Homework Set 10

Floer Homology 2019

Problem 1
From the lecture:
(i) Prove that $\mathrm{ind}(L_1) = k_- - k_+$ for the cases $k_- \equiv n \mod 2, k_+ \equiv (n-1) \mod 2$, $k_- \equiv n-1 \mod 2, k_+ \equiv n \mod 2$ and $k_- \equiv k_+ \equiv (n-1) \mod 2$.
(ii) Complete the proof of the index formula for the case $n = 1$ and $k_- \neq 0$ or $k_+ \neq 0$ and even. One way of doing it is to take the direct sum of the original L with $L' = \partial_s + J_0 \partial t + S'$ where S' is a small diagonal metric (whether definite or indefinite does not matter).

Problem 2
Functional analysis from lecture revisited.
Consider the operator $J_0 \partial t : W^{1,2}(S^1, \mathbb{R}^{2n}) \to L^2(S^1, \mathbb{R}^{2n})$
(i) Show that its image is given by

$$\mathrm{im}(J_0 \partial t) = \{\sigma \in L^2(S^1, \mathbb{R}^{2n}) \mid \int_{S^1} \sigma(t) dt = 0\}$$

while its kernel consists of the constant maps. Deduce that this operator is Fredholm, with index $\mathrm{ind}(J_0 \partial t) = 0$
(ii) For $A = J_0 \partial_t + S(.)$ with smooth $S : S^1 \to \mathrm{Symm}(2n)$ explain why A is Fredholm. What is its index?
(iii) Let S be non-degenerate. Show that A is an isomorphism.
(iv) Explain why A^{-1} has a discrete spectrum which accumulates with multiplicity only at 0.

Problem 3
Let $\{\lambda_k\}_{k \in \mathbb{Z} \setminus \{0\}}$ be the set of eigenvalues of A from Problem 2 (ordered, such that $\lambda_k > 0$ iff $k > 0$ and φ_k the corresponding system of eigenvectors, orthonormal w.r.t. the L^2-metric. Then $Y \in L^2$ iff

$$Y = \sum_{k \in \mathbb{Z} \setminus \{0\}} y_k \varphi_k$$

with the sequence of real numbers satisfying

$$\sum_{k \in \mathbb{Z} \setminus \{0\}} |y_k|^2 < \infty.$$

$(i)^*$ Show that $Y \in W^{1,2}$ iff

$$\sum_{k \in \mathbb{Z} \setminus \{0\}} \lambda_k^2 |y_k|^2 < \infty.$$

(ii) Show that for $L = \partial_s + A : W^{1,2}(\mathbb{R} \times S^1; \mathbb{R}^{2n}) \to L^2(\mathbb{R} \times S^1; \mathbb{R}^{2n})$ $\mathrm{Ker} L = 0$ (see lecture)
(iii) For $Z \in L^2(\mathbb{R} \times S^1; \mathbb{R}^{2n})$ $Z(s,t) = \sum_{k \in \mathbb{Z} \setminus \{0\}} z_k(s) \varphi_k(t)$ we constructed

$$y_k(s) := \int_{-\infty}^{s} e^{\lambda_k (s' - s)} z_k(s') ds'$$

for $\lambda_k > 0$ and

$$y_k(s) := -\int_{s}^{\infty} e^{\lambda_k (s' - s)} z_k(s') ds'$$

for $\lambda_k < 0$. Show that $Y(s,t) := \sum_{k \in \mathbb{Z} \setminus \{0\}} y_k(s) \varphi(t)$ defines an element in $L^2(\mathbb{R} \times S^1; \mathbb{R}^{2n})$.
(iv) Show that Y from (iii) is an element of $W^{1,2}(\mathbb{R} \times S^1; \mathbb{R}^{2n})$. Using this, conclude that $DY = Z$.

Problem 4
Motivational functional analysis for the proof of property (1) of Fredholm operators.
Let B_0 be a Banach space of infinite dimension. Let $\alpha : B_0 \to \mathbb{R}$ be linear. Equip $B = B_0 \oplus \mathbb{R}$ with the product norm which turns N into a Banach space. The graph of α, $\Gamma := \{(v, \alpha(v) \mid v \in B_0\} \subset B$, is a linear subspace of codimension 1: $\Gamma + (\{0\} \times \mathbb{R}) = B$ and $\Gamma \cap (\{0\} \times \mathbb{R}) = \{0\}$.
(i) Show that Γ is closed iff α is continuous.
(ii) Construct an example of a Banach space B_0 and a linear map $\alpha : B_0 \to \mathbb{R}$ which is not continuous.
(iii) Wonder about it. ☺