Homework Set 4

Floer Homology 2019

Problem 1
(a) Sketch a Morse function and (the flow of) a gradient-like vector field on the 2–torus, Klein bottle, real projective plain using the combinatorial description of these spaces.
(b) How can one see in (a) whether the vector field is Morse-Smale? Determine the Morse-Smale-Witten complex and compute its homology in this case.

Problem 2
(a) Recall the definition of the projective space $\mathbb{C}P^n$.
(b) Fix real numbers $a_0 < a_1 < ...a_n$. Show that the function

$$f(z_0, ..., z_n) := \frac{a_0|z_0|^2 + a_1|z_1|^2 + ... + a_n|z_n|^2}{|z_0|^2 + |z_1|^2 + ... + |z_n|^2}$$

defines a Morse function on $\mathbb{C}P^n$. Determine all its critical points and their Morse indices. What does it mean for its homology?
$(c)^*$ Can you sketch the arguments leading to the Morse-Smale-Witten chain complex for a similar function on $\mathbb{R}P^n$ and an appropriate gradient-like vector field?

Problem 3
(a) Recall the notion of a chain map of chain complexes. Show that it induces a map of homologies.
(b) Recall the notion of a chain homotopy between chain maps. Show that chain homotopic chain maps induce the same homomorphisms between homologies.
(c) In the lecture it is claimed that for any two Morse-Smale pairs (f_k, X_k) $k = 0, 1$ to a homotopy $\{(f_t, X_t)\}_{t\in[0,1]}$ between them which satisfies another Morse-Smale condition one can assign a chain map between their Morse-Smale-Witten complexes with the following properties: (i) Any two such chain maps are chain homotopic. (ii) For three Morse-Smale pairs (f_k, X_k) $k \in \{0; 1; 2\}$ and Morse-Smale homotopies $\{(f_t, X_t)\}_{t\in[0,1]}$ and $\{(f_t, X_t)\}_{t\in[1,2]}$ between them there is a Morse-Smale homotopy $\{(g_t, Y_t\}_{t\in[0,1]}$ with $(g_0, Y_0) = (f_0, X_0)$ and $(g_1, Y_1) = (f_2, X_2)$ whose assigned chain map coincides with the composition of the chain maps assigned to the two former homotopies.
Show that this implies that for any two Morse-Smale pairs $(f_0, X_0), (f_1, X_1)$ there is a unique isomorphism of the homologies of their assigned Morse-Smale-Witten complexes.

Problem 4
(a) Determine the Hessian of the Hamiltonian action functional $\mathcal{A}_{H_t}$ at a critical point. This was discussed in the lecture.
(b) Show that this Hessian is of the following form: If $\gamma : [0, 1] \to M$ with $\gamma(0) = \gamma(1)$ is a solution of $\gamma'(t) = X_{H_t}(\gamma(t))$, i.e. a critical point of $\mathcal{A}_{H_t}$ then w.r.t. a trivialization of the symplectic vector bundle $\gamma^*TM \cong [0, 1] \times (\mathbb{R}^{2n}, \omega_0)$ (coinciding at 0 and 1), the Hessian takes the form

$$H := \mathrm{H}ess_p(\xi, \eta) = \int_0^1 \langle \mathrm{i}\xi'(t) - S(t)\xi(t), \eta(t)\rangle dt$$

where ω_0 denotes the standard symplectic form, i denotes the complex multiplication after identifying $\mathbb{R}^{2n} \cong \mathbb{C}^n$ in such a way that $\langle v, \mathrm{i}w\rangle = -\omega_0(v, w)$ and $S : [0, 1] \to M(2n, \mathbb{R})$ is a periodic family of symmetric matrices. The vector fields ξ, η along γ are treated as 1–periodic maps $\xi, \eta : [0, 1] \to \mathbb{R}^{2n}$ using the trivialisation.
(c) Show for an arbitrary family S as in (b), that the bilinear form is symmetric, i.e. do not use that it was the Hessian of $\mathcal{A}_{H_t}$ at a critical point.
(d) Explain, why $\{\dim V \mid V \subset C^1_{\mathrm{per}}([0, 1], \mathbb{R}^{2n}), H|_V$ is negative definite $\}$ is unbounded and likewise for replacing "negative definite" by "positive definite". Hint: What are (the) eigenfunctions if $\xi \mapsto \mathrm{i}\xi'(t)$? Take H to be the span of an appropriate set of such eigenfunctions.