Differential Geometry II

Vector Bundles

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Cocycle Condition

$\pi^{-1}(U_i) \to U_i \times F$

Lemma 43: (1) Let $\Phi_i : U_i \times F \to \pi^{-1}(U_i)$ $i = 1, 2, 3$ be three trivializations of the bundle over open subsets U_i and denote by $g_{ij} : U_i \cap U_j \to \mathrm{Homeo}(F)$ the transition function defined by $\Phi_i \circ \Phi_j^{-1}(p, f) = (p, g_{ij}(p, f))$.

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\begin{aligned}
g_{ii}(p, .) &= \text{id}_F \quad \forall p \in U_i \\
g_{ij}(p, .) \circ g_{jk}(p, , .) \circ g_{ik}(p, .) &= \text{id}_F \quad \forall p \in U_1 \cap U_2 \cap U_3.
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(2) Let $(U_i)_{i \in I}$ be an open covering of the topological space B and $\{g_{ij} : U_i \cap U_j \to \mathrm{Homeo}(F)\}_{i,j \in I}$ be a family of maps to the homeomorphism group of a topological space F which satisfy the condition of (1).

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(2) Let $(U_i)_{i \in I}$ be an open covering of the topological space B and $\{g_{ij} : U_i \cap U_j \to \mathrm{Homeo}(F)\}_{i,j \in I}$ be a family of maps to the homeomorphism group of a topological space F which satisfy the condition of (1). Then there exists a fibre bundle (E, B, π, F) which admits trivializations over U_i whose transition functions are exactly given by $\{g_{ij}\}_{ij \in I}$. Any two such bundles are isomorphic.

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$$g_{ii}(p, .) = \mathrm{id}_F \quad \forall p \in U_i$$
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(3) Everything remains valid if we consider in (1) and (2) fibre bundles of manifolds and replace continuous by smooth, homeomorphisms by diffeomorphism.

Proof of Lemma 43

$$E := \coprod_{i \in I} (U_i \times F) \big/ \sim \;\xrightarrow{\pi}\; B$$

$$[(x,f)_i] \longmapsto x$$

$\sim$: $\forall\, i,j \in I$ $\forall\, x \in U_i \cap U_j$, $\forall\, f \in F$:

$$(x,f)_j \sim (x, g_{ij}(x,f))_i$$

Quotient topology : X top. space, $\sim$ equivalence rel. on X

quotient topology on $X/\sim$: finest topology s.t.

$p : X \to X/\sim \quad p(x) := [x]$ is continuous.

in short: $U \subset X/\sim$ open $\iff$ $p^{-1}(U) \subset X$ is open.

$U \subset B$ open $\quad p^{-1}(\pi^{-1}(U)) = \coprod_{i \in I} (U \cap U_i) \times F$ open!

$\Rightarrow \pi^{-1}(U) \subset E$ is open $\Rightarrow \pi$ is continuous.

Proof of Lemma 43

$$\pi^{-1}(U_i) = p(U_i \times F) \subset E \quad \text{open}$$

$$p|_{U_i \times F} : U_i \times F \to p(U_i \times F) \subset E \quad \text{bijection!}$$

$$(p|_{U_i \times F})^{-1} : \pi^{-1}(U_i) \to U_i \times F \quad \text{continuous.}$$

$$W \subset U_i \times F \quad \text{open}$$

need to show $\quad p(W) \subset E$ is open

$$\Leftrightarrow p^{-1}(p(W)) = \coprod_{j \in I} \underbrace{\Phi_{ij}^{-1}(W)}_{\text{open}} \quad \text{open.}$$

$$\Phi_{ij}(x, f) = (x, g_{ij}(x, f))$$

Proof of Lemma 43

Pf uniqueness: Let $\tilde{E} \xrightarrow{\tilde{\pi}} B$ be another bundle with fibre F & local trivializations

$$\tilde{\Phi}_i : \tilde{\pi}^{-1}(U_i) \to U_i \times F \quad \text{s.t.}$$

$$\to \quad \tilde{\Phi}_i \circ \tilde{\Phi}_j^{-1}(x, f) = (x, g_{ij}(x, f))$$

Define an isomorphism $\psi : \tilde{E} \to E$

$$\begin{array}{ccc} \tilde{\pi}^{-1}(U_i) & \xrightarrow{\tilde{\Phi}_i} & U_i \times F \\ \cong \downarrow \psi|_{\tilde{\pi}^{-1}(U_i)} & \swarrow \Phi_i^{-1} & \\ \pi^{-1}(U_i) & & \end{array}$$

$\square$

Vector Bundles

Definition 45: A (real) **vector bundle** of rank k is a fibre bundle $(E, B, \pi, \mathbb{R}^k)$ (of manifolds), such that each fibre is a vector space and the trivializations can be chosen to be $\mathbb{R}$–linear on each fibre.

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$$g : U \cap V \longrightarrow Gl(k; \mathbb{R}).$$

$E_p := \pi^{-1}(b)$

Remark: For two vector bundles over the same base $(E_i, B, \pi_i, \mathbb{R}^{k_i})$ denote by $E_1 \oplus E_2$ their fibre product. Then the addition defines a morphism of fibre bundles (of manifolds)

$$E \oplus E \to E$$

as well as the scalar multiplication

$$\mathbb{R} \times E \to E.$$

$E \xrightarrow{\pi} B$ vector bdle.

$0_E \subset E$

$= \{ 0 \in E_p \mid p \in B \} \overset{\pi}{\cong} B$

The Tangent Bundle

Proposition 46: (1) The disjoint union of the family $\{T_pM\}_{p\in M}$ of tangent spaces of a smooth manifold of dimension n forms a smooth vectorbundle of rank n over that manifold where the projection is given by

$$\pi(X) = p \text{ for } X \in T_pM.$$

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(2) Given a chart (U, φ, V) the map

$$X \in \pi^{-1}(U) \mapsto (\pi(X), (X_1, ..., X_n)) \in U \times \mathbb{R}^n$$

where

$$X = \sum_{j=1}^{n} X_j \frac{\partial}{\partial x_j}(\pi(X)),$$

is a trivialization of this bundle.

The Tangent Bundle

Proof: Let $\phi := \tilde{\varphi}^{-1} \circ \varphi : \varphi^{-1}(U \cap \tilde{U}) \to \tilde{\varphi}^{-1}(U \cap \tilde{U})$ be the transition map between two charts from the differentiable atlas (U, φ, V) and $(\tilde{U}, \tilde{\varphi}, \tilde{V})$.

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$$\frac{\partial}{\partial \tilde{x}_i} = \sum_{j=1}^{n} \frac{\partial \phi_i}{\partial x_j} \frac{\partial}{\partial x_j}.$$

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Hence the transition function of the prospective trivializations are given by $g : U \cap \tilde{U} \to Gl(n; R)$

$$g(p) = \left(\frac{\partial \phi_i}{\partial x_j}\right)_{i,j=1}^{n} = D_{\varphi^{-1}(p)}\phi$$

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M was a smooth manifold, the transition map ϕ is smooth, hence g is smooth.

The Tangent Bundle

$\Phi : V \times \mathbb{R}^n \to \pi^{-1}(U)$ given by

$$\Phi(x, v) := \left(\varphi(x), \sum_{j=1}^{n} v_j \frac{\partial}{\partial x_j}\right)$$

defines a bijective map such that the transition maps between any two of them are differentiable. Therefore they form a differentiable atlas of the **tangent space**

$$TM := \coprod_{p \in M} T_p M.$$

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$$TM := \coprod_{p \in M} T_p M.$$

The trivializations $\{\Phi_\iota\}$ define a topological vector bundle. By Lemma 40 its total space TM admits a metric.
Hence TM is a manifold. The projection map $\pi : TM \to M$ w.r.t. any of the charts $(\pi^{-1}(U), \Phi, V \times \mathbb{R}^n)$ takes the form

$$\varphi^{-1} \circ \pi \circ \Phi(x, v) = x$$

and is hence smooth.

Fibres, Sections and Vector Fields

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Example: Consider $T(TM)$ – the tangent space of the tangent space of a manifold. For each $v \in TM$ there is a canonical subspace $T_{\pi(v)}M \cong T_v(T_{\pi(v)}M) \subset T_v(TM)$ – the tangent space to the fibre $T_{\pi(v)}M$.

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Then $X : TM \to T(TM)$ defined by

$$X(v) := v$$

is a smooth vector field on TM, called the **Euler field**.

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Similarly, the cotangent spaces $\{T_p^*M\}_{p\in M}$ form the **cotangent bundle**, T^*M, of M

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$\pi : T^*M \to M$ is given by

$$\pi(\alpha) = p \text{ for } \alpha \in T^*M.$$

With a differentiable chart (U, φ, V)

$$\alpha \in \pi^{-1}(U) \mapsto (\pi(\alpha), (\alpha(\frac{\partial}{\partial x_1}), ..., \alpha(\frac{\partial}{\partial x_n})))$$

provides the local trivializations.

The Cotangent Bundle

With $\phi = \tilde{\varphi}^{-1} \circ \varphi$ for differentiable charts as before the transition map $g^* : U \cap \tilde{U} \to Gl(n; \mathbb{R})$ from the trivialization related to (U, φ, V) to the trivialization related to $(\tilde{U}, \tilde{\varphi}, \tilde{V})$ is given by

$$g^*(p) = (g(p)^{-1})^T.$$

The **tautological one form**, $\theta \in \Omega^1(T^*M)$ is defined by

$$\theta_\alpha(X) := \alpha(d_p(X))$$

where $\alpha \in T_p^*M$ and $X \in T_\alpha(T^*M)$.

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Exercise: Express θ in coordinates of T^*M around α provided by a chart of M and compute its exterior derivative $d\theta$.

Subbundles

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Definition 47: Let $E \xrightarrow{\pi} M$ be a smooth vector bundle.
(1) A **subbundle of** E is a submanifold $F \subset E$ such that $\pi(F) = M$ and for any $p \in M$, $\pi^{-1}(p) \cap F \subset E_p$ is a linear subspace.
(2) The **dual (bundle)**, $E^* \xrightarrow{\pi^*} M$ of E is given by

$$E^* := \coprod_{p \in M} (E_p)^*$$

and the obvious projection map π^* together with the following trivializations: Let $\Phi : \pi^{-1}(U) \to U \times \mathbb{R}^n$ be a local trivialization of E then $(\Phi^{-1})^* : (\pi^*)^{-1}(U) \to U \times (\mathbb{R}^k)^*$ assigns

$$\alpha \in T_p^* M \mapsto \alpha \circ \Phi(p, .)^{-1} \in (\mathbb{R}^k)^*$$

Tensor Products

Definition 48: Given to vector bundles $E_i \stackrel{\pi_i}{\to} M$ their **tensor product** $E_1 \otimes E_2 \stackrel{\pi}{to} M$ is given by

$$E_1 \otimes E_2 = \coprod_{p \in M} B((E_1)^*_p, (E_2)^*_p)$$

where $B((E_1)^*_p, (E_2)^*_p)$ denotes the vector space of bilinear forms $\alpha : (E_1)^*_p \times (E_2)^*_p \to \mathbb{R}$.

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Remark: This definition is only good if the rank of the bundles is finite! Then $(E_i)_p^{**} \cong (E_i)_p$. Notice that the fibre of $E_1^* \otimes E_2^*$ is given by bilinear maps $B(E_1, E_2)$.