Differential Geometry II

Applications of Stokes'Theorem

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Brouwer's Fixed Point Theorem

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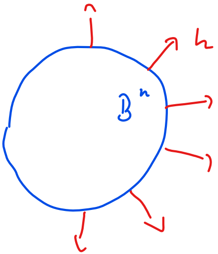
the **outer normal** given by $\mathbf{n}(x) = x$.



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$$\alpha := \mathbf{n} \lrcorner (dx^1 \wedge ... \wedge dx^n)$$

defines a positive ~~mass~~ measure on S^{n-1}. Hence $d\alpha = 0$ and

$$\int_{S^{n-1}} \alpha > 0.$$

Proof of Brouwer's Fixed Point Theorem

But by Stokes' Theorem and $f|_{S^{n-1}} = id_{S^{n-1}}$

$$0 = \int_{B^n} d(f^*\alpha) = \int_{S^{n-1}} f^*\alpha = \int_{S^{n-1}} \alpha > 0$$

and we arrive at a contradiction.

$d(f^*\alpha) = f^*(d\alpha) = 0$

$= 0$

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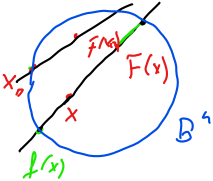
Proof of Theorem 31: (1) Assume there is a smooth function $F : B^n \to B^n$ without fixed points.
Define $f : B^n \to S^{n-1}$ which assigns to $x \in B^n$ the intersection of the well-defined line through x and $F(x)$, such that $x \in [f(x), F(x)]$. Then

$$f|_{S^{n-1}} = \mathrm{id}_{S^{n-1}}.$$



Exercise: f is smooth.
This contradicts Proposition 32.

Proof of Brouwer's Fixed Point Theorem

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Approximate F by smooth functions $F_n : B^n \to B^n$ which converge uniformly to F (How?).

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Let $x_n \in B$ be fixed points: $F_n(x_n) = x_n$.
B compact: subsequence converges in B^n: $x_n \to x_* \in B^n$. Then

$$F(x_*) = x_*. \quad \square$$

(Exercise).

Riemannian Metrics

Recall: A **Riemannian metric** on a smooth manifold with boundary is a smooth family $\{g_p\}_{p\in M}$ of symmetric positive definite bilinear forms on T_pM.

$\{\frac{\partial}{\partial x_k}\} \rightsquigarrow (g_{ij}) : V \to M(n, \mathbb{R})$

Proposition 33: Any separable smooth manifold with boundary admits a Riemannian metric.

For this we will discuss the partition of unity.

Partition of Unity

Lemma 34: Let M be a smooth manifold with boundary. Let $\{U_\iota\}_{\iota \in I}$ be an open covering of M. There exist a countable family $\{\lambda_k\}_{k \in \mathbb{N}}$ of non-negative smooth functions with compact support, such that

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(iii) **partition of unity:**

$$\sum_{k=1}^{\infty} \lambda_k \equiv 1.$$

We will discuss the proof later, possibly.

Construction of Riemannian Metric

Proof of Proposition 33: Let $\{(U_\iota, \varphi_\iota, V_\iota)\}_{\iota \in I}$ be a smooth atlas of M, $\dim M = n$ and let $\{\lambda_k\}_{k \in \mathbb{N}}$ be a partition of unity w.r.t. covering. Let $\iota_k \in I$, such that $\operatorname{supp}(\lambda_k) \subset U_{\iota_k}$.

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Define for any $p \in M$

$$g_p := \sum_{k=1}^{\infty} \lambda_k(p)(\varphi_{\iota_k}^{-1})_p^* \langle .,. \rangle$$

where $\langle .,. \rangle$ denotes the standard scalar product in $\mathbb{R}^n$.

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$(\varphi_{\iota_k}^{-1})^* \langle ., . \rangle$ is symmetric positive definite.
g_p is is finite convex linear combination of such, hence symmetric and positive definite. $\square$

The Volume Form

Let M be an *oriented* n–dimensional manifold equipped with a Riemannian metric g.
Definition 35: The **volume form** of (M, g) is the n–form, dM, which is given at any $p \in M$ by the volume form of the oriented euclidean vector space

$$dM_p := d(T_pM, g_p).$$

See lecture 1.

dM is not (in general) the differential of a $(n-1)$-form on M : $dM \neq d(M)$

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Let (U, φ, V) be an oriented chart. Then

$$\varphi^*(dM) = \sqrt{\det(g)}dx^1 \wedge \dots \wedge dx^n.$$

>0

(Exercise).

The Volume Form

In particular dM defines a positive measure. We define the **volume** of (M, g) as

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Corollary: Assume $\partial M = \emptyset$, and compact, and oriented. Then the de Rham chomology class of dM is non-zero

$$[dM] \neq 0 \in H^n_{DR}(M).$$

Proof: Assume $dM = d\alpha$ for some $\alpha \in \Omega^{n-1}(M)$

$$0 < \int_M dM = \int_M d\alpha \underset{\text{Stokes}}{=} \int_{\partial M = \emptyset} \alpha = 0 \quad \text{↯}.$$

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If **n** is the outward normal vector field of M along ∂M, then

$n_p \perp T_p(\partial M)$

$\|n_p\|_{g_p} = 1$

$$d(\partial M) = n \lrcorner dM.$$

(Exercise).

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Theorem 36: Let M be a compact, oriented manifold with boundary. Then there is no smooth map $\varphi : M \to \partial M$ which restricts to the identity on the boundary.

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$$\begin{aligned}\int_{\partial M} d(\partial M) &= \int_{\partial M} \varphi^* d(\partial M) \\ &= \int_M d(\varphi^*(d(\partial M)) \\ &= \int_M \varphi^* d(d(\partial M)) = 0.\end{aligned}$$

$\varphi|_{\partial M} = id_{\partial M}$

Now the volume form defines a positive measure on ∂M and hence the first integral is positive. Contradiction. □

Codifferential and Laplacian

Let (M, g) be an oriented Riemannian manifolds. The **Hodge-$*$-operator**

$$* : \Omega^k(M) \longrightarrow \Omega^{n-k}(M)$$

is defined by the Hodge-$*$-operator on each $\Lambda^*(T_pM)$.

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The **Laplaceoperator** is the differentiall operator

$$\Delta : \Omega^k(M) \longrightarrow \Omega^k(M)$$
$$\Delta := d\delta + \delta d$$

with $\delta = 0$ on $\Omega^0(M)$ and $d = 0$ on $\Omega^n(M)$.

The Laplace–Beltrami Operator

$g = (g_{ij})$ Riemann tensor in oriented coordinates, g^{ij} its inverse, f smooth function. In coordinates we obtain $\Delta f = \delta(df)$

$$\Delta_g f = -\frac{1}{\sqrt{\det g}} \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(\sqrt{\det g}\, g^{ij} \frac{\partial f}{\partial x_j}\right)$$

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$$\int_M \langle \Delta\alpha, \alpha \rangle \, dM \geq 0$$

$M = \mathbb{R}^n$, $g = \langle .,. \rangle$

$$\Delta\Big(\sum_{I=\{1\leq i_1<\ldots<i_k\leq n\}} \alpha_I dx^{i_1} \wedge \ldots \wedge dx^{i_k}\Big) = \sum_{I=\{1\leq i_1<\ldots<i_k\leq n\}} (\Delta\alpha_I) dx^{i_1} \wedge \ldots \wedge dx^{i_k}$$

where $\Delta\alpha_I$ is the classical Laplace-operator on functions on $\mathbb{R}^n$.

$$\Delta\alpha_I = -\sum_{k=1}^{n} \frac{\partial^2 \alpha_I}{\partial x_k}$$

Gauss' Divergence Theorem

Definition 37: Let X be a smooth vector field on an oriented Riemannian manifold (M, g). The **divergence** of X is the smooth real function $\mathrm{div}X$ defined by

$$\mathrm{div}X := \delta\big(\underline{g(X,.)}\big)$$

$\in \Omega^1(M)$

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Theorem 38: With the notation above

$$\int_M \text{div}X \, dM = - \int_{\partial M} g(X, \mathbf{n}) d(\partial M).$$

where $\mathbf{n}$ is the outward normal along ∂M.

Gauss' Divergence Theorem

Proof: We have

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Let $\{e_1, ..., e_{n-1}\}$ be an oriented orthonormal basis of $T_p(\partial M)$. Now with $e_0 := \mathbf{n}_p$ and $X_k := g_p(X, e_k)$ and evaluating the left side we obtain

$$(*g(X,.))(e_1,...,e_{n-1}) = \Big(\sum_{k=0}^{n-1} X_i e^0 \wedge ... \wedge \widehat{e^k} \wedge ... \wedge e^{n-1}\Big)(e_1,...,e_{n-1}) = X_0$$

which is equal to the right hand side. $\square$