Problems for BMS Basic Course "Commutative Algebra"

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Hand in November 15th, after the 2nd lecture 4.45 p.m.

Please solve each problem on a different sheet of paper, and sign each sheet with your name and student ID

4th Problem Set (30 points)

Problem 1 (10 pts)

Let X be a non-empty topological space. X is called *irreducible* if every non-empty open subset is dense in X.

(a) Show that X is irreducible iff every pair of non-empty open sets in X has a non-empty intersection.

(b) Let $Y \subseteq X$ be a topological subspace which is irreducible. Show that the closure $\overline{Y}$ of Y in X is also irreducible.

(c) Show that every irreducible topological subspace Y of X is contained in a maximal irreducible topological subspace.

(d) Show that the maximal irreducible topological subspaces of X are closed and cover X. They are called the *irreducible components of* X.

(e) Let A be a commutative ring with 1. Show that the irreducible components of $\mathrm{Spec}(A)$ are the closed sets $V(\mathfrak{p})$, where $\mathfrak{p}$ is a minimal prime ideal of A.

Problem 2 (10 pts)

Does the following statement hold true?

"Let $\mathbf{C}$ and $\mathbf{D}$ be two chain complexes such that $H_n(\mathbf{C}) \cong H_n(\mathbf{D})$ for all $n \in \mathbb{Z}$. Then there exists either a morphism of chain complexes $\mathbf{f} : \mathbf{C} \longrightarrow \mathbf{D}$ or a morphism of chain complexes $\mathbf{f} : \mathbf{D} \longrightarrow \mathbf{C}$ such that $H_n(\mathbf{f})$ is an isomorphism for all $n \in \mathbb{Z}$."

Give a proof or a counter example.

Problem 3 (10 pts)

Denote by $C^0 := C^\infty(\mathbb{R}^n, \mathbb{R})$ the ring of smooth real-valued functions on $\mathbb{R}^n$. A differential form ω on $\mathbb{R}^n$ of degree r is given by

$$\omega = \sum_{\{i_1,\dots,i_r\}\subseteq\{1,\dots,n\}} f_{i_1,\dots,i_r}(x_1,\dots,x_n)dx_{i_1}\dots dx_{i_r} \quad (f_{i_1,\dots,i_r} \in C^0)$$

where the differentials $dx_1, \dots, dx_n$ are subject to the relation

$$dx_j dx_k = -dx_k dx_j \quad (j, k \in \{1, \dots, n\}).$$

Therefore ω can be rewritten as

$$\omega = \sum_{1\leq i_1<\dots<i_r\leq n} g_{i_1,\dots,i_r}(x_1,\dots,x_n)dx_{i_1}\dots dx_{i_r} \quad (g_{i_1,\dots,i_r} \in C^0).$$

Consider the set

$$C^r := \{\omega \mid \omega \text{ is a differential form of degree } r\}$$

which carries a natural structure of an $\mathbb{R}$-vector space, being trivial for $r > n$.

Furthermore, consider the $\mathbb{R}$-linear map $d : C^r \longrightarrow C^{r+1}$ given by

$$d\omega := \sum_{1\leq i_1<\dots<i_r\leq n} \sum_{j=1}^{n} \frac{\partial g_{i_1,\dots,i_r}}{\partial x_j} dx_j dx_{i_1}\dots dx_{i_r}.$$

(a) Show that

$$\mathbf{C} : 0 \longrightarrow C^0 \xrightarrow{d} C^1 \xrightarrow{d} C^2 \xrightarrow{d} \dots$$

is a cochain complex (of $\mathbb{R}$-vector spaces), i.e., $d^2 = d \circ d = 0$.

(b) Compute the cohomology groups $H^r(\mathbf{C})$ in the case $n = 2$ for $r \in \mathbb{N}$.