Problems for BMS Basic Course "Commutative Algebra"

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Hand in Nov 1st, after the 2nd lecture 4.45 p.m.

Please sign each sheet of paper with your name and student ID

2nd Problem Set (30 points)

Problem 1 (10 pts)

Let A be a commutative ring with 1. We consider the set

$$X = \mathrm{Spec}(A) := \{\mathfrak{p} \subsetneq A \mid \mathfrak{p} \text{ is a prime ideal}\}.$$

For a subset $E \subseteq A$, we define

$$V(E) := \{\mathfrak{p} \in X \mid \mathfrak{p} \supseteq E\}.$$

a) Let $\mathfrak{a}$ be an ideal generated by E in A. Show that $V(\mathfrak{a}) = V(E)$.

b) Check that the sets $V(E) \subseteq X$ fulfil the axioms of closed sets and make X into a topological space.

Problem 2 (10 pts)

a) Use the classification of finitely generated abelian groups to derive a classification of finitely generated $\mathbb{Z}$-modules up to isomorphism.

b) Let k be a field and V a k-vector space. Fix a k-endomorphism $\varphi \in \mathrm{End}_K(V)$. Then we can define on V a left multiplication with elements of $k[t]$ by setting

$$(\lambda(t), v) \mapsto \lambda(\varphi)(v) \quad (\lambda(t) \in k[t], v \in V).$$

Show that this endows V with the structure of a $k[t]$-module.

Problem 3 (10 pts)

Let A be a commutative ring with 1.

a) Show that tfae.

   i) A sequence of A-modules

   $$0 \longrightarrow N' \longrightarrow N \longrightarrow N''$$

   is exact.

   ii) For all A-modules M, the induced sequence

   $$0 \longrightarrow \operatorname{Hom}_A(M, N') \longrightarrow \operatorname{Hom}_A(M, N) \longrightarrow \operatorname{Hom}_A(M, N'')$$

   is exact.

b) Give a counterexample to the analogous claim to a) if an arrow $\longrightarrow 0$ is added on all right-hand sides.

c) Give a counterexample to the claim that if

$$0 \longrightarrow N' \longrightarrow N$$

is exact, then for all A-modules M, the induced sequence

$$\operatorname{Hom}_A(N, M) \longrightarrow \operatorname{Hom}_A(N', M) \longrightarrow 0$$

is exact.