Background material for Lecture 5

Open book decompositions of 3-manifolds

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Consider $\pi : M \setminus B \to S^1$ such that:

- $B \subset M$ is an oriented link (**“binding”**)
- $M \backslash B \xrightarrow{\pi} S^1$ is a fibration (fibres $=$ **“pages”**),

$$\text{nbhd}(B) \cong \coprod (S^1 \times \mathbb{D}^2) \xrightarrow{\pi} S^1$$
$$(\theta, (r, \phi)) \mapsto \phi$$

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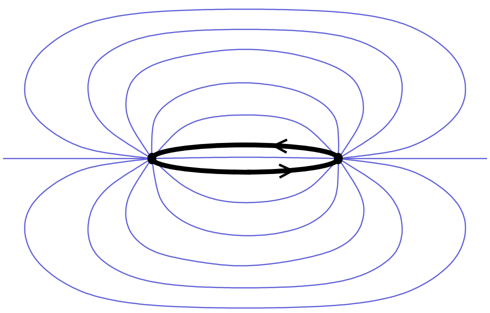
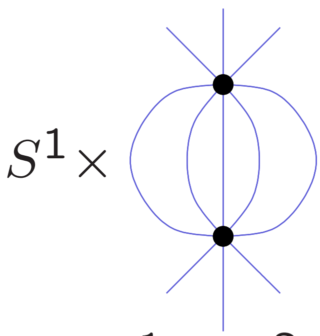


$S^3 = \mathbb{R}^3 \cup \{\infty\}$



$S^1 \times S^2$

The Giroux correspondence

A contact structure ξ is supported by an open book $\pi : M \setminus B \to S^1$ if $\xi = \ker \alpha$ for some contact form α such that

$$\alpha|_{TB} > 0 \qquad \text{and} \qquad d\alpha|_{\text{pages}} > 0.$$

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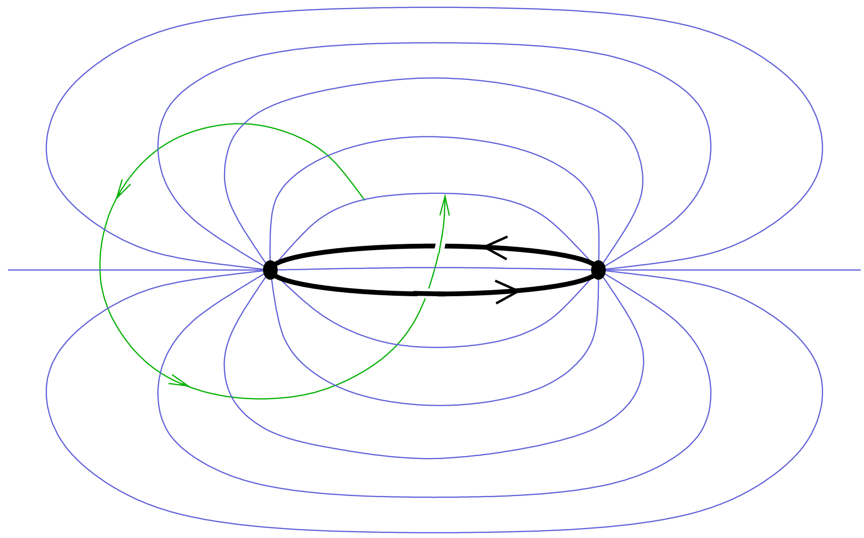

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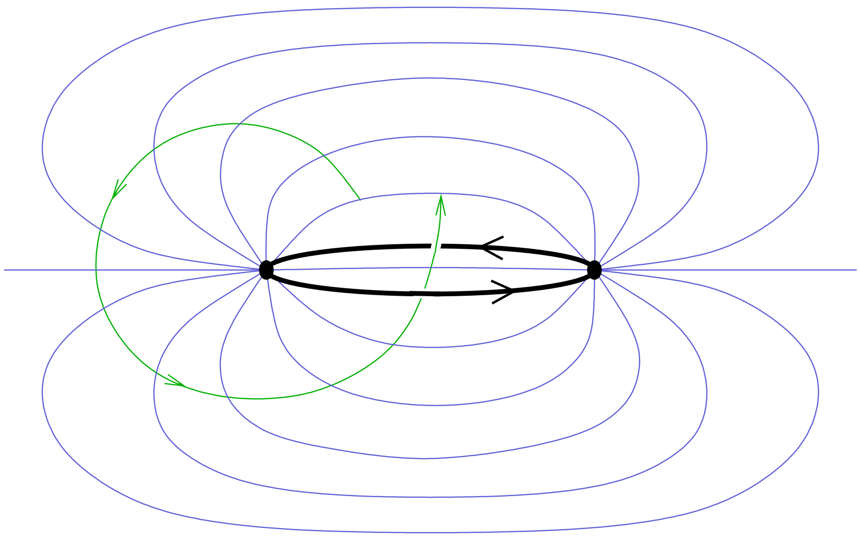

Thurston-Winkelnkemper:

$$\{\text{OBDs}\} \longrightarrow \{\text{ctct strs}\}\big/\text{isotopy}$$

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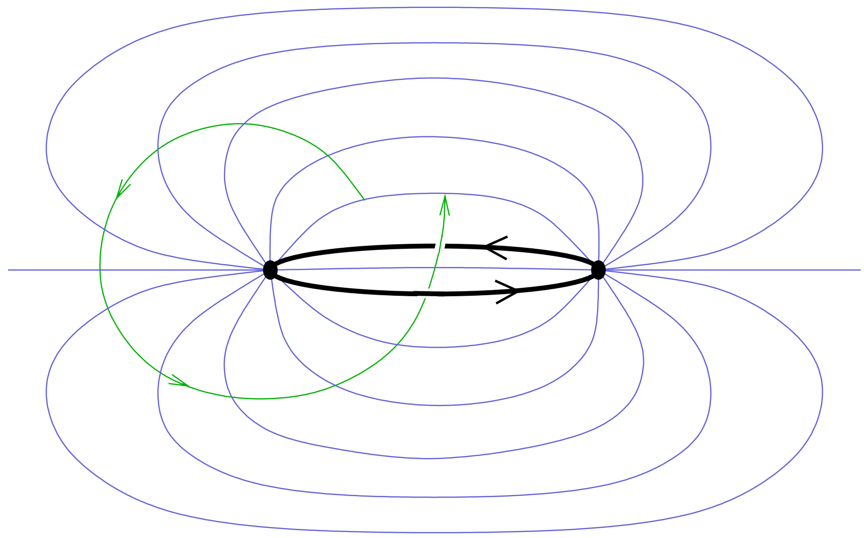

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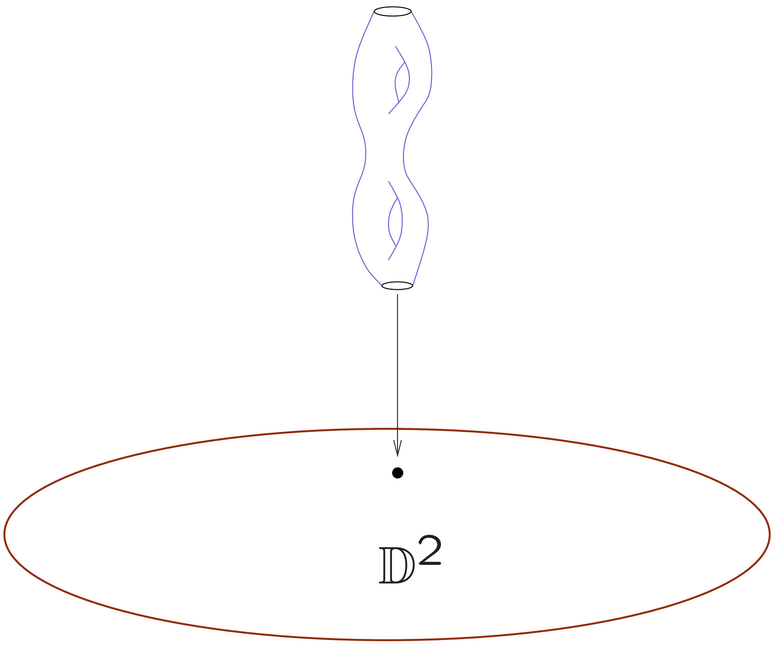
Giroux:

$$\{\text{ctct strs}\}\big/\text{isotopy} \overset{1:1}{\longleftrightarrow} \{\text{OBDs}\}\big/\text{stabilisation}$$

Bordered Lefschetz fibrations

Lefschetz fibration $W^4 \xrightarrow{\pi} \mathbb{D}^2$ with interior critical points and fibres with boundary:



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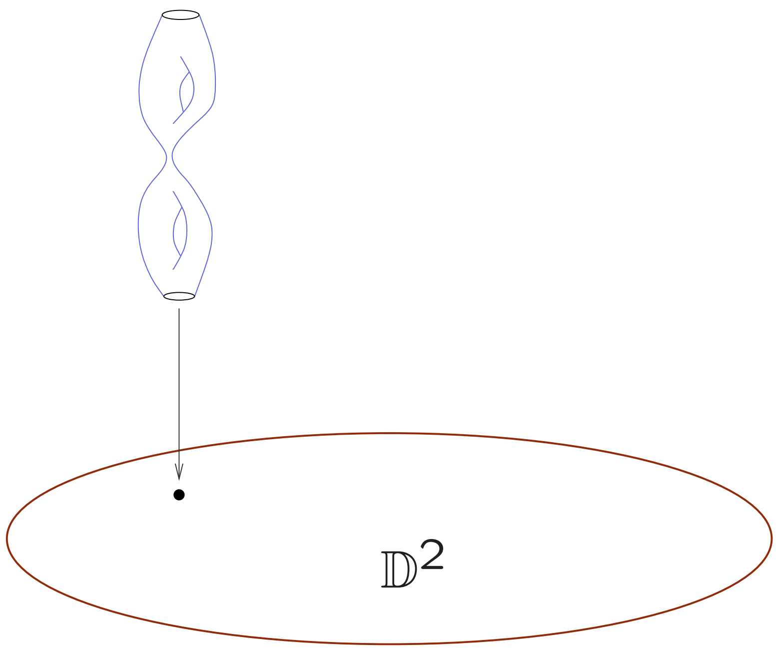

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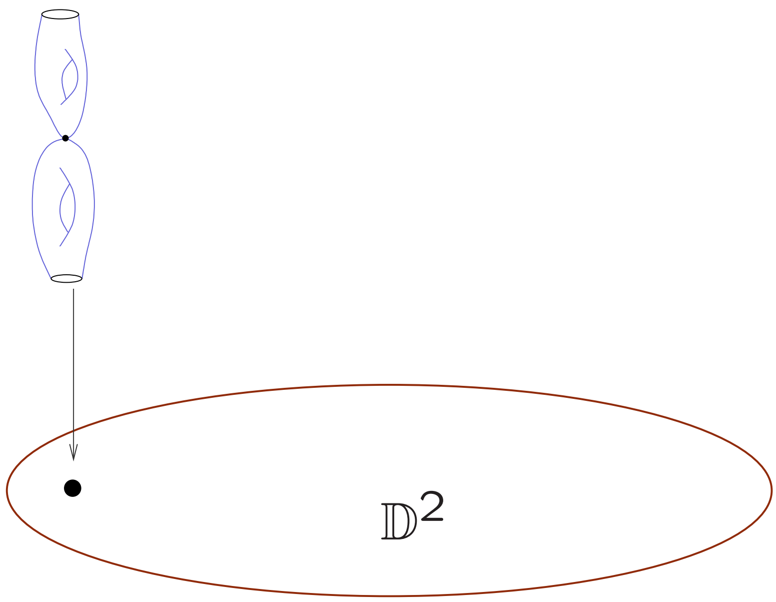

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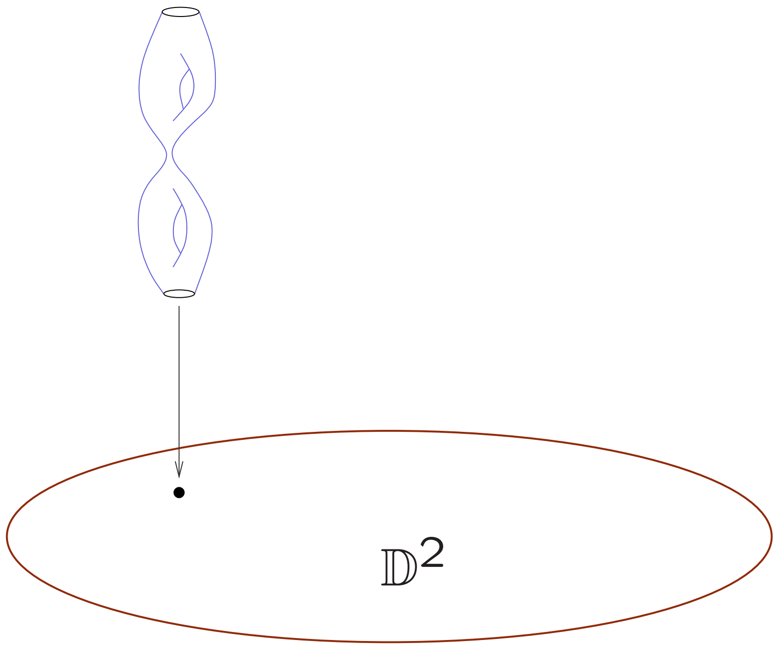

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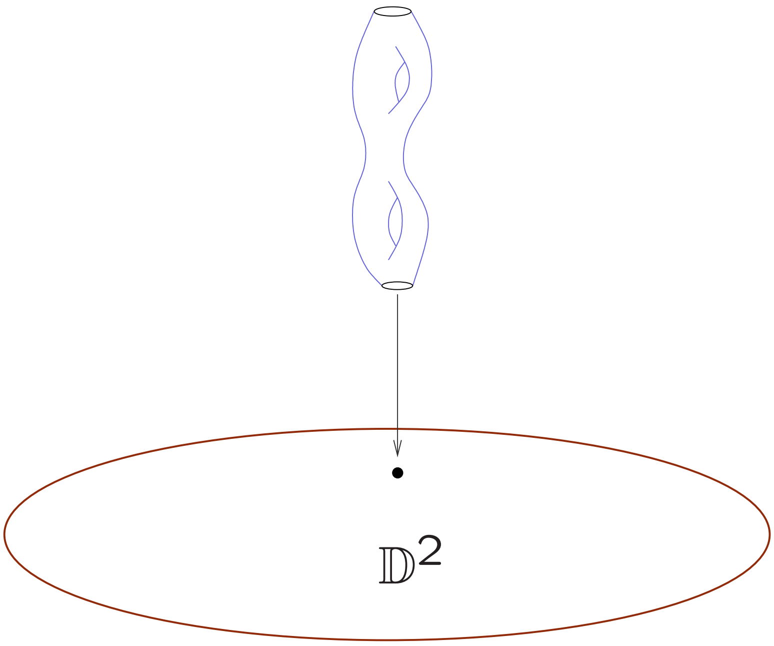

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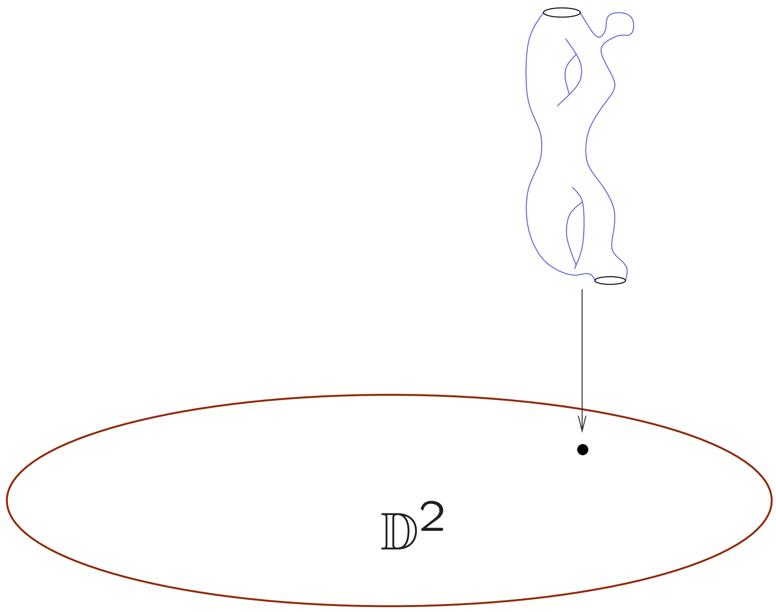

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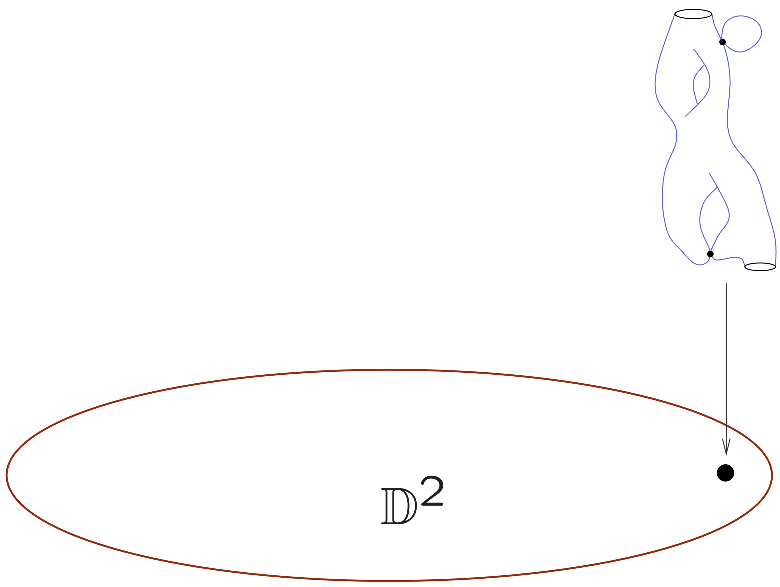

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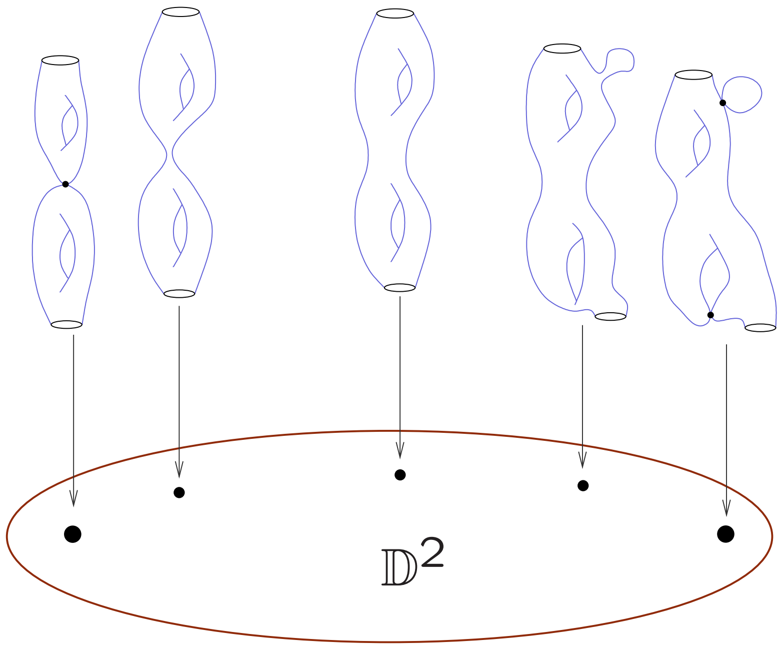

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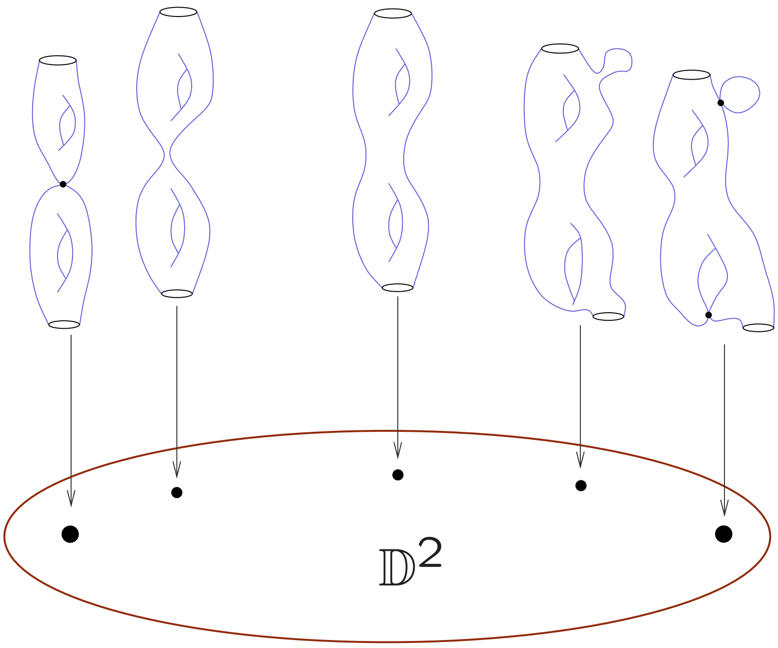

W has boundary and corners: smooth faces

$$\partial W = \partial_v W \cup \partial_h W,$$

where

$$\partial_v W := \pi^{-1}(\partial \mathbb{D}^2) \xrightarrow{\text{fibration}} \partial \mathbb{D}^2 = S^1,$$

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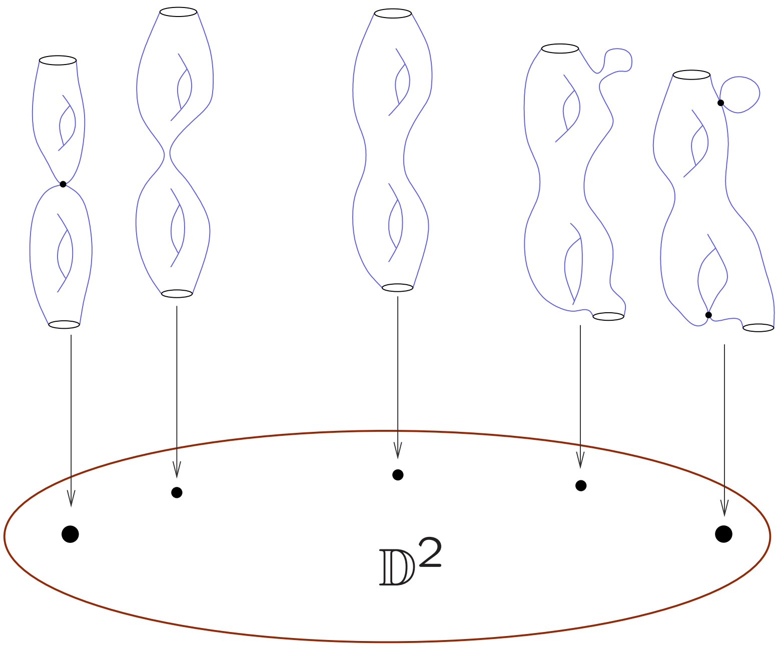

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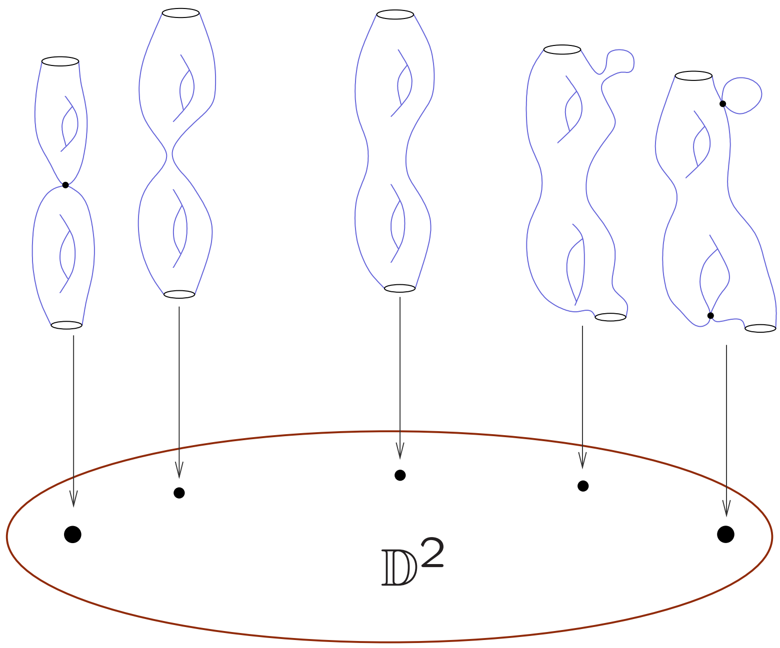
$\Longrightarrow$ ∂W inherits an open book.

Theorem

Any bordered Lefschetz fibration $W \xrightarrow{\pi} \mathbb{D}^2$ admits (canonically up to deformation) a symplectic form ω such that fibres are symplectic and (W, ω) has convex boundary (M, ξ) supported by the induced open book.



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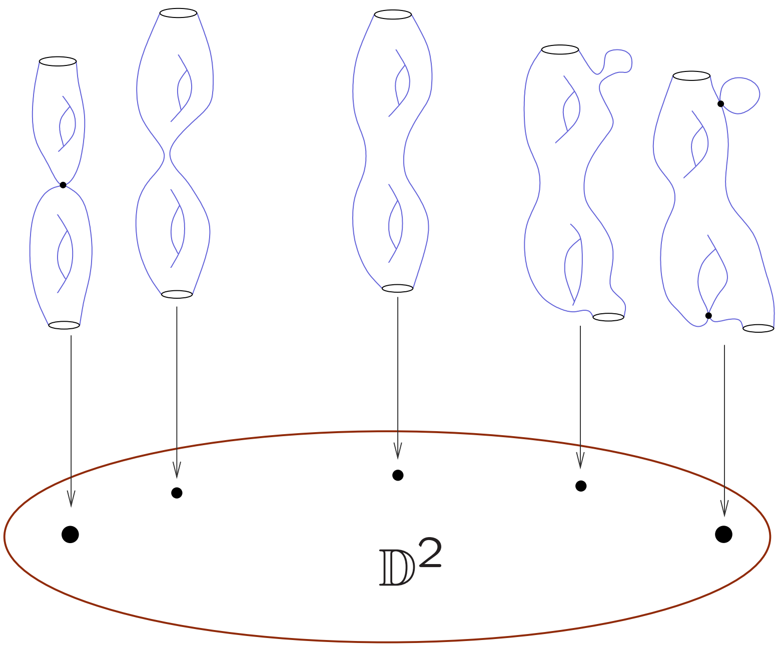
If no irreducible components of singular fibres are closed (i.e. $W \xrightarrow{\pi} \mathbb{D}^2$ is "allowable"), then one can make (W, ω) a Stein filling of (M, ξ).



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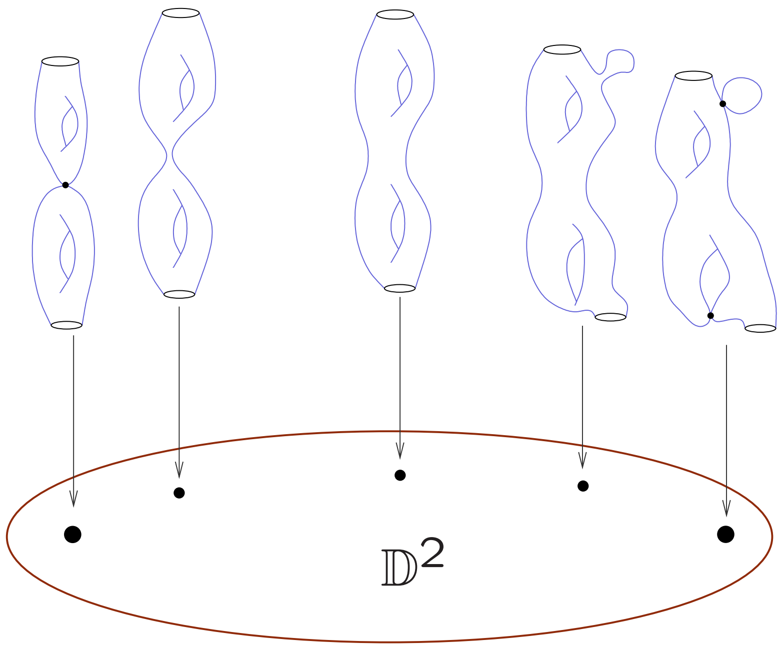

Proposition
The monodromy of the open book on ∂W is a composition of positive Dehn twists, one for each critical point of $W \xrightarrow{\pi} \mathbb{D}^2$.