PRACTICE EXAM / PROBEKLAUSUR

Instructions

You have 2 hours and 45 minutes in total, though the exam is designed to be doable in significantly less time than that. For reference, you may use any notes or books that you bring with you, but nothing electronic, i.e. no calculators or smartphones.

Answers can be written in German or English, and all answers require justification (within reason) in order to receive full credit. You may use any result that was proved in the lectures or on problem sets without reproving it, *except* in cases where the point of the problem is to prove that result. When you use a result that you learned in the course, state clearly which result it is. If you would like to use a result that you've found in a book but it was not covered in the class, then you need to explain the proof.

Keep in mind that if you get stuck on one part of a problem, it may sometimes be possible to skip it and do the next part.

Problems [100 pts total]

1. [10 pts] Suppose $f : X \to Y$ is a continuous map between topological spaces. Prove that if X is connected, then the image of f is a connected subset of Y.
   Note: The statement says "connected," not "path-connected".

2. Consider the quotient space $X = \mathbb{R}/(0,1)$.
   (a) [10 pts] Find two distinct points $x, y \in X$ such that every neighborhood of x intersects every neighborhood of y.
   (b) [15 pts] Find a compact subset $K \subset X$ that is not closed.
   Hint: Proving compactness should be easy—remember that continuous images of compact sets are also compact.

3. [20 pts] Let $\mathbb{D}^2 \subset \mathbb{C}$ denote the closed unit disk, and consider the quotient space $X = \mathbb{D}^2/\!\sim$ where $\sim$ is the smallest equivalence relation such that for all $z, w \in \partial\mathbb{D}^2$, $z \sim w$ whenever $z^3 = w^3$. Compute $\pi_1(X)$.

4. [12 pts] Prove that every continuous map $f : \mathbb{RP}^2 \to \mathbb{T}^2$ is homotopic to a constant map.
   Hint: Can f be lifted to the universal cover of $\mathbb{T}^2$?

5. In the following, Σ_g denotes the closed connected and oriented surface with genus $g \geq 0$, and $\Sigma_{g,1} = \Sigma_g \setminus \mathring{\mathbb{D}}^2$ for some embedding of the disk $\mathbb{D}^2 \hookrightarrow \Sigma_g$.
   (a) [15 pts] Compute $H_1(\Sigma_g; \mathbb{Z})$ and $H_1(\Sigma_{g,1}; \mathbb{Z})$.
   Hint: Use what you know about the fundamental group.
   (b) [8 pts] Prove that if $h > g$, then there exists no retraction of $\Sigma_{g,1}$ to a subset $A \subset \Sigma_{g,1}$ homeomorphic to $\bigvee_{i=1}^{2h} S^1$.
   Note: You may use without proof the following algebraic fact about abelian groups. If $G \subset \mathbb{Z}^m$ is a subgroup generated by $k < m$ elements, then $G \neq \mathbb{Z}^m$.

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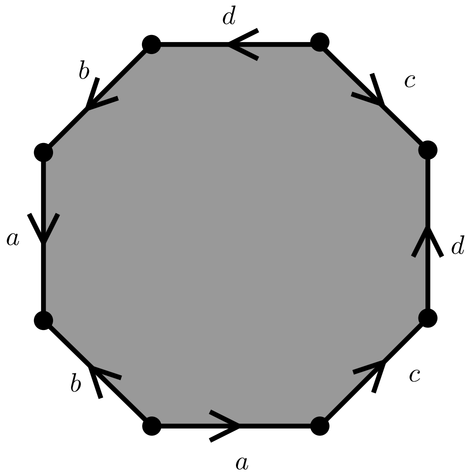
6. [10 pts] The following diagram represents a closed connected surface $\Sigma = P/\sim$. As usual, P denotes the region in $\mathbb{R}^2$ bounded by the polygon, and the equivalence relation identifies all vertices to a point and also identifies all pairs of edges labeled by matching letters via homeomorphisms that match their arrows.



According to the classification of surfaces, Σ is homeomorphic to one of the surfaces Σ_g for some $g \geq 0$ (the oriented surface of genus g) or $\#_{k=1}^{g} \mathbb{RP}^2$ for some $g \geq 1$ (the g-fold connected sum of the projective plane). Which one is it? Explain.