Differential Geometry II

Connections of Vector Bundles

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The fibre is given by

$$Hom(E, F)_p := Hom(E_p, F_p) = E_p^* \otimes F_p.$$

The Covariant Derivative

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Definition 49: A **covariant derivative** or **connection**, ∇, assigns to a smooth section $\sigma : M \to E$ and a tangent vector $X \in T_pM$ a vector $\nabla_X\sigma \in E_p = \pi^{-1}(p)$ such that for all smooth sections $\sigma, \tau : M \to E$ and functions $f : M \to \mathbb{R}$, tangent vectors $X, Y \in T_pM$ and $\lambda \in \mathbb{R}$

$\pi \circ \sigma = id_M$

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(i) $\nabla_{X+\lambda Y}\sigma = \nabla_X \sigma + \lambda \nabla_Y \sigma \quad \in E_p$

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(ii) $\nabla_X(\sigma + \tau) = \nabla_X\sigma + \nabla_Y\tau$

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(ii) $\nabla_X(\sigma + \tau) = \nabla_X \sigma + \nabla_Y \sigma$
(iii) $\nabla_X(f\sigma) = X(f)\sigma(p) + f(p)\nabla_X \sigma$ (Leibniz' rule)

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Moreover, if X is smooth vector field on M, then $\nabla_X \sigma$ is a smooth section. $\nabla_{fX}\sigma = f \cdot \nabla_X \sigma$

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Moreover, if X is smooth vector field on M, then $\nabla_X \sigma$ is a smooth section.

A section $\sigma : M \to E$ satisfying $\nabla \sigma \equiv 0$ is called **parallel section**.

$\nabla_X \sigma = 0 \quad \forall X \in TM$

The Covariant Derivative

$\tilde{\sigma} : M \to E$
$\tilde{\sigma}(p) = (p, \sigma(p))$

Examples: (1) If $E = M \times \mathbb{R}^k$, i.e. the trivial bundle, $\sigma : M \to \mathbb{R}^k$ smooth defines a section and $X \in T_pM$:

$$\nabla_X \sigma := X(\sigma) = (d\sigma_j(X))_{j=1}^k$$

defines the **trivial connection** on E.

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(2) Any $A \in \Omega^1(M, M(n; \mathbb{R}))$ by

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(3) A connection ∇ on the tangent bundle TM of a smooth manifold M is called an **affine connection**. If M is a Riemannian manifold and ∇ metric and torsion free, then it is the uniquely determined **Levi Civita connection**.

Induced Covariant Derivatives

Lemma 50: (1) Let ∇ be a connection on the vector bundle $E \xrightarrow{\pi} M$ over a manifold M. It induces a unique connection on the dual E^* via

$$\nabla^*_X \alpha(\sigma(p)) := X(\alpha(\sigma)) - \alpha_p(\nabla_X \sigma)$$

for any smooth section $\alpha : M \to E^*$, $\sigma : M \to E$, $p \in M$ and $X \in T_pM$.

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(2) Let ∇^k be connections on the vector bundles $E_k \xrightarrow{\pi_k} M$ $(k = 1, 2)$. They induce a unique connection $\nabla^\oplus$ on $E_1 \oplus E_2$ by

$$\nabla^\oplus_X(\sigma_1, \sigma_2) = (\nabla^1_X \sigma_1, \nabla^2_X \sigma_2)$$

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$$\nabla^\otimes_X(\sigma_1 \otimes \sigma_2) = (\nabla^1_X \sigma_1) \otimes \sigma_2(p) + \sigma_1(p) \otimes (\nabla^2_X \sigma_2).$$

Induced Covariant Derivatives

Proof: (3) Any smooth section $\sigma : M \to E_1 \otimes E_2$ is given:

$\forall\, p \in M \;\; \exists\, \mathcal{U} \subset M$ open, $p \in \mathcal{U}$ &

$\sigma_{1j} : \mathcal{U} \to E_1, \;\; \sigma_{2j} : \mathcal{U} \to E_2$ smooth $j = 1, \dots, N$

$$\sigma|_{\mathcal{U}} = \sum_{j=1}^{N} \sigma_{1j} \otimes \sigma_{2j}$$

(1) $E^* \to M$, ∇^* def. as above, $k := \operatorname{rk} E$

$p \in M$, $\mathcal{U} \subset M$ open, $p \in \mathcal{U}$ & $\Phi : \pi^{-1}(\mathcal{U}) \to \mathcal{U} \times \mathbb{R}^k$

$\rightsquigarrow$ $\sigma_1, \dots, \sigma_k : \mathcal{U} \to E|_{\mathcal{U}}$ sections, $\{\sigma_1(x), \dots, \sigma_k(x)\} \subset E_x$ basis $\forall\, x \in \mathcal{U}$

$\nabla^*_X \alpha \in E^*_p$ is uniquely determined by its value on

$\sigma_1(p), \dots, \sigma_k(p)$

Induced Covariant Derivatives

Splittings of TE

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Thus for $p \in M$

$$T_p M \oplus E_p \cong T_p E$$

where $E_p \cong T_{0_p} E_p$.

The Connection 1–Form

Definition 51: Let ∇ be a connection on a vector bundle $E \xrightarrow{\pi} M$ of rank k, $\Phi : \pi^{-1}(U) \to U \times \mathbb{R}^k$ a smooth trivialization, $U \subset M$, open. Then

$$A := \Phi \circ \nabla \circ \Phi^{-1} \in \Omega^1(U, M(k, \mathbb{R})).$$

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A is called the **connection 1–form** of ∇ w.r.t. the trivialization Φ.

Given a section $\sigma : U \to E$ we have

$$\Phi(\nabla_X \sigma) = X(\Phi \circ \sigma)) + A(X)(\Phi(\sigma(p)))$$

for any tangent vector $X \in T_p M$ or

$$\Phi(\nabla \sigma) = d(\Phi \circ \sigma)) + A(\Phi \circ \sigma)).$$

The Connection 1–Form

Lemma 52: Given another trivialization $\Psi : \pi^{-1} \to V \times \mathbb{R}^k$, and $\Psi \circ \Phi^{-1} : (U \cap V) \times \mathbb{R}^k \to (U \cap V) \times \mathbb{R}^k$ given by the transition function $g : U \cap V \to Gl(n; \mathbb{R})$, $\Psi \circ \Phi^{-1}(p, v) = (p, g(p)v)$.

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Lemma 52: Given another trivialization $\Psi : \pi^{-1} \to V \times \mathbb{R}^k$, and $\Psi \circ \Phi^{-1} : (U \cap V) \times \mathbb{R}^k \to (U \cap V) \times \mathbb{R}^k$ given by the transition function $g : U \cap V \to Gl(n; \mathbb{R})$, $\Psi \circ \Phi^{-1}(p, v) = (p, g(p)v)$.

Then for the two connection 1–forms A_Φ and A_Ψ we have

$$A_\Phi = g^{-1} dg + g^{-1} A_\Psi g.$$

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Proof:

Pull-Backs

Denote by (E, ∇) a vector bundle of rank k over a manifold M equipped with a connection ∇. Let $g : P \to M$ be a smooth map between manifolds (with boundary).

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Definition 53: (1) The **pull back**, g^*E, of the bundle E is the vector bundle

$$g^*E = \coprod_{p \in P} E_{g(p)}$$

where a trivialization $\Phi : \pi^{-1}(U) \to U \times \mathbb{R}^k$ of E over $U \subset M$ induces a trivialization $\Phi_g : g^{-1}(\pi^{-1}(U)) \to g^{-1}(U) \times \mathbb{R}^k$ via

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for $e \in (g^*E)_p$.

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$$\Phi_g(e) = (p, \mathrm{pr}_{\mathbb{R}^k} \Phi(e))$$

for $e \in (g^*E)_p$.

(2) The **pull back**, ∇^g, of the connection ∇ is given w.r.t. the trivialization by the connection 1–form

$$A_\Phi^g := g^* A_\Phi .$$

Parallel Transport

∇^g is well–defined, i.e. independent of the local trivialization Φ of E.

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Let $\gamma : [a, b] \to M$ be a smooth curve connecting $p = \gamma(a)$ and $q = \gamma(b)$.

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Let $\gamma : [a, b] \to M$ be a smooth curve connecting $p = \gamma(a)$ and $q = \gamma(b)$.

Proposition 54: For any $v \in E_p$ there is a unique section $\sigma : [a, b] \to \gamma^* E$, with $\sigma(a) = v$ which is parallel:

$$\nabla^\gamma \sigma \equiv 0.$$

Parallel Transport

Horizontal Tangent Spaces