Asymptotic Enumeration of Unlabelled Outerplanar Graphs

Diploma Thesis

Stefan Vigerske

Advisor: Dr. Mihyun Kang

Forschungsseminar “Algorithmen und Komplexität”

Institut für Informatik
Humboldt-Universität zu Berlin

11.11.2005

Unlabelled outerplanar graphs

Definitions and Notations

Definition (Outerplanar graph)

A graph is outerplanar when it can be embedded in the plane, such that every vertex lies on the outer face.

$\Rightarrow$ Two-connected outerplanar = Dissection of a convex polygon

Unlabelled outerplanar graphs

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Unlabelled graphs

- $\Gamma(G)$: automorphism group of the unlabelled graph G
- vertices v, w in G similar $\Leftrightarrow \exists \alpha \in \Gamma(G)$: $v = \alpha(w)$
- orbits: equivalence classes of similar vertices

Unlabelled outerplanar graphs

Questions to answer here...

Exact numbers

The number of (two-connected, connected) outerplanar graphs on n vertices is exactly ... (computable in polynomial time)

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Growth constants

The number of (two-connected, connected) outerplanar graphs on n vertices is approximately ...

Unlabelled outerplanar graphs

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Typical properties of a random outerplanar graph

- Connectedness, nr. of components, nr. of isolated vertices
- Chromatic number
- Number of edges

Unlabelled outerplanar graphs

What is known...

Generating outerplanar uniformly at random [Bodirsky, Kang '03]

recursive counting formulas for labelled outerplanar graphs and unlabelled rooted connected outerplanar graphs

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On general dissections of a polygon [Read '78]

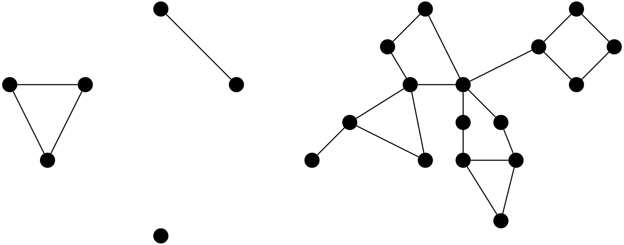
generating functions and exact numbers for unlabelled dissections

Overview

Gliederung

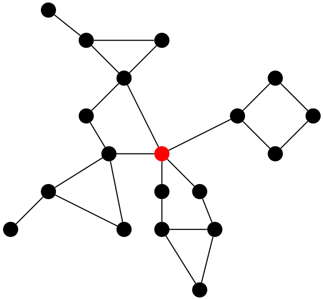
Outerplanar graphs: $\mathcal{G}$

An outerplanar graph is a collection (multiset) of connected outerplanar graphs...



(Rooted) connected outerplanar graphs: $\mathcal{C}$ ($\hat{\mathcal{C}}$)

Cutvertex-rooted connected outerplanar graph



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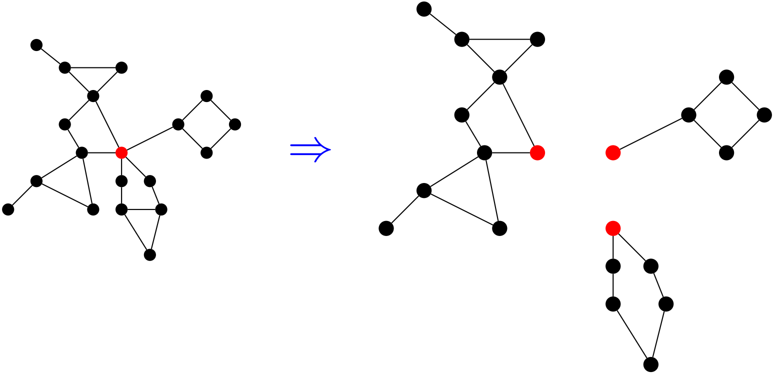
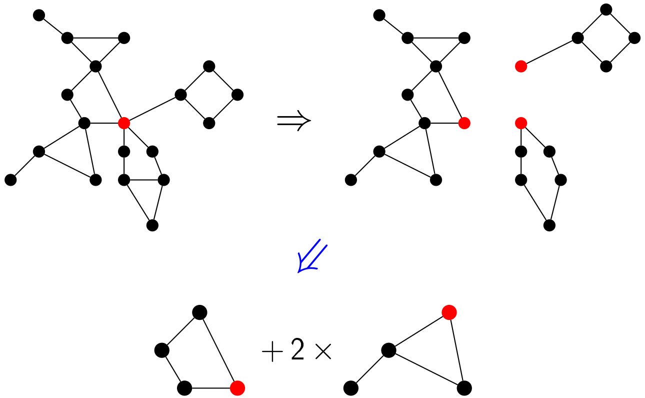
Decomposition in non-cutvertex-rooted outerplanar graphs



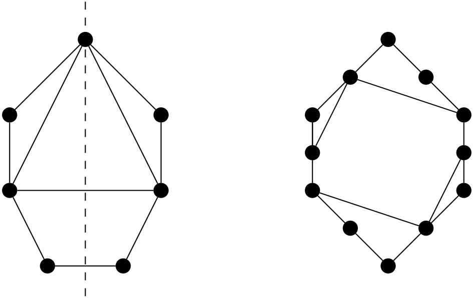
(Rooted) connected outerplanar graphs: $\mathcal{C}$ ($\hat{\mathcal{C}}$)



Decomposition in rooted dissection and rooted connected outerplanar graphs

(Rooted) two-connected outerplanar graphs: $\mathcal{D}$ ($\hat{\mathcal{D}}$)

- Unique Hamiltonian cycle ⇒ Unique embedding ⇒ Dissection of a convex polygon
- Automorphisms: reflections and rotations



- R. C. Read 1978: On general dissections of a polygon

The cycle index

- Ordinary generating function: $G(x) = \sum_n g_n x^n = \sum_{G \in \mathcal{G}} x^{|G|}$
- Idea: integrate information about the automorphism group $\Gamma(G)$ into the "generating function"

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- Idea: integrate information about the automorphism group $\Gamma(G)$ into the "generating function"

Definition (Polya 1937)

Let A be a group of permutations on n objects. The *cycle index* of A in the formal variables $s_1, s_2, \ldots, s_n$ is

$$Z(A) := \frac{1}{|A|} \sum_{\alpha \in A} \prod_{k=1}^{n} s_k^{j_k(\alpha)}.$$

$j_k(\alpha)$ is the number of cycles of length k in α.

$$\underbrace{(1\,2)}_{s_2}\underbrace{(3\,6)}_{s_2}\underbrace{(4\,5\,7)}_{s_3}\underbrace{(8)}_{s_1} \quad \Rightarrow \quad s_1\, s_2^2\, s_3$$

Cycle index $\Rightarrow$ Generating function

Definition (Polya 1937)

Let A be a group of permutations on n objects. The *cycle index* of A in the formal variables s_1, s_2, ..., s_n is

$$Z(A) := Z(A; s_1, s_2, \ldots) := \frac{1}{|A|} \sum_{\alpha \in A} \prod_{k=1}^{n} s_k^{j_k(\alpha)}.$$

$j_k(\alpha)$ is the number of cycles of length k in α.

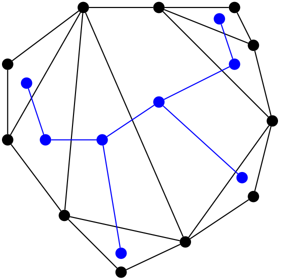
- cycle index sum of $\mathcal{G}$: $Z(\mathcal{G}) := \sum_{G \in \mathcal{G}} Z(\Gamma(G))$
- substitute $s_k \mapsto x^k$: $Z(\Gamma(G)) \mapsto x^{|G|}$

$$\Rightarrow Z(\mathcal{G}; x, x^2, x^3, \ldots) = \sum_{G \in \mathcal{G}} x^{|G|} = G(x)$$

Unrooted dissections

How to count unrooted dissections?

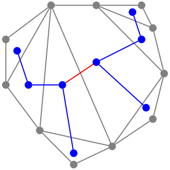
Observation: The dual graph of a dissection is a tree.



Unrooted dissections

Dissimilarity characteristic theorem for trees

1 = #dissimilar vertices - #dissimilar edges + #symmetry-edges



$1 = 4 - 4 + 1$

Unrooted dissections

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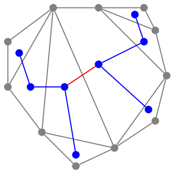

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Dissimilarity characteristic theorem for trees (cycle index version)

Z(tree) = Z(vertex-rooted trees with root-vertex $\notin$ symmetry-edge) - Z(edge-rooted trees) + 2 Z(symmetry-edge-rooted trees)

Unrooted dissections

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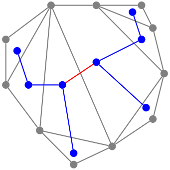

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Cycle index sum for unrooted dissections

$Z(\mathcal{D})$ = Z(face-rooted diss. with no symmetry-edge at root-face) - Z(inner-edge-rooted diss.) + 2 Z(symmetry-edge-rooted diss.)

Rooted dissections

- $\mathcal{H}$ a family of labelled graphs, $H(x)$ the e.g.f.:

$$\hat{H}(x) = x \frac{\partial}{\partial x} H(x)$$

- Unlabelled graphs $\Rightarrow$ double counting

Rooted dissections

- $\mathcal{H}$ a family of labelled graphs, $H(x)$ the e.g.f.:

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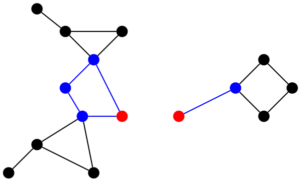
- Unlabelled graphs $\Rightarrow$ double counting
- But, for the cycle index sum:

$$Z(\hat{\mathcal{D}}) = s_1 \frac{\partial}{\partial s_1} Z(\mathcal{D})$$

Rooted connected outerplanar graphs ($\hat{\mathcal{C}}$)



Rooted at a non-cutvertex:

- Take a rooted dissection: $Z(\hat{\mathcal{D}})$
- Replace each vertex other than the root by a rooted connected graph: $Z(\hat{\mathcal{C}})$

$$\frac{s_1}{Z(\hat{\mathcal{C}})} Z\left(\hat{\mathcal{D}}; Z\left(\hat{\mathcal{C}}\right), Z\left(\hat{\mathcal{C}}; s_2, s_4, \dots\right), \dots\right)$$

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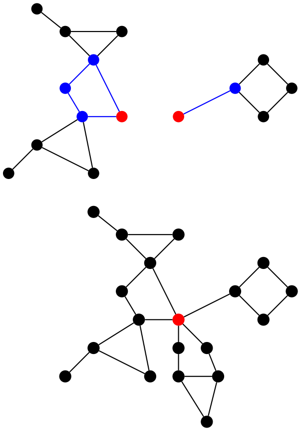
Rooted at a cutvertex:

- Join k non-cutvertex-rooted graphs on their root: $s_1 Z\left(S_k; \frac{1}{Z(\hat{\mathcal{C}})} Z\left(\hat{\mathcal{D}}; Z\left(\hat{\mathcal{C}}\right)\right)\right)$
  S_k ... symmetric group on k objects



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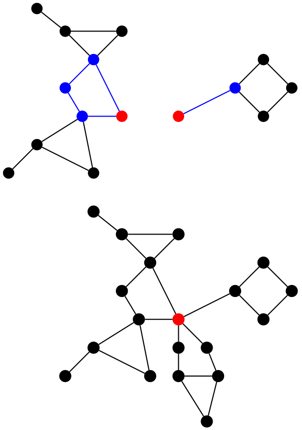

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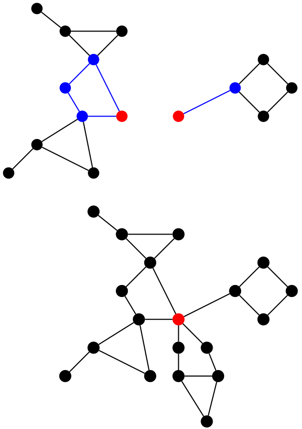
$$\Rightarrow Z(\hat{\mathcal{C}}) = \frac{s_1}{Z(\hat{\mathcal{C}})} Z\left(\hat{\mathcal{D}}; Z(\hat{\mathcal{C}})\right) + s_1 \sum_{k \geq 2} Z\left(S_k; \frac{1}{Z(\hat{\mathcal{C}})} Z\left(\hat{\mathcal{D}}; Z(\hat{\mathcal{C}})\right)\right)$$

Rooted connected outerplanar graphs ($\hat{\mathcal{C}}$)



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$$\Rightarrow Z(\hat{\mathcal{C}}) = s_1 \exp \sum_{k \geq 1} \frac{1}{k} \frac{1}{Z(\hat{\mathcal{C}}; s_k)} Z\left(\hat{\mathcal{D}}; Z(\hat{\mathcal{C}}; s_k)\right)$$

Connected outerplanar graphs ($\mathcal{C}$)

$$Z(\hat{\mathcal{C}}) = s_1 \frac{\partial}{\partial s_1} Z(\mathcal{C})$$

Hence,

$$Z(\mathcal{C}) = \int_0^{s_1} \frac{1}{s_1} Z(\hat{\mathcal{C}}) ds_1 + Z(\mathcal{C})\,|_{s_1=0}\ .$$

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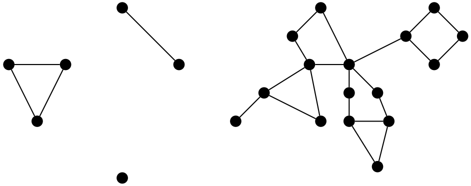
Using the known (implicit) formula for $Z(\hat{\mathcal{C}})$, we can show

$$Z(\mathcal{C}) = Z(\hat{\mathcal{C}}) + Z\left(\mathcal{D}; Z(\hat{\mathcal{C}})\right) - Z\left(\hat{\mathcal{D}}; Z(\hat{\mathcal{C}})\right).$$

Outerplanar graphs ($\mathcal{G}$)



Outerplanar graph on k components = multiset of k connected outerplanar graphs:

$$Z\left(S_k; Z(\mathcal{C})\right)$$

Hence,

$$Z\left(\mathcal{G}\right) = \sum_{k\geq 1} Z\left(S_k; Z\left(\mathcal{C}\right)\right) = \exp\left(\sum_{k\geq 1} \frac{1}{k} Z\left(\mathcal{C}; s_k, s_{2k}, \ldots\right)\right) - 1$$

Ordinary generating functions

$$\hat{D}(x) = \frac{1}{8x^2}\left(1 + x - 3x^2 - 2x^3 + x^4 - (1+x)\sqrt{x^4 - 6x^2 + 1} - x^3\sqrt{x^2 - 6x + 1}\right)$$

$$D(x) = -\frac{1}{2}\sum_{d}\frac{\varphi(d)}{d}\log\left(\frac{1}{4}\left(3 - x^d + \sqrt{x^{2d} - 6x^d + 1}\right)\right) + \frac{x^2}{8} - \frac{x}{4} - \frac{5}{16}$$
$$+ \frac{1}{8x} + \frac{1}{16x^2} + \frac{3-x}{16}\sqrt{x^2 - 6x + 1} - \frac{1 + 2x + x^2}{16x^2}\sqrt{x^4 - 6x^2 + 1}$$

$$\hat{C}(x) = x\exp\left(\sum_{k\geq 1}\frac{Z\left(\hat{\mathcal{D}}; \hat{C}\left(x^k\right)\right)}{k\,\hat{C}\left(x^k\right)}\right)$$

$$C(x) = \hat{C}(x) + Z\left(\mathcal{D}; \hat{C}(x)\right) - Z\left(\hat{\mathcal{D}}; \hat{C}(x)\right)$$

$$G(x) = \left(\exp\sum_{k\geq 1}\frac{1}{k}C\left(x^k\right)\right) - 1$$

Exact numbers

n	dissections	rooted connected	connected	outerplanar
1	0	1	1	1
2	1	1	1	2
3	1	3	2	4
4	2	10	5	10
5	3	40	13	25
6	9	181	46	80
7	20	918	172	277
8	75	5039	777	1150
9	262	29313	3783	5291
10	1117	177773	20074	26918
11	4783	1110517	111604	145744
12	21971	7093110	646409	828856
13	102249	46079944	3846640	4872771
14	489077	303422249	23410035	29395784
15	2370142	2020011842	144965988	180857382
16	11654465	13572042838	910898943	1130700488
17	57916324	91901417553	5794179218	7163245811
18	290693391	626498987843	37248630398	45895629266
19	1471341341	4296043185620	241676806702	296937363511
20	7504177738	29611845187329	1580880366039	1937625709854
21	38532692207	205050800043677	10416314047854	12739784808937
22	199076194985	1425764035634376	69080674190341	84331837321404
23	1034236705992	9950544932379738	460841447382976	561647630439975
24	5400337050086	69679887165423882	3090747326749823	3761221057579892
25	28329240333758	489438852540945514	20829976038652612	25314597326376883

Gliederung

Singularities and growth constants

Reference: Flajolet, Sedgewick (2005): Analytic combinatorics

- z_0 is singularity of $f(z)$ if f is not analytically continuable at z_0
- z_0 is dominant singularity of $f(z)$ if f is analytic for all z with $|z| < z_0$
- exponential growth: z_0 dominant singularity of $f(z)$

$$f_n = [x^n] f(z) \sim \theta(n)\, |z_0|^{-n} \qquad \left(\limsup_{n\to\infty} |\theta(n)|^{1/n} = 1 \right)$$

- subexponential factor: $f(z) = \left(1 - \frac{z}{z_0}\right)^{-\alpha}, \alpha \notin \{-1, -2, \ldots\}$

$$f_n = \binom{n+\alpha-1}{n} |z_0|^{-n} \sim \frac{n^{\alpha-1}}{\Gamma(\alpha)} |z_0|^{-n}$$

$\Gamma(\alpha)$ ... Gamma function

Dissections

$$\begin{aligned} D(x) &= -\frac{1}{2}\sum_{d}\frac{\varphi(d)}{d}\log\left(\frac{1}{4}\left(3-x^d+\sqrt{x^{2d}-6x^d+1}\right)\right) \\ &\quad +\frac{3-x}{16}\sqrt{x^2-6x+1}-\frac{1+2x+x^2}{16x^2}\sqrt{x^4-6x^2+1} \\ &\quad +\frac{x^2}{8}-\frac{x}{4}-\frac{5}{16}+\frac{1}{8x}+\frac{1}{16x^2} \\ &= \left(-\frac{(1-\delta x)^{3/2}}{16(x-3)}+\frac{(1-\delta x)^{3/2}}{6(x-3)^3}\right)\left(1-\frac{x}{\delta}\right)^{3/2}+A(x) \end{aligned}$$

$\delta = 3-2\sqrt{2}$ dominant singularity, $A(x)$ analytic at δ

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$$\begin{aligned} \Rightarrow d_n &\sim \left(-\frac{\left(1-\delta^2\right)^{3/2}}{16(\delta-3)}+\frac{\left(1-\delta^2\right)^{3/2}}{6(\delta-3)^3}\right)\frac{n^{-5/2}}{\Gamma\left(-\frac{3}{2}\right)}\delta^{-n} \\ &\approx 0.00596026\, n^{-5/2}\, 5.82843^n \end{aligned}$$

Rooted connected outerplanar graphs

$$\hat{C}(x) = x \exp\left(\sum_{k\geq 1} \frac{Z\left(\hat{\mathcal{D}}; \hat{C}\left(x^k\right)\right)}{k\,\hat{C}\left(x^k\right)}\right)$$

- ρ dominant singularity of $\hat{C}(x)$, $\sigma := \lim_{x\to\rho^-} \hat{C}(x)$
- dominant singularity ρ caused by non-uniqueness of inversion

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$$H(x,y) := x \exp\left(\frac{Z\left(\hat{\mathcal{D}}; y, \hat{C}\left(x^2\right)\right)}{y} + \sum_{k\geq 2} \frac{Z\left(\hat{\mathcal{D}}; \hat{C}\left(x^k\right)\right)}{k\, \hat{C}\left(x^k\right)} \right) - y$$

- $H(x,y)$ analytic for $|x|^2 < \rho$ and $|y| < \delta$, $\quad H(x, \hat{C}(x)) = 0$
- (ρ, σ) satisfy $H(\rho,\sigma) = 0$ and $\frac{\partial}{\partial y} H(\rho,\sigma) = 0$

Rooted connected outerplanar graphs

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- (ρ, σ) satisfy $H(\rho,\sigma) = 0$ and $\frac{\partial}{\partial y} H(\rho,\sigma) = 0$

$$\rho \approx 0.133269 \qquad \sigma \approx 0.170756$$

Rooted connected outerplanar graphs

$$H(x, \hat{C}(x)) = 0 \qquad H(\rho, \sigma) = 0 \qquad \frac{\partial}{\partial y} H(\rho, \sigma) = 0$$

$$\Rightarrow \quad \hat{C}(x) = \sigma - \sqrt{\frac{2\rho \frac{\partial}{\partial x} H(\rho, \sigma)}{\frac{\partial^2}{\partial y^2} H(\rho, \sigma)}} \sqrt{1 - \frac{x}{\rho}} + O\left(1 - \frac{x}{\rho}\right)$$

Rooted connected outerplanar graphs

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$$\Rightarrow \quad \hat{C}(x) = \sigma - \sqrt{\frac{2\rho \frac{\partial}{\partial x} H(\rho, \sigma)}{\frac{\partial^2}{\partial y^2} H(\rho, \sigma)}} \sqrt{1 - \frac{x}{\rho}} + O\left(1 - \frac{x}{\rho}\right)$$

Hence,

$$\hat{c}_n \sim \sqrt{\frac{2\rho \frac{\partial}{\partial x} H(\rho, \sigma)}{\frac{\partial^2}{\partial y^2} H(\rho, \sigma)}} \frac{n^{-3/2}}{\Gamma\left(-\frac{1}{2}\right)} \rho^{-n} \approx 0.00721895\, n^{-3/2}\, 7.50360^n.$$

Connected outerplanar graphs

$$C(x) = \hat{C}(x) + Z\left(\mathcal{D}; \hat{C}(x)\right) - Z\left(\hat{\mathcal{D}}; \hat{C}(x)\right)$$

asymptotic expansion of $\hat{C}(x)$ (constants $\hat{C}_1$, $\hat{C}_2$, ... computable):

$$\hat{C}(x) = \sigma + \hat{C}_1\sqrt{1-\frac{x}{\rho}} + \hat{C}_2\left(1-\frac{x}{\rho}\right) + \hat{C}_3\left(1-\frac{x}{\rho}\right)^{3/2} + \dots$$

insert asymp. expansion of $\hat{C}(x)$ into analytic expansion of $C(x)$:

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$$C_1 = 0 \quad \Rightarrow \quad c_n \sim C_3 \frac{n^{-5/2}}{\Gamma\left(-\frac{3}{2}\right)} \rho^{-n} \approx 0.00760471\, n^{-5/2}\, 7.50360^n$$

Outerplanar graphs

$$G(x) = \left(\exp \sum_{k\geq 1} \frac{1}{k} C\left(x^k\right)\right) - 1$$

$$C(x) = C(\rho) + C_2\left(1 - \frac{x}{\rho}\right) + C_3\left(1 - \frac{x}{\rho}\right)^{3/2} + \dots$$

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$$g_n \sim G_3 \frac{n^{-5/2}}{\Gamma\left(-\frac{3}{2}\right)} \rho^{-n} \approx 0.00909941\, n^{-5/2}\, 7.50360^n.$$

Quality of estimation of g_n

$$G(x) = G(\rho) + G_2\left(1 - \frac{x}{\rho}\right) + G_3\left(1 - \frac{x}{\rho}\right)^{3/2} + \ldots$$

higher accuracy by more subexponential factors in expansion of g_n

n	$s = 1$	$s = 2$	$s = 3$	exact
15	160209818	178630202	183557038	180857382
16	1014313867	1122297556	1148869016	1130700488
17	6491736667	7135183190	7281787242	7163245811
18	41945902094	45835255493	46660298692	45895629266
19	273331278755	297139390631	301863941625	296937363511
20	1794596586205	1941979912637	1969454532633	1937625709854
21	11862841341954	12784428807549	12946410448754	12739784808937
22	78898746539258	84713818025456	85680649129918	84331837321404
23	527673859045748	564667858381048	570503168710670	561647630439975
24	3547008662079210	3784115104767780	3819691077051302	3761221057579892
25	23953788441774176	25483858290731958	25702755427821773	25314597326376883

s = number of subexponential factors $G_3, G_5, G_7, \ldots$

Gliederung

Connectedness, components, isolated vertices

G a random outerplanar graph drawn uniformly at random from $\mathcal{G}_n$

Connectedness:

$$\mathbb{P}\left[G \text{ connected}\right] = \frac{c_n}{g_n} \sim \frac{C_3 n^{-5/2} \rho^{-n}}{G_3 n^{-5/2} \rho^{-n}} \approx 0.845721$$

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Number of components: $G(x,u) = \exp\left(\sum_{k\geq 1} \frac{1}{k} u^k C\left(x^k\right)\right) - 1$

$$\mathbb{E}\left[\# \text{ components in } G\right] = \frac{1}{g_n}\left[x^n\right]\frac{\partial}{\partial u} G(x,1) \sim 1 + \sum_{r\geq 1} C\left(\rho^r\right) \approx 1.17847$$

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Nr. of isolated vertices: geometrically distributed with parameter ρ (for $n \to \infty$)

$$\mathbb{P}[\# \text{ of i.v.} = k] \sim \frac{1}{1-\rho}\rho^k \qquad \mathbb{E}[\# \text{ of i.v.}] \sim \frac{\rho}{1-\rho} \approx 0.153716$$

Chromatic number

- Every outerplanar graphs is three-colourable.
- asymptotic enumeration for bipartite outerplanar graphs:

$$b_n \sim b\, n^{-5/2}\, \rho_{\mathrm{bipartite}}^{-n}, \qquad \rho_{\mathrm{bipartite}}^{-1} \approx 4.57717$$

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- Hence,

$$\rho_{\text{bipartite}}^{-1} < \rho^{-1} \quad \Rightarrow \quad \mathbb{P}\left[G \text{ is bipartite}\right] \sim \frac{b_n}{g_n} \to 0$$

- The chromatic number of a random outerplanar graph is three asymptotically almost sure.

Number of edges

- Enumeration in terms of vertices and edges: $G(x, y)$
- Asymptotic expansion:

$$G(x,y) = \sum_{k \geq 0} G_k(y) \left(1 - \frac{x}{\rho(y)}\right)^{k/2}$$

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- Perturbation of singularities $\Rightarrow$ Gaussian distribution with

$$\mathbb{E}[\#\text{ of edges}] \sim -\frac{\rho'(1)}{\rho(1)} n + O(1) \approx 1.54894n$$

$$\mathbb{V}[\#\text{ of edges}] \sim \left(-\frac{\rho''(1)}{\rho(1)} - \frac{\rho'(1)}{\rho(1)} + \left(\frac{\rho'(1)}{\rho(1)}\right)^2\right) n + O(1)$$

$$\approx 0.227504n$$

- Dissections: $\mathbb{E}[\#\text{ of edges}] \sim \left(1 + 1/\sqrt{2}\right) n \approx 1.70711n$, $\mathbb{V}[\#\text{ of edges}] \sim \sqrt{2}/8\, n \approx 0.176777n$

Conclusions

- cycle index sums and ordinary generating functions
- exact numbers in polynomial time
- growth constants and asymptotic estimates
- properties of random outerplanar graphs

	dissections	outerplanar graphs	
	(un)labelled	unlabelled	labelled
growth constant	$\delta^{-1} \approx 5.82843$	$\rho^{-1} \approx 7.50360$	$\lambda^{-1} \approx 7.32098$
$\mathbb{P}$ [connectivity]	1	0.845721	0.861667
$\mathbb{E}$ [nr. of components]	1	1.17847	-
distr. of nr. of isolated vert.	Dirac	Geom(ρ)	Po(λ)
$\mathbb{E}$ [nr. of isolated vertices]	0	0.153761	0.136593
chromatic number	3	3	3
distr. of nr. of edges	Gaussian	Gaussian	Gaussian
$\mathbb{E}$ [nr. of edges]	$1.70711n$	$1.54894n$	$1.56251n$
$\mathbb{V}$ [nr. of edges]	$0.176777n$	$0.227504n$	$0.223992n$