PROBLEM SET 2
Due: 04.05.2023

Instructions

Problems marked with $(*)$ will be graded.[1] Solutions may be written up in German or English and should be handed in before the Übung on the due date. For problems without $(*)$, you do not need to write up your solutions, but it is highly recommended that you think through them before the next Thursday lecture.

Problems

1. Given a collection $\mathcal{B}$ of subsets of a fixed set X, define $\mathcal{T}$ to be the collection of all sets of the form

$$\bigcup_{\alpha \in I} (\mathcal{U}_\alpha^1 \cap \ldots \cap \mathcal{U}_\alpha^{N_\alpha}) \subset X,$$

where I is an arbitrary (possibly empty) set, and for each $\alpha \in I$, $N_\alpha \geq 1$ is an integer and $\mathcal{U}_\alpha^i \in \mathcal{B}$ for $i = 1, \ldots, N_\alpha$.[2] Prove the following statement mentioned in lecture: $\mathcal{T}$ is a topology on X if and only if $\bigcup_{A \in \mathcal{B}} A = X$.

2. Each of the following collections $\mathcal{B}$ of subsets in $\mathbb{R}$ is a subbase of either the *standard*, *discrete*, or *cofinite* topology on $\mathbb{R}$. For each subbase, say which topology it generates. In which cases is the subbase also a base?

   (a) $\mathcal{B} = \{\mathbb{R} \setminus \{x\} \mid x \in \mathbb{R}\}$
   (b) $\mathcal{B} = \{(a, b) \mid -\infty \leq a < b \leq \infty\}$
   (c) $(*)$ $\mathcal{B} = \{\{x, y\} \mid x, y \in \mathbb{R} \text{ with } x \neq y\}$
   (d) $\mathcal{B} = \{\mathbb{R} \setminus \{x, y\} \mid x, y \in \mathbb{R} \text{ with } x \neq y\}$
   (e) $\mathcal{B} = \{\{x\} \mid x \in \mathbb{R}\}$

3. Suppose $\mathcal{B}$ is a subbase for a topology $\mathcal{T}$ on a set X.

   (a) Show that a sequence $x_n \in X$ converges to $x \in X$ if and only if for every $\mathcal{U} \in \mathcal{B}$ containing x, $x_n \in \mathcal{U}$ for all n sufficiently large.
   (b) $(*)$ Given another topological space Y, show that a map $f : Y \to X$ is continuous if and only if for every $\mathcal{U} \in \mathcal{B}$, $f^{-1}(\mathcal{U})$ is open in Y.

   Now suppose $\{(X_\alpha, \mathcal{T}_\alpha)\}_{\alpha \in I}$ is a (possibly infinite) collection of topological spaces, $(X, \mathcal{T})$ is $\prod_{\alpha \in I} X_\alpha$ with the product topology, and the subbase $\mathcal{B} \subset \mathcal{T}$ is taken to consist of all sets of the form

$$\left\{ \{x_\alpha\}_{\alpha \in I} \;\middle|\; x_\beta \in \mathcal{U}_\beta \right\} \subset \prod_\alpha X_\alpha$$

   for arbitrary $\beta \in I$ and $\mathcal{U}_\beta \in \mathcal{T}_\beta$. Use parts (a) and (b) to prove the following two statements mentioned in lecture:

   (c) $(*)$ A sequence $\{x_\alpha^n\}_{\alpha \in I} \in X$ converges to $\{x_\alpha\}_{\alpha \in I} \in X$ as $n \to \infty$ if and only if $x_\alpha^n \to x_\alpha$ for every $\alpha \in I$.

[1] For the first few problem sets in this semester we do not yet have a grader, so for each starred problem you will be given a pass/fail mark based on whether an obvious effort has been made.

[2] A student pointed out at the end of Tuesday's lecture that if I allowed $N_\alpha = 0$ here, then one could argue that $\mathcal{T}$ automatically contains X (and would therefore be a topology) because, technically, the intersection of an empty collection of subsets of X is X. But I find this confusing, so I am requiring $N_\alpha \geq 1$.

(d) For any other topological space Y, a map $f : Y \to X$ is continuous if and only if $\pi_\alpha \circ f : Y \to X_\alpha$ is continuous for every $\alpha \in I$, where $\pi_\alpha : X \to X_\alpha$ denotes the natural projection $\{x_\beta\}_{\beta \in I} \mapsto x_\alpha$.

4. Let $\mathbb{R}_{\text{std}}$ and $\mathbb{R}_{\text{cof}}$ denote topological spaces consisting of the set $\mathbb{R}$ with the standard or cofinite topology respectively; recall that for the latter, subsets other than $\mathbb{R}$ are closed if and only if they are finite.

   (a) Show that a map $f : \mathbb{R}_{\text{cof}} \to \mathbb{R}_{\text{std}}$ is continuous if and only if it is constant.
   Hint: If f is not constant, what can you say about $f^{-1}([n, n+1])$ for each $n \in \mathbb{Z}$?

   (b) What does it mean for a sequence $x_n \in \mathbb{R}_{\text{cof}}$ to converge to $x \in \mathbb{R}_{\text{cof}}$?

5. Assume I is an infinite set and $\{(X_\alpha, \mathcal{T}_\alpha)\}_{\alpha \in I}$ is a collection of topological spaces. In addition to the usual product topology on $\prod_\alpha X_\alpha$, one can define the so-called *box topology*, which has a base of the form

$$\left\{ \prod_{\alpha \in I} \mathcal{U}_\alpha \;\middle|\; \mathcal{U}_\alpha \in \mathcal{T}_\alpha \text{ for all } \alpha \in I \right\}.$$

   (a) Compared with the usual product topology, is the box topology stronger, weaker, or neither?

   (b) $(*)$ What does it mean for a sequence in $\prod_\alpha X_\alpha$ to converge in the box topology? In particular, consider the case where all the X_α are a fixed space X and $\prod_\alpha X$ is identified with the space of all functions $X^I = \{f : I \to X\}$; what does it mean for a sequence of functions $f_n : I \to X$ to converge in the box topology to a function $f : I \to X$?

6. For any topological space X, a subset $A \subset X$ is called *dense* if its closure $\bar{A}$ is X. We say that X is *separable* if it contains a dense subset that is countable.

   (a) Show that if X is a metric space and $A \subset X$ is a dense subset, then the balls $B_{1/n}(x)$ for $n \in \mathbb{N}$ and $x \in A$ form a base for the topology of X.
   Hint: If the statement is not true, then there exists an open set $\mathcal{U} \subset X$ containing a point y which is not in any ball of the form $B_{1/n}(x) \subset \mathcal{U}$ for $n \in \mathbb{N}$ and $x \in A$. Use the fact that since A is dense, $A \cap B_\epsilon(y) \neq \emptyset$ for every $\epsilon > 0$. Can you find an $\epsilon > 0$ such that $B_{2\epsilon}(y) \subset \mathcal{U}$ and some $x \in A \cap B_\epsilon(y)$ must satisfy $y \in B_{1/n}(x)$ for some $n \in \mathbb{N}$?

   (b) Deduce from part (a) that every topological space that is separable and metrizable is also second countable.
   Remark: As we've seen in lecture, metrizable spaces are always first countable, but they need not be second countable in general, e.g. any uncountable set with the discrete topology is a counterexample.

7. $(*)$ Here is a statement that is **not true**.

   If X is a topological space in which every open cover has a finite subcover, then every sequence in X has a convergent subsequence.

   You learned in analysis that this is true for metric spaces in general, and it will turn out to be true for a somewhat larger class of topological spaces, but not all of them—we will see a counterexample next week. Nonetheless, you might agree that the following "proof" looks very plausible at first glance. Find the error in the proof. Can you fix it by adding an extra assumption about X? (Think of the countability axioms...)

   "Proof"*: Arguing by contradiction, suppose every open cover of X has a finite subcover, but $x_n \in X$ is a sequence with no convergent subsequence. In particular, for every $x \in X$, no subsequence of x_n converges to x, which means that x_n cannot enter arbitrary neighborhoods of x for arbitrarily large values of n, i.e. there exists $N_x \in \mathbb{N}$ and an open neighborhood $\mathcal{U}_x \subset X$ of x such that $x_n \notin \mathcal{U}_x$ for every $n \geq N_x$. The collection $\{\mathcal{U}_x\}_{x \in X}$ is then an open cover of X, so by assumption, there exists a finite subset $I \subset X$ such that $X = \bigcup_{x \in I} \mathcal{U}_x$. But each of these finitely many subsets contains at most finitely many terms of x_n, and this is impossible since there are infinitely many terms.*