Problems for BMS Basic Course "Commutative Algebra"

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Hand in November 29th, after the 2nd lecture 4.45 p.m.

Please solve each problem on a different sheet of paper, and sign each sheet with your name and student ID

6th Problem Set (30 + 10 points)

Problem 1 (10 pts)

Let A be a commutative ring with 1 and let M, N, P be A-modules. Construct the following isomorphisms by using the universal property of the tensor product:

(a) $M \otimes_A N \cong N \otimes_A M$.

(b) $(M \otimes_A N) \otimes_A P \cong M \otimes_A (N \otimes_A P)$.

(c) $(M \oplus N) \otimes_A P \cong (M \otimes_A P) \oplus (N \otimes_A P)$.

(d) $A \otimes_A M \cong M$.

Problem 2 (10 pts)

Let A be commutative ring with 1 and $\mathfrak{a}$ an ideal in A. Let M be an A-module and $N \subseteq M$ a submodule.

(a) Construct an isomorphism $M \otimes_A A/\mathfrak{a} \cong M/\mathfrak{a}M$.

(b) Show that $\mathfrak{a}N \subseteq \mathfrak{a}M \cap N$.

(c) Show that $\mathfrak{a}N = \mathfrak{a}M \cap N$ iff the map $N \otimes_A A/\mathfrak{a} \longrightarrow M \otimes_A A/\mathfrak{a}$ is injective.

Problem 3 (10 pts)

(a) Show that $\mathbb{Q}/\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Q} = 0$ and that $\mathbb{Q}/\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} = 0$.

(b) Give a counter-example to the claim that $(- \otimes_A M)$ is an exact functor.

Problem 4*

(a) Show that $\mathbb{C}[X, Y] \cong \mathbb{C}[X] \otimes_{\mathbb{C}} \mathbb{C}[Y]$.

(b) Consider the natural embedding $i : \mathbb{C}[X] \longrightarrow \mathbb{C}[X, Y]$. Describe the induced embedding $i^c : \mathrm{Max}(\mathbb{C}[X, Y]) \longrightarrow \mathrm{Max}(\mathbb{C}[X])$ of the maximal ideals.

(c) Does the Zariski topology on $\mathrm{Spec}(\mathbb{C}[X, Y])$ coincide with the product of the Zariski topologies on $\mathrm{Spec}(\mathbb{C}[X])$ and $\mathrm{Spec}(\mathbb{C}[Y])$?