# Homework 1

## Topology II

### Winter 2016/17

**Problem 1 (Five-Lemma)**
Consider the following commutative diagram of abelean groups:

$$\begin{array}{ccccccccc}
A & \xrightarrow{f} & B & \xrightarrow{g} & C & \xrightarrow{h} & D & \xrightarrow{k} & E \\
\downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle\epsilon} \\
A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' & \xrightarrow{h'} & D' & \xrightarrow{k'} & E'
\end{array}$$

Show that if the rows are exact, i.e. $\mathrm{Ker}g = \mathrm{Im}f$, $\mathrm{Ker}h = \mathrm{Im}g$, $\mathrm{Ker}k = \mathrm{Im}h$ (and analogously for die lower row) and $\beta$ und $\delta$ are isomorphisms, $\alpha$ is surjectiv and $\delta$ is injectiv, then $\gamma$ is also an isomorphism.

**Problem 2**
Let $f : X \to Y$ be a continuous map between topological spaces $X, Y$. Let $f_n : C_n(X) \to C_n(Y)$ be the homomorphism between the chain groups of $X$ and $Y$, respectively, defined on the generators $\sigma : \Delta^n \to X$

$$f_n(\sigma) = f \circ \sigma : \Delta^n \to Y.$$

(1) Show that $f_n$ is a chain map, i.e. $f_n \circ \partial_{n+1} = \partial_{n+1} \circ f_{n+1}$ for all $n \in \mathbb{N}$.
(2) Explain how this implies, that $(f_n)$ induce a well-defined homomorphism $f_{*n} : H_n(X) \to H_n(Y)$.
(3) Derive the functoriality of this construction, i.e. $(f \circ g)_* = f_* \circ g_*$, where $g : X \to Y$ and $f : Y \to Z$ are continuous maps.
(4) Show that homeomorphic spaces $X$ and $Y$ have isomorphic homologies.

**Problem 3** Show the inclusion $p \mapsto q$ for $q \in U$ induces an isomorphism (on all degrees)

$$H_*(\{p\}) \cong H_*(U)$$

for any starshaped set $U \subset \mathbb{R}^n$ without using the homotopy property of singular homology.
Hint: Consider the homomorphism $c_n : C_n(U) \to C_{n+1}(U)$ given on a generator $\sigma : \Delta^n \to U$ by $c_n(\sigma) : \Delta^{n+1} \to U$ via

$$c_n(\sigma)(t_0, t_1, ..., t_{n+1}) := t_{n+1}q + (1 - t_{n+1})\sigma(\frac{t_0}{1 - t_{n+1}}, ..., \frac{t_n}{1 - t_{n+1}})$$

if $t_{n+1} < 1$ and $c(\sigma)(0, ..., 0, 1) = q$. Check that $c_n(\sigma)$ is continous. Derive the formulas

$$\partial_{n+1} \circ c_n - c_{n-1} \circ \partial_n = \mathrm{id}.$$

for $n > 0$ and

$$\partial_1 \circ c_0 = \mathrm{id} - \sigma_q$$

where by $\sigma_q : \Delta^0 \to U$ we denote the map $\sigma(1) = q$. Conclude the claim.