Problems for BMS Basic Course "Commutative Algebra"

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Hand in November 8th, after the 2nd lecture 4.45 p.m.

Please sign each sheet of paper with your name and student ID

3rd Problem Set (30 points)

Problem 1 (10 pts)

Consider a commutative diagram of A-modules and A-homomorphisms with exact rows:

$$\begin{array}{ccccccc} & & M' & \longrightarrow & M & \longrightarrow & M'' & \longrightarrow & 0 \\ & & {\scriptstyle f}\downarrow & & {\scriptstyle g}\downarrow & & {\scriptstyle h}\downarrow & & \\ 0 & \longrightarrow & N' & \longrightarrow & N & \longrightarrow & N'' & & \end{array}$$

Prove:

(i) If f, h are monomorphisms, then g is a monomorphism.

(ii) If f, h are epimorphisms, then g is an epimorphism.

(iii) If, in addition, $0 \longrightarrow M' \longrightarrow M$ and $N \longrightarrow N'' \longrightarrow 0$ are exact, show that if any two of f, g, h are isomorphisms, then so is the third.
(*Hint:* Use the snake lemma.)

Problem 2 (10 pts)

Let A be a commutative ring with unit element 1. For each $f \in A$, let $D(f)$ denote the complement of the set $V(f)$ in $\operatorname{Spec}(A)$. Show that the sets $D(f)$ $(f \in A)$ are open and that they form a basis of open sets for the Zariski topology of $\operatorname{Spec}(A)$. Furthermore, show that

(i) $D(f) \cap D(g) = D(f \cdot g) \quad (f, g \in A)$.

(ii) $D(f) = \emptyset \Longleftrightarrow f$ is nilpotent.

(iii) $D(f) = X \Longleftrightarrow f \in A^{\times}$.

(iv) $D(f) = D(g) \Longleftrightarrow r(f) = r(g) \quad (f, g \in A)$.
Recall that $r(f)$ denotes the radical of the principal ideal (f).

Problem 3 (10 pts)

Let $\varphi : A \longrightarrow B$ be a homomorphism of rings, $X := \mathrm{Spec}(A)$ and $Y := \mathrm{Spec}(B)$. By the assignment $\mathfrak{p} \mapsto \varphi^{-1}(\mathfrak{p})$ $(\mathfrak{p} \in Y)$ we get a natural map $\varphi^* : Y \longrightarrow X$.
Show that "Spec" is a contravariant functor from the category of commutative rings with 1 to the category of topological spaces, i.e.:

(i) The map φ^* ist continuous (w.r.t. the topology given in problem 1 of the 2nd set). (*Hint:* Show that $(\varphi^*)^{-1}(D(f)) = D(\varphi(f))$ $(f \in A)$, cf. problem 2.)

(ii) Let $\psi : B \longrightarrow C$ be another homomorphism of rings. Then $(\psi \circ \varphi)^* = \varphi^* \circ \psi^*$ holds.

Does an analogous result hold for "Max" instead of "Spec"?