PROBLEM SET 6
Due: 8.06.2023

Instructions

Problems marked with $(*)$ will be graded. Solutions may be written up in German or English and should be handed in before the Übung on the due date. For problems without $(*)$, you do not need to write up your solutions, but it is highly recommended that you think through them before the next Thursday lecture.

Note: You may continue to treat the computation $\pi_1(S^1) \cong \mathbb{Z}$ as a known fact, even though we have not proved it in full detail yet.

Problems

1. Consider the finitely-presented group $G = \{x, y \mid x^2 = y^3\}$. Let us recall quickly what this notation means: literally, G is defined as the quotient group

$$G = F_{\{x,y\}} \big/ \langle \{x^{-2}y^3\} \rangle_N,$$

where for any set S, F_S denotes the free group on S, and for any group H with a subset $A \subset H$, $\langle A \rangle_N \subset H$ denotes the smallest normal subgroup of H containing A. Thus G consists of all equivalence classes of reduced words made up of the letters x, y, x^{-1} and y^{-1}, subject to an equivalence relation generated by the equation $x^2 = y^3$. (It is conventional to denote elements of groups defined in this way simply as words, the same as with elements of $F_{\{x,y\}}$, but we must always keep in mind that different reduced words may represent the same element of G, e.g. in the present case, $x^{-1}y^4x^2 = x^{-1}yx^4 = x^{-1}yx^2y^3$, and also $x^{-1}y^4x^2 = xyx^2$.)

We will show next week that $G \cong \pi_1(\mathbb{R}^3 \backslash K)$ for the trefoil knot K. The main goal of this problem is to show that G is not abelian. A corollary will be that K cannot be deformed continuously in $\mathbb{R}^3$ to the unknot K_0, because we will also show next week that $\pi_1(\mathbb{R}^3 \backslash K_0) \cong \mathbb{Z}$, which is abelian.

   (a) $(*)$ Denoting the identity element by e, consider the related group

$$H = \{x, y \mid x^2 = y^3,\ y^3 = e,\ xyxy = e\}.$$

   Show that every element of H is equivalent to one of the six elements $e, x, y, y^2, xy, xy^2 \in H$. This proves that H has order at most six, though in theory it could be less, since some of those six elements might still be equivalent to each other. To prove that this is not the case, construct (by writing down a multiplication table) a nonabelian group H' of order six that is generated by two elements a, b satisfying the relations $a^2 = b^3 = e$ and $abab = e$. Show that there exists a surjective homomorphism $H \to H'$, which is therefore an isomorphism since $|H| \leqslant 6$.
   Remark: You don't need this fact, but you might in any case notice that H is isomorphic to the dihedral group (Diedergruppe) of order 6.

   (b) $(*)$ Show that H is a quotient of G by some normal subgroup, and deduce that G is also nonabelian.

2. Given a group G, the *commutator subgroup* $[G, G] \subset G$ is the subgroup generated by all elements of the form $xyx^{-1}y^{-1}$ for $x, y \in G$.

   (a) Show that $[G, G] \subset G$ is always a normal subgroup, and it is trivial if and only if G is abelian.

   (b) The *abelianization* (Abelisierung) of G is defined as the quotient group $G/[G, G]$. Show that this group is always abelian, and it is equal to G if G is already abelian.[1]

[1]Note that if $G = \{S \mid R\}$ is a finitely-presented group with generators S and relations R, then its abelianization is $\{S \mid R'\}$ where R' is the union of R with all relations of the form "$ab = ba$" for $a, b \in S$.

(c) Given any two abelian groups G, H, find a natural isomorphism from the abelianization of the free product $G * H$ to the Cartesian product $G \times H$.

(d) ($*$) Prove that the abelianization of $\{x, y \mid x^2 = y^3\}$ is isomorphic to $\mathbb{Z}$.
Hint: An isomorphism φ from the abelianization to $\mathbb{Z}$ will be determined by two integers, $\varphi(x)$ and $\varphi(y)$. If φ exists, how must these two integers be related to each other?

(e) For any $g \in \mathbb{N}$, consider the group

$$G_g := \left\{ x_1, y_1, \ldots, x_g, y_g \;\middle|\; x_1 y_1 x_1^{-1} y_1^{-1} \ldots x_g y_g x_g^{-1} y_g^{-1} = e \right\}.$$

Show that the abelianization of G_g is isomorphic to the additive group $\mathbb{Z}^{2g}$.
Hint: By definition, G_g is a particular quotient of the free group on $2g$ generators. Notice that the abelianization of that free group is exactly the same group as the abelianization of G_g. (Why?)
Remark: We will show next week that G_g is the fundamental group of Σ_g, the closed oriented connected surface of genus g. The classification of finitely generated abelian groups implies that $\mathbb{Z}^{2g}$ is not isomorphic to $\mathbb{Z}^{2h}$ for $g \neq h$, so a corollary of this exercise is that Σ_g and Σ_h cannot be homeomorphic for $g \neq h$.

3. ($*$) Suppose X and Y are two spaces that both contain subspaces homeomorphic to the closed n-dimensional unit disk $\mathbb{D}^n \subset \mathbb{R}^n$, and let

$$\iota_X : \mathbb{D}^n \hookrightarrow X, \qquad \iota_Y : \mathbb{D}^n \hookrightarrow Y$$

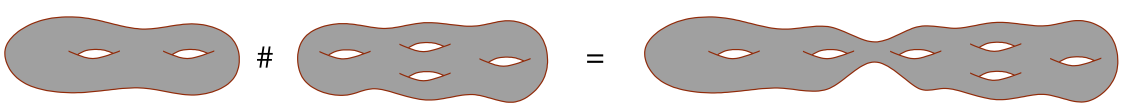
denote the inclusions of these subspaces. For $r > 0$, let $\mathbb{D}^n_r \subset \mathbb{R}^n$ denote the disk of radius r, and $\mathring{\mathbb{D}}^n_r$ its interior. We can then form the *connected sum* (zusammenhängende Summe)

$$X \# Y := \left(X \backslash \iota_X(\mathring{\mathbb{D}}^n_{1/2}) \right) \cup_{\partial \mathbb{D}^n_{1/2}} \left(Y \backslash \iota_Y(\mathring{\mathbb{D}}^n_{1/2}) \right)$$

by removing open disks of radius $1/2$ from both X and Y and then gluing together what remains via the obvious homeomorphism of the resulting boundary spheres $\partial \mathbb{D}^n_{1/2} \cong S^{n-1}$. The effect is to cut disk-like "holes" in both X and Y but then connect them to each other via a "tube"

$$\left(\mathbb{D}^n \backslash \mathring{\mathbb{D}}^n_{1/2} \right) \cup_{\partial \mathbb{D}^n_{1/2}} \left(\mathbb{D}^n \backslash \mathring{\mathbb{D}}^n_{1/2} \right) \cong [-1, 1] \times S^{n-1}.$$

This operation is most interesting when performed on n-dimensional manifolds, which we'll talk about later in the semester: if X and Y are both connected n-manifolds, then the required embeddings of $\mathbb{D}^n$ into X and Y always exist. For example, in the case $n = 1$, $S^1 \# S^1$ is always homeomorphic to S^1. (Draw a few pictures to convince yourself that this is true.) A more interesting example with $n = 2$ is shown in the following picture: the connected sum of two closed oriented surfaces of genus g and h respectively is homeomorphic to a closed oriented surface of genus $g + h$.



Show that if $n \geqslant 3$, then $\pi_1(X \# Y) \cong \pi_1(X) * \pi_1(Y)$ whenever X and Y are both path-connected. Where does your proof fail in the cases $n = 1$ and $n = 2$?
Remark: It is obvious that the formula is false for $n = 1$ since $S^1 \# S^1 \cong S^1$ but $\mathbb{Z} * \mathbb{Z}$ is not isomorphic to $\mathbb{Z}$. We will see next week that the formula is also not true for $n = 2$, but we will still be able to compute π_1 for surfaces of genus g via a more careful application of the Seifert-van Kampen theorem.

4. ($*$) Show that if X is the complement of $n \geqslant 0$ points in $\mathbb{R}^2$, then $\pi_1(X)$ is a free group with n generators. Draw a picture for the case $n = 3$ indicating three specific choices of loops that represent a complete set of generators for $\pi_1(X)$.
Hint: X admits a deformation retraction to a wedge sum of n circles. Prove this with a picture. You might find the picture easier to draw if you replace $\mathbb{R}^2$ with an open 2-disk—that is of course fine since they are homeomorphic.