Problems for BMS Basic Course "Commutative Algebra"

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Hand in Feb 7th, after the 2nd lecture 4.45 p.m.

**Please solve each problem on a different sheet of paper,
and sign each sheet with your name and student ID**

Voluntary Problem Set - Exam preparation (0 + 48 points)

Problem 1 (10 pts)

Let $(A, \mathfrak{m})$ be a noetherian local ring.

(a) Show that $\bigcap_{n \geq 0} \mathfrak{m}^n = (0)$.

(b) Assume that $\mathfrak{m} = (x)$ with x being not a zero divisor. Show that each ideal of A apart from (0) is generated by a power of x.

(c) Denote the residue field as $k := A/\mathfrak{m}$. Assume that there is an injective homomorphism of rings $\varphi : k \longrightarrow A$ which endows A with the structure of a finite-dimensional k-vector space. Assume further that $\mathfrak{m}$ is generated by $x \in A$.

Show that the homomorphism $\psi : k[X] \longrightarrow A$ given by $\psi(X) = x$ is surjective and $\ker(\psi) = (X^n)$ with $n = \dim_k A$.

Problem 2 (10 pts)

Compute the kernel and the cokernel of the map given by the multiplication by n : $\mathbb{Q}/\mathbb{Z} \xrightarrow{\cdot n} \mathbb{Q}/\mathbb{Z}$ by lifting this map to the sequence $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$ and applying the Snake Lemma.

Problem 3 (minimal 0, maximal 8 pts, -1 pt for a wrong answer)

Are the following statements true or false? If false, give a counter-example.

(a) The map $A \to A[X]/(bX - 1)$, $a \mapsto \overline{a}$ is the zero map iff b is nilpotent.

(b) If $\varphi : A \longrightarrow B$ is a homomorphism of rings, then the preimage of a maximal ideal is maximal.

(c) In a principal ideal domain, every non-zero prime ideal is maximal.

(d) Any commutative ring has a quotient ring without non-zero nilpotent elements.

(e) In a factorial domain, every non-zero prime ideal is maximal.

(f) In a principal ideal domain, every primary ideal is the power of a prime ideal.

(g) If A is a ring and $a \in A$ not a zero divisor, then there is ring B and a homomorphism $\psi : A \longrightarrow B$ such that $h(a)$ is a unit.

(h) If $\varphi : A \longrightarrow B$ is a homomorphism of k-algebras, then the preimage of a maximal ideal is maximal.

Problem 4 (10 pts)

Compute $\operatorname{Ext}^k(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/m\mathbb{Z})$ for $n, m, k \in \mathbb{N}$, $n, m \geq 2$. (Hint: In terms of $\gcd(n, m)$!)

Problem 5 (10 pts)

Let A be an integral domain and $\emptyset \neq S \subseteq A$, $0 \notin S$, a multiplicatively closed set. Show that $S^{-1}A$ is not integral over A.