Homework Set 9

Floer Homology 2019

Problem 1
(i) With u, v, ξ as in Problem 3 of Set 8 show that

$$\frac{\partial}{\partial \tau}|_{\tau=0} P^{-1}_{u,\exp_u(\tau\xi)}(\frac{\partial}{\partial s}\exp_u(\tau\xi)) = \nabla^u_s \xi$$

where the covariant derivative ∇^u on u^*TM is the pull-back of the Levi-Civita connection.
(ii) Derive the following formula:

$$D_u\overline{\partial}_{H,J}\xi := \frac{\partial}{\partial \tau}|_{\tau=0} P^{-1}_{u,\exp_u(\tau\xi)}(\overline{\partial}_{H,J}\exp_u(\tau\xi)) = \nabla^u_s\xi + (J\circ u)(\nabla^u_t\xi - \nabla_\xi X_H) + (\nabla_\xi J\circ u)(\partial_t u - X_H).$$

(iii) Let $u \in \mathcal{B}^p(x,y)$ and x and y are periodic solutions of Hamilton's equation. Show that after (suitable?) trivialization $u^1 * TM \cong \mathbb{R}\times S^1 \times \mathbb{C}^n$ as complex vector bundles the linear operator in (ii) takes the form

$$D_u\overline{\partial}_{H,J}\xi(s,t) = \frac{\partial \xi}{\partial_s}(s,t) + J_0\frac{\partial\xi}{\partial_t}(s,t) + S(s,t)\xi(s,t)$$

where J_0 denotes the multiplication by i, $S : \mathbb{R}\times S^1 \to M(2n,\mathbb{R})$ with the identification $\mathbb{C}^n \cong \mathbb{R}^{2n}$. Moreover, $\lim_{s\to\pm\infty} S(s,t) = S^{\pm}(t)$ uniformly in t and the solutions $R^{\pm} : [0,1] \to M(2n;\mathbb{R})$ of $\dot{R}^{\pm} = J_0 S^{\pm}(t)R^{\pm}(t)$ define paths of symplectic matrices conjugate to the linearizations of the Hamiltonian flow Φ_t at $x(0)$ and $y(0)$. In particular, if x and y are non-degenerate, then $1 \notin \mathrm{spec}(R^{\pm}(1))$.

Problem 2
Remember the definitions $\mathbb{E} := \mathcal{E}\times\mathcal{J}$, $\mathbb{B} := \mathcal{B}\times\mathcal{J}$ and recall the section $\overline{\partial}_H(u,J) := \overline{\partial}_{H,J}u$ of the obvious bundle $\mathbb{E}\to\mathbb{B}$. Then we defined $\mathbb{M} := \overline{\partial}_H^{-1}(0)$ and the differentiable map $\pi : \mathbb{M}\to\mathcal{J}$ given by $\pi(u,J) = J$. Let $(u,J)\in\mathbb{M}$.

(i) Show that $\mathrm{Ker}D_{u,J}\pi \cong \mathrm{Ker}D_u\overline{\partial}_{H,J}$
(ii) Construct an isomorphism $\mathrm{Coker}(D_{u,J}\pi) \cong \mathrm{CoKer}(D_u\overline{\partial}_{H,J})$
(iii) Discuss the connection between the Fredholm property of $D_u\overline{\partial}_{H,J}$ and of $D_{u,J}\pi$ and the relation of their indices.
(iv) What can be concluded if $D_{u,J}\pi$ is surjective?

Problem 3
(i) Let $L : E \to F$ be a bounded linear map. Show: If there is a bounded linear map $L' : F \to E$ such that $LL' - \mathrm{id}_F$ and $L'L - \mathrm{id}_E$ are finite-dimensional, then L is Fredholm.
(ii) Let $v \in E$. Find an example of a subspace $E_0 \subset E$ such that $E = E_0 \oplus \mathbb{R}v$ algebraically but E_0 is not closed or prove the opposite statement.
(iii) Let $L, L_0 : E \to F$, and $L'_0 : F \to E$ as in (i) for L_0 instead of L. Suppose $\|L - L_0\| < \frac{1}{\|L'_0\|}$. Explain why $(\mathrm{id}_F + (L-L_0)L'_0)^{-1}$ and $(\mathrm{id}_E + L'_0(L-L_0))^{-1}$ exist. Moreover, deduce that $L' := L'_0(\mathrm{id}_F + (L-L_0)L'_0)^{-1}$ satisfies (i) for L.

Problem 4

(i) Let $H : S^2 \to \mathbb{R}$ be the height function with exactly twp critical points: minimum x and maximum y. The Hamiltonian flow is simply given by rotation of S^2, the time-one map is the rotation whose size is determined by H and the symplectic form ω (the volume form). For all but a discrete set of c's the only 1-periodic solutions to Hamilton's equation fro cH are the stationary x and y. If $c > 0$ and sufficiently small the Conley-Zehnder index of x is $\mu(x) = -1$ and $\mu(y) = 1$. This changes for $c <=$ and $c > 0$ and large. How does this fit with claimed invariance of Floer homology from the data given.

(ii) Let $f : T^2 \to \mathbb{R}$ be a Morse function with exactly one maximum x, two non-extremal critical points y_1, y_2 and one minimum z. How do the Conley-Zehnder indices of these considered as 1–periodic solutions of Hamilton's equation for cf change for varying c? What does it mean for teh Hamiltonian system?