Problems for BMS Basic Course "Commutative Algebra"

Prof. Dr. J. Kramer

Hand in December 20th, after the 2nd lecture 4.45 p.m.

Please solve each problem on a different sheet of paper, and sign each sheet with your name and student ID

9th Problem Set (30 points)

Problem 1 (10 pts)

Let A be a commutative ring with 1. Show the following:

(a) Let
$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$
be a short exact sequence of A-modules. Then M is noetherian iff M' and M'' are noetherian.

(b) If M_i $(i = 1, \dots, n)$ are noetherian A-modules, then $\bigoplus_{i=1}^{n} M_i$ is a noetherian A-module.

Problem 2 (10 pts)

(a) Show that every surjective endomorphism of a noetherian module is an isomorphism.

(b) Give an example of an injective endomorphism of a noetherian module which is not an isomorphism.

Problem 3 (10 pts)

Let A be a commutative ring with 1. Recall that the sets $D(f)$ $(f \in A)$ form a basis of the topology of $\operatorname{Spec}(A)$ (cf. Set 3, Problem 2).
We define $\mathcal{O}(D(f)) := A_f$ (A_f being the localization of A with respect to $S = \{1, f, f^2, \dots\}$), and for an open set $U = \bigcup_{j \in J} D(f_j)$, we put
$$\mathcal{O}(U) := \varinjlim_{j \in J} \mathcal{O}(D(f_j)).$$

Show that $\mathcal{O}$ defines a presheaf on $\operatorname{Spec}(A)$.