Problem 1
(a) Let \((M, \omega)\) be a closed symplectic manifold. Show that for all positive integers \(k \leq \dim M/2\)

\[\omega^k = \omega \wedge \ldots \wedge \omega\]

is not exact. What does it mean for the homology of \(M\)?

(b) Which of the following closed manifolds cannot carry a symplectic structure: spheres, tori, Klein-bottle, \(S^3 \times S^1\), \(\mathbb{C}P^2 \mathbb{C}P^2\), \(\mathbb{C}P^2 \mathbb{C}P^2\), \(\mathbb{C}P^2 \mathbb{C}P^2\). \(\sharp\) denotes the connected sum of manifolds, \(\mathbb{C}P^2\) is the complex projective plane oriented as a complex manifold, \(\overline{\mathbb{C}P^2}\) is the same space with the opposite orientation. Remark: Try to answer the question for as many spaces as possible. The hard cases are interesting and you will get a hint in class what kind of mathematics is needed to decide it.

Problem 2
(a) Discuss that \(\tilde{\Omega}_0(M)\) as defined in class is the universal cover of the space of contractible loops \(\Omega_0(M)\). Explain why the fundamental group of the latter is \(\pi_1(\Omega_0(M)) \cong \pi_2(M)\).

(b) Prove that the Hamiltonian action functional \(A_H: \tilde{\Omega}_0(M) \to \mathbb{R}\) is well-defined.

(c) Generalise \(A_H\) to the universal covers of all connected components of the space of loops in \(M, \Omega(M)\).

Problem 3
Show the following identity from the lecture

\[\frac{d}{dt} \left( A_H(\gamma, u) - \int_{D^2} u^* \omega \right) = \frac{d}{dt} \left( \int_{-1}^1 \int_0^1 \omega(\frac{d}{ds} \gamma_s(t), \gamma'_s(t)) dt ds - \int_0^1 H(\gamma(t), z) dt \right).\]

Problem 4
(a) Give examples of symplectically aspherical, closed symplectic manifolds.

(b) Explain why the action functional \(A_H: \Omega_0(M) \to \mathbb{R}\) descends to a functional on \(\Omega_0(M)\) in the case of a symplectically aspherical manifold.

(c) Show that even in this case, \(A_H\) is unbounded from above and below. Hence minimising sequences cannot converge to a critical point in any reasonable way.

Problem 5
(a) Let \(f: M \to \mathbb{R}\) be a differentiable function. Recall the definition of the Hessian of \(f\) in a critical point \(p\). Explain that this is only well-defined since \(p\) is critical.

(b) Let \(f\) be a Morse function and \(X\) be a differentiable vector–field on \(M\) such that \(X(f) \geq 0\) and equality holds exactly at the critical points of \(f\) (in the lecture this was called gradient-like, but for these it is usually asked for stronger conditions). Check the following hypothesis: \(X\) vanishes exactly at the critical points of \(f\).

Problem 6
(a) Let \(U, S \subset M\) be two differentiable submanifolds of \(M\) which intersect transversely. Show that the intersection \(U \cap S\) is a differentiable submanifold of \(M\) of dimension

\[\dim U \cap S = \dim U + \dim S - \dim M.\]

Find a formula for the codimension, where \(\text{codim} X = \dim M - \dim X\) for a submanifold of \(M\).

(b) Show the identity from the lecture by providing the bijection

\[U_p \cap S_q = \{ \gamma: \mathbb{R} \to M | \gamma'(t) = -X(\gamma(t)), \lim_{t \to \infty} \gamma(t) = q, \lim_{t \to -\infty} \gamma(t) = p \}.\]