$$\overset{(n-1)\text{ transpositions}}{\{-e_n, e_1, \dots, e_{n-1}\}}$$

agrees with the standard orientation of $\mathbb{R}^{n\cancel{-1}}$ if n is even and disagrees if n is odd hence

$$\int_{\partial\mathbb{H}^n} \alpha = (-1)^n \int_{\mathbb{R}^{n-1}} \alpha_n(., 0) d\lambda^{n-1}.$$

For $k = 1, \dots, n-1$

$$\begin{aligned}
\int_{\mathbb{H}^n} \frac{\partial \alpha_k}{\partial x_k} dx^1 \wedge \dots \wedge dx^n &\overset{Fubini}{=} \int_{\mathbb{H}^{n-1}} \left(\int_{-R}^{R} \frac{\partial \alpha_k}{\partial x_k} dx^k \right) d\lambda^{n-1} \\
&\overset{part.Int.}{=} \int_{\mathbb{H}^{n-1}} (\alpha_k(., R, .) - \alpha_k(., -R, .)) d\lambda^{n-1} \\
&= 0,
\end{aligned}$$

Proof of Stokes' Theorem

hence

$$\int_{\mathbb{H}^n} d\alpha = \int_{\mathbb{H}^n} (-1)^{n-1} \frac{\partial \alpha_n}{\partial x_n} dx^1 \wedge \dots \wedge dx^n$$

$$\overset{Fubini}{=} (-1)^{n-1} \int_{\mathbb{R}^{n-1}} \int_0^R \frac{\partial \alpha_n}{\partial x_n} dx^n d\lambda^{n-1}$$

$$\overset{part.Int.}{=} (-1)^{n-1} \int_{\mathbb{R}^{n-1}} (\alpha_n(., R) - \alpha_n(., 0)) \; d\lambda^{n-1}$$

$$= (-1)^n \int_{\mathbb{R}^{n-1}} \alpha_n(., 0)) d\lambda^{n-1}.$$

Proof of Stokes' Theorem

(2) $p \in M$, (U_p, φ_p, V_p) chart around p, $x_p := \varphi_p^{-1}(p) \in \mathbb{H}^n$.
$r_p > 0$ such that $B(x_p, 2r_p) \subset V_p$.

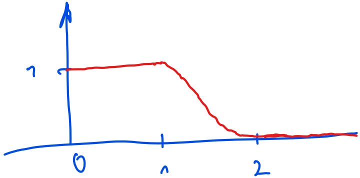
$$\operatorname{supp}(\alpha) \subset \bigcup_{p \in \operatorname{supp}(\alpha)} \varphi_p(B(x_p, r_p)).$$

$\operatorname{supp}(\alpha)$ compact: there are $p_1, \ldots, p_N \in \operatorname{supp}(\alpha)$ such that

$$\operatorname{supp}(\alpha) \subset \bigcup_{k=1}^{N} \varphi_{p_k}(B(x_{p_k}, r_{p_k})).$$

$[0,\infty)$

Let $f : \cancel{\mathbb{R}} \to [0,1]$ smooth, $f|_{[0,1]} \equiv 1$, $f|_{[2,\infty)} \equiv 0$. (Exercise: Prove existence!)



Proof of Stokes' Theorem

Define smooth $\tilde{\lambda}_k : M \to [0,1]$

$$\tilde{\lambda}_k(q) := f(\|\varphi_k^{-1}(q)\|/r_{p_k})$$

for $q \in U_{p_k}$ and 0 otherwise. We set $\lambda_k := \tilde{\lambda}_k \Big/ \sum_{j=1}^{N} \tilde{\lambda}_j$. Then

on supp(α)

$$\sum_{k=1}^{N} \lambda_k \Big|_{\mathrm{supp}(\alpha)} \equiv 1.$$

$(\lambda_k \alpha) \in \Omega^{n-1}(M)$ by setting $(\lambda_k \alpha)_q = 0$ if $q \notin \mathrm{supp}(\alpha)$

Proof of Stokes' Theorem

(3) Notice

$$d\alpha = \sum_{k=1}^{N} d(\lambda_k \alpha).$$

$\mathrm{supp}(\lambda_k \alpha) \subset U_{p_k}$. If $U_{p_k} \cap \partial M = \emptyset$ then by partial integration and compactness of support we obtain

$$\int_M d(\lambda_k \alpha) = 0 = \int_{\partial M} \lambda_k \alpha. \qquad \partial M \cap \mathrm{supp}(\lambda_k \alpha) = \emptyset$$

Else

$$\begin{aligned}
\int_M d(\lambda_k \alpha) &\overset{\text{Def.}}{=} \int_{\underline{V_{p_k}}} \varphi_k^*(d(\lambda_k \alpha)) \\
&= \int_{\mathbb{H}^n} d(\varphi_k^*(\lambda_k \alpha)) \\
&= \int_{\mathbb{R}^{n-1} \times \{0\}} \varphi_k^*(\lambda_k \alpha) \\
&\overset{\text{Def}}{=} \int_{\partial M} \lambda_k \alpha.
\end{aligned}$$

Proof of Stokes' Theorem

Summing over k gives the result:

$$\int_M d\alpha = \int_M d\Big(\sum_{k=1}^{N} \lambda_k \alpha\Big)$$

$$= \sum_{k=1}^{N} \int_M d(\lambda_k \alpha)$$

$$= \sum_{k=1}^{N} \int_{\partial M} \lambda_k \alpha \quad = \int_{\partial M} \Big(\sum_{k=1}^{N} \lambda_k\Big) \alpha$$

$$= \int_{\partial M} \alpha. \qquad \square$$

Remark: The collection of functions $\{\lambda_k\}$ is called a **partition of unit**.