Differential Geometry II

Fibre Bundles

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Recall the definition of a *topological space*.

Set X, $\mathcal{O} \subset \mathcal{P}(X)$ is called topological structure:

1) $X, \emptyset \in \mathcal{O}$

2) $(U_i)_{i \in I} \subset \mathcal{O}$ family, then $\bigcup_{i \in I} U_i \in \mathcal{O}$

3) $U_1, \dots, U_k \in \mathcal{O}$, then $\bigcap_{j=1}^{k} U_j \in \mathcal{O}$

$(X, \mathcal{O})$ topological space, $U \in \mathcal{O}$ is called open subset.

Ex: (X, d) metric space.

$$\mathcal{O}_{(X,d)} := \left\{ U \subset X \;\middle|\; \begin{array}{l} \forall p \in U \; \exists r > 0 : \\ B(p, r) \subset U \end{array} \right\}$$

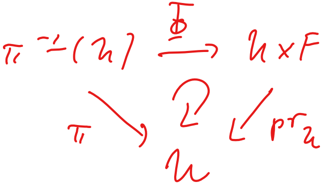
Continuity: preimages of open subsets are open!

Fibre Bundles

Definition 39: A **fibre bundle** (E, B, π, F) of topological spaces consists of a continuous map $\pi : E \to B$ such that there for each point $p \in B$ there is an open neighbourhood $U \subset B$ and a homeomorphism

$$\Phi : \pi^{-1}(U) \to U \times F$$

so that $\mathrm{pr}_U(\Phi(e)) = \pi(e)$.



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Lemma 40: Let $\pi : E \to B$ be a topological fibre bundle over a separable metric space B with fibre (homeomorphic to) a metric space F. Then E is a metrizable space.

The proof is left as an exercise.

Fibre bundles

Example: (1) The product $B \times F$ is called the **trivial** F–bundle over B. $\pi = pr_B$, $\mathcal{U} = B$ $\Phi : B \times F \to B \times F$
" id.

Fibre bundles

Example: (1) The product $B \times F$ is called the **trivial** F–bundle over B.

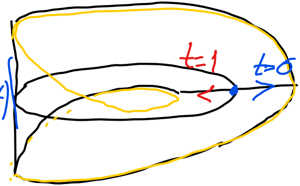
(2) The compact Moebius strip M^2 is a non-trivial $[-1,1]$–bundle over S^1.

$$M^2 \subset \mathbb{R}^3$$

$$= \{(2\cos(2\pi t) + s\cos(\pi t),\ 2\sin(2\pi t),\ s\sin(\pi t)),\ t \in [0,1],\ s \in [-1,1]\} \subset \mathbb{R}^3$$

$$\downarrow \pi$$

$$S^1$$



$$\pi(2\cos(2\pi t) + s\cos(\pi t), \dots) = (\cos(2\pi t), \sin(2\pi t))$$

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(2) The compact Moebius strip M^2 is a non-trivial $[-1,1]$–bundle over S^1.
(3) If F is equipped with the discrete topology then (E, B, π, F) is also called a **covering (space)** of B. E.g. ∂M^2 of (2) is a (non-trivial) covering of S^1.

Fibre Bundles of Manifolds

Definition 41: A **fibre bundle of manifolds** is a fibre bundle (E, B, π, F) where E, B, F are manifolds, $\pi : E \to B$ is smooth and the local trivializations $\Phi : \pi^{-1}(U) \to U \times F$ can be chosen to be diffeomorphisms.

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Remark: The projection π of a fibre bundle of manifolds is always a surjection, i.e. for every $e \in E$ its differential $d_e\pi : T_eE \to T_{\pi(e)}B$ is surjective.

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Examples: (1) $(B \times F, B, \mathrm{pr}_B, F)$ is the trivial bundle of manifolds.

(2) $(\mathbb{R}, \mathbb{R}, \pi, \{*\})$, where $\pi(x) = x^3$ is topological bundle but *not* a bundle of manifolds.

The Hopf Bundle

Examples: (3) Consider the 3–sphere as

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$$\{[z_1, z_2] | (z_1, z_2) \in \mathbb{C}^2 \setminus \{0\}\}$$

where $[z_1, z_2]$ denotes the equivalence class of (z_1, z_2) for the relation

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$$(z_1, z_2) \sim \lambda(z_1, z_2)$$

for any $\lambda \in \mathbb{C} \setminus \{0\}$. Then $\pi : S^3 \to S^2$

$$\pi(z_1, z_2) := [z_1, z_2]$$

is a fibre bundle of manifolds with fibre π. (Exercise)

Bundle Morphisms

Definition 42: Let (E_i, B_i, π_i, F_i), $i_1, 2$ be two fibre bundles (of manifolds), $\varphi : B_1 \to B_2$ be a continuous map of their bases.

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Example: A trivialization $\Phi : \pi^{-1}(U) \to U \times F$ is a isomorphism of $(\pi^{-1}(U), U, \pi, F)$ and $(U \times F, U, \mathrm{pr}_U, F)$.

Transition Functions

Let (E, B, π, F) be a topological fibre bundle.

$U, V \subset B$ open sets, $\Phi : \pi^{-1}(U) \to U \times F$ and $\Psi : \pi^{-1}(V) \to V \times F$ trivializations.

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On $U \cap V$ the composition

$$\Psi \circ \Phi^{-1} : (U \cap V) \times F \to U \cap V \times F$$

has the form

$$\Psi \circ \Phi^{-1}(p, f) = (p, g(p, f))$$

where $g(p, .) : F \to F$ is a continuous family of homeomorphisms. $g : U \cap V \to \text{Homeo}(F)$ is called **transition function**.

Cocycle Condition

Lemma 43: (1) Let $\Phi_i : U_i \times F \to \pi^{-1}(U_i)$ $i = 1, 2, 3$ be three trivializations of the bundle over open subsets U_i and denote by $g_{ij} : U_i \cap U_j \to \text{Homeo}(F)$ the transition function defined by $\Phi_i \circ \Phi_j^{-1}(p, f) = (p, g_{ij}(p, f))$.

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$$\begin{aligned} g_{ii}(p, .) &= \text{id}_F \quad \forall p \in U_i \\ g_{ij}(p, .) \circ g_{jk}(p, , .) \circ g_{ik}(p, .) &= \text{id}_F \quad \forall p \in U_1 \cap U_2 \cap U_3. \end{aligned}$$

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(2) Let $(U_i)_{i \in I}$ be an open covering of the topological space B and $\{g_{ij} : U_i \cap U_j \to \text{Homeo}(F)\}_{i,j \in I}$ be a family of maps to the homeomorphism group of a topological space F which satisfy the condition of (1).

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(3) Everything remains valid if we consider in (1) and (2) fibre bundles of manifolds and replace continuous by smooth, homeomorphisms by diffeomorphism.

Proof of Lemma 43

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Fibre Products

Given two fibre bundles (of manifolds) (E_i, B_i, π_i, F_i) the cartesian product $(E_1 \times E_2, B_1 \times B_2, \pi_1 \times \pi_2, F_1 \times F_2)$ where

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If $C \subset B$ is a subset (submanifold), then the restriction of the bundle to C, $(E|_C := \pi^{-1}(C), C, \pi|_{E|_C}, F)$ is a fibre (of manifolds) over C.

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Definition 44: Given two fibre bundles (of manifolds) over the same base (E_i, B, π_i, F_i), their **fibre product** is defined by the restriction of their cartesian product to the diagonal

$$\Delta_B := \{(b, b) | b \in B\} \cong B$$

naturally identified with B by $(b, b) \mapsto b$.

Vector Bundles

Definition 45: A (real) **vector bundle** of rank k is a fibre bundle $(E, B, \pi, \mathbb{R}^k)$ (of manifolds), such that each fibre is a vector space and the trivializations can be chosen to be $\mathbb{R}$–linear on each fibre.

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$$g : U \cap V \longrightarrow Gl(k; \mathbb{R}).$$

Remark: For two vector bundles over the same base $(E_i, B, \pi_i, \mathbb{R}^{k_i})$ denote by $E_1 \oplus E_2$ their fibre product. Then the addition defines a morphism of fibre bundles (of manifolds)

$$E \oplus E \to E$$

as well as the scalar multiplication

$$\mathbb{R} \times E \to E.$$

Vector Bundles

Morphisms of vector bundles are morphisms of bundles (of manifolds) which restricts to each fibre as a homomorphism of vector bundles.

Examples: (1) The trivial vector bundle $B \times \mathbb{R}^k$.

(2) The Moebius bundle:

Tangent Bundles