PROBLEM SET 2
To be discussed: 1.11.2017

Instructions

This homework will not be collected or graded, but it is highly advisable to at least think through all of the problems before the next Wednesday lecture after they are distributed, as they will often serve as mental preparation for the material in that lecture. We will discuss the solutions in the Übung afterwards.

Advice*: If you have time for nothing else this week, be sure to have a look at Problems 3 and 4.*

1. Suppose $\mathscr{A}$ is a category whose objects form a set X, such that for each pair $x, y \in X$, the set of morphisms $\mathrm{Mor}(x, y)$ contains either exactly one element or none. We can turn this into a binary relation by writing $x \bowtie y$ for every pair such that $\mathrm{Mor}(x, y) \neq \varnothing$.

   (a) What properties does the relation $\bowtie$ need to have in order for it to define a category in the way indicated above?

   (b) If $\mathscr{B}$ is another category whose objects form a set Y with morphisms determined by a binary relation $\bowtie$ as indicated above, what properties does a map $f : X \to Y$ need to have in order for it to define a functor from $\mathscr{A}$ to $\mathscr{B}$?

2. In any category $\mathscr{C}$, each object X has an **automorphism group** (also called **isotropy** group) $\mathrm{Aut}(X)$, consisting of all the isomorphisms in $\mathrm{Mor}(X, X)$. A **groupoid** is a category in which all morphisms are also isomorphisms.

   (a) Show that if $\mathscr{G}$ is a groupoid and Grp denotes the usual category of groups with homomorphisms, there exists a contravariant functor from $\mathscr{G}$ to Grp that assigns to each object X of $\mathscr{G}$ its automorphism group $\mathrm{Aut}(X)$. How does this functor act on morphisms $X \to Y$? Could you alternatively define it as a *covariant* functor? Conclude either way that whenever X and Y are isomorphic objects in $\mathscr{G}$ (meaning there exists an isomorphism in $\mathrm{Mor}(X, Y)$), the groups $\mathrm{Aut}(X)$ and $\mathrm{Aut}(Y)$ are isomorphic.

   (b) Given a topological space X and two points x, y, let $\mathrm{Mor}(x, y)$ denote the set of homotopy classes (with fixed end points) of paths $[0, 1] \to X$ from x to y, and define a composition function $\mathrm{Mor}(x, y) \times \mathrm{Mor}(y, z) \to \mathrm{Mor}(x, z) : (\alpha, \beta) \mapsto \alpha \cdot \beta$ by the usual notion of concatenation of paths. Show that this notion of morphisms defines a groupoid whose objects are the points in X.[1] In this case, what are the automorphism groups $\mathrm{Aut}(x)$ and the isomorphisms $\mathrm{Aut}(y) \to \mathrm{Aut}(x)$ given by the functor in part (a)?

3. Consider the categories Short and Long, defined as follows. Objects in Short are short exact sequences of chain complexes $0 \to A_* \xrightarrow{f} B_* \xrightarrow{g} C_* \to 0$, with a morphism from this object to another object $0 \to A'_* \xrightarrow{f'} B'_* \xrightarrow{g'} C'_* \to 0$ defined as a triple of chain maps $A_* \xrightarrow{\alpha} A'_*$, $B_* \xrightarrow{\beta} B'_*$ and $C_* \xrightarrow{\gamma} C'_*$ such that the following diagram commutes:

$$\begin{tikzcd} 0 \arrow[r] & A_* \arrow[r, "f"] \arrow[d, "\alpha"] & B_* \arrow[r, "g"] \arrow[d, "\beta"] & C_* \arrow[r] \arrow[d, "\gamma"] & 0 \\ 0 \arrow[r] & A'_* \arrow[r, "f'"] & B'_* \arrow[r, "g'"] & C_* \arrow[r] & 0 \end{tikzcd}$$

   The objects in Long are long exact sequences of $\mathbb{Z}$-graded abelian groups $\ldots \to C_{n+1} \xrightarrow{\delta} A_n \xrightarrow{F} B_n \xrightarrow{G} C_n \xrightarrow{\delta} A_{n-1} \to \ldots$, with morphisms from this to another object $\ldots \to C'_{n+1} \xrightarrow{\delta'} A'_n \xrightarrow{F'} B'_n \xrightarrow{G'} C'_n \xrightarrow{\delta'}$

[1] It is called the **fundamental groupoid** of X.

$A'_{n-1} \to \ldots$ defined as triples of homomorphisms $A_* \xrightarrow{\alpha} A'_*$, $B_* \xrightarrow{\beta} B'_*$ and $C_* \xrightarrow{\gamma} C'_*$ that preserve the $\mathbb{Z}$-gradings and make the following diagram commute:

$$\begin{tikzcd}
\ldots \arrow[r] & C_{n+1} \arrow[r, "\delta"] \arrow[d, "\gamma"] & A_n \arrow[r, "F"] \arrow[d, "\alpha"] & B_n \arrow[r, "G"] \arrow[d, "\beta"] & C_n \arrow[r, "\delta"] \arrow[d, "\gamma"] & A_{n-1} \arrow[r] \arrow[d, "\alpha"] & \ldots \\
\ldots \arrow[r] & C'_{n+1} \arrow[r, "\delta'"] & A'_n \arrow[r, "F'"] & B'_n \arrow[r, "G'"] & C'_n \arrow[r, "\delta'"] & A'_{n-1} \arrow[r] & \ldots
\end{tikzcd}$$

Recall also the category $\mathsf{Top}_{\mathrm{rel}}$, whose objects are pairs (X, A) of topological spaces X with subsets A, with a morphism $(X, A) \to (Y, B)$ being a continuous map of pairs.

(a) Show that there is a covariant functor $\mathsf{Top}_{\mathrm{rel}} \to \mathsf{Short}$ assigning to each pair (X, A) its short exact sequence of singular chain complexes.

(b) Show that there is also a covariant functor $\mathsf{Short} \to \mathsf{Long}$ assigning to each short exact sequence of chain complexes the corresponding long exact sequence of their homology groups. (Note that this can be composed with the functor in part (a) to define a functor $\mathsf{Top}_{\mathrm{rel}} \to \mathsf{Long}$.)

(c) Let $\mathsf{Top}^h_{\mathrm{rel}}$ and Short^h denote categories with the same objects as in $\mathsf{Top}_{\mathrm{rel}}$ and Short respectively, but with morphisms of $\mathsf{Top}^h_{\mathrm{rel}}$ consisting of homotopy classes of maps of pairs, and morphisms of Short^h consisting of triples of chain homotopy classes of chain maps. Show that the functors in parts (a) and (b) also define functors $\mathsf{Top}^h_{\mathrm{rel}} \to \mathsf{Short}^h$ and $\mathsf{Short}^h \to \mathsf{Long}$, which then compose to define a functor $\mathsf{Top}^h_{\mathrm{rel}} \to \mathsf{Long}$.

4. The goal of this problem is to develop some intuition as to what singular homology cycles represent geometrically. The general idea is that for any closed k-dimensional manifold M, a continuous map $f : M \to X$ defines a homology class $[f] \in H_k(X; \mathbb{Z}_2)$ that vanishes whenever f can be extended to $F : W \to X$ for some compact manifold W with $\partial W = M$; moreover, all of this can also be done with integer coefficients if the manifolds in question are oriented. Understanding this requires a short digression on simplicial complexes and triangulations.

We denote the **standard n-simplex** by $\Delta^n = \{(t_0, \ldots, t_n) \in [0,1]^{n+1} \mid t_0 + \ldots + t_n = 1\}$, and call any subset of the form $\{t_{i_1} = \ldots = t_{i_\ell} = 0\} \subset \Delta^n$ a **face** of Δ^n. Note that such a face can naturally be identified with Δ^m for $m := n - \ell$ by throwing out the coordinates $t_{i_1}, \ldots, t_{i_\ell}$ and keeping the others. In particular for each $k = 0, \ldots, n$, the **kth boundary face** of Δ^n is $\partial_{(k)}\Delta^n = \{(t_0, \ldots, t_n) \in \Delta^n \mid t_k = 0\}$, and is naturally identified with Δ^{n-1}.

Define the term **geometric n-simplex** to mean a topological space X together with an equivalence class of "parametrizations" $\varphi : \Delta^n \to X$, meaning homeomorphisms, where two parametrizations $\varphi, \psi : \Delta^n \to X$ are considered equivalent whenever $\psi^{-1} \circ \varphi : \Delta^n \to \Delta^n$ is the restriction to Δ^n of a linear isomorphism on $\mathbb{R}^{n+1}$. Notice that since $\psi^{-1} \circ \varphi$ must map vertices to vertices, the corresponding linear map on $\mathbb{R}^{n+1}$ simply permutes the standard basis vectors, and it is orientation preserving if and only if this permutation is even. Given a geometric n-simplex X with parametrization $\varphi : \Delta^n \to X$, the **faces of** X are defined to be the images of the faces of Δ^n under φ, and each of these inherits the structure of a geometric m-simplex for some $m = 0, \ldots, n-1$ in a natural way. An **orientation** of a geometric n-simplex X is then an equivalence class of orderings of its vertices, where two orderings are equivalent if they are related by an even permutation. Each boundary face $\partial_{(k)}X \subset X$ now inherits a **boundary orientation**, defined by choosing an ordering of the vertices of X that is compatible with the orientation and places the one vertex not in $\partial_{(k)}X$ first, then eliminating this first vertex and keeping the order of the others. (I recommend taking a moment to think about the case $n = 2$ in order to understand what this orientation convention means.)

Finally, a **finite simplicial complex** is a topological space M together with a finite collection $\mathcal{K}$ of subsets $M_i \subset M$, each endowed with the structure of a geometric simplex, such that the subsets $M_i \in \mathcal{K}$ cover M, and for any two distinct $M_i, M_j \in \mathcal{K}$, $M_i \cap M_j$ is either empty or is a common face of both simplices and also belongs to $\mathcal{K}$. This extra data attached to M is called a **triangulation** of M. A closed subset $A \subset M$ is called a **subcomplex** if every simplex in the triangulation of M is either disjoint from or contained in A, hence there is an induced triangulation of A. If M is a topological manifold with boundary, we shall always require triangulations to have the additional property that the boundary ∂M is a subcomplex. Call the triangulation **oriented** if each of its n-simplices M_i is additionally endowed with an orientation such that whenever $M_i \cap M_j$ is an $(n-1)$-simplex, the two induced boundary orientations of $M_i \cap M_j$ as a boundary face of M_i or M_j differ.

(a) Draw pictures of oriented triangulations for the 2-sphere S^2, the 2-torus $\mathbb{T}^2 := \mathbb{R}^2/\mathbb{Z}^2$, the 2-disk $\mathbb{D}^2$, the closed oriented surface Σ_2 of genus 2, and the torus $\Sigma_{1,2}$ with two holes cut out.

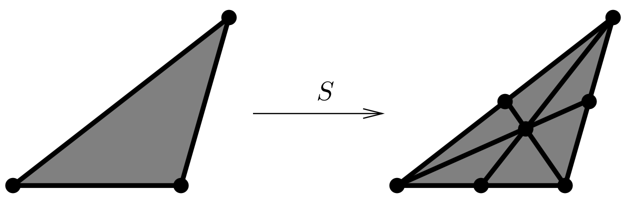
(b) Draw pictures of triangulations for the Möbius band, $\mathbb{RP}^2$, and the Klein bottle. (Can you make them oriented?)

(c) Suppose X is a topological space, M is a closed[2] (but not necessarily connected) topological k-manifold with an oriented triangulation, and $f : M \to X$ is a continuous map. Show that there exists a singular k-cycle in X of the form $\sum_i \varepsilon_i \langle f \circ \varphi_i \rangle$, where the sum ranges over the geometric k-simplices M_i in the triangulation of M, with suitably chosen parametrizations $\varphi_i : \Delta^k \to M_i$ and signs $\varepsilon_i = \pm 1$. We shall denote the homology class represented by this k-cycle by $[f] \in H_k(X)$. As a special case, taking f to be the identity map $M \to M$ defines the **fundamental class**

$$[M] := [\mathrm{Id}] \in H_k(M),$$

so that for any $f : M \to X$ we can write $[f] = f_*[M] \in H_k(X)$.
Hint: The fact that this singular k-chain can be made to satisfy $\partial \left(\sum_i \varepsilon_i \langle f \circ \varphi_i \rangle\right) = 0$ depends on the assumption that M is a manifold. If M_i is a $(k-1)$-simplex in the triangulation, how many k-simplices can it be a boundary face of?
Second hint: For bookkeeping purposes, you may find it helpful to fix an ordering of all the vertices in the triangulation.

The notation for $[M] \in H_k(M)$ and $[f] \in H_k(X)$ is meant to suggest the important fact that these homology classes do not depend on the choice of triangulation for M, except for its orientation. We will not be able to prove this until later in the semester, but the following remarks may help make it sound plausible. As we discussed in our proof of the excision theorem (see pp. 119–124 of Hatcher or §IV.17 of Bredon), the singular chain complex $C_*(X)$ admits a chain map $S : C_*(X) \to C_*(X)$ whose action on each singular k-simplex $\sigma : \Delta^k \to X$ is given by a finite sum of the form $S\langle\sigma\rangle = \sum_i \varepsilon_i \langle \sigma_i \rangle$, where $\epsilon_i = \pm 1$ and the $\sigma_i : \Delta^k \to X$ are restrictions of σ to the simplices in the **barycentric subdivision** of Δ^k, decomposing it into smaller k-simplices as shown (for the $k = 2$ case) in the picture below.



In the same manner, one can always subdivide a triangulation of the manifold M to produce a *finer* triangulation, and this changes the singular k-cycle we defined above by operating on it with the chain map S. But S is also chain homotopic to the identity, thus it acts on $H_*(X)$ as the identity map, implying that barycentric subdivision does not change the definition of $[f] \in H_k(X)$.

(d) Show that if the triangulation of M in part (c) is not oriented, one can still use the same idea to define a k-cycle in the singular chain complex of X with $\mathbb{Z}_2$-coefficients, and thus homology classes $[f] \in H_k(X; \mathbb{Z}_2)$ and $[M] \in H_k(M; \mathbb{Z}_2)$.

(e) Modify parts (c) and (d) for the case where M is a compact manifold with boundary and $f(\partial M) \subset A \subset X$ so that f defines a relative homology class $[f]$ in $H_k(X, A)$ or $H_k(X, A; \mathbb{Z}_2)$, and the fundamental class $[M]$ becomes a relative class in $H_k(M, \partial M)$ or $H_k(M, \partial M; \mathbb{Z}_2)$.

[2]Recall: a **closed** manifold is one that is compact with empty boundary.

(f) Show that if $M = S^1$ with an oriented triangulation, with $t_0 \in S^1$ as one of its vertices and $f(t_0) = p$, then the class $[f] \in H_1(X)$ is the image under the Hurewicz[3] homomorphism $h : \pi_1(X, p) \to H_1(X)$ of the element in $\pi_1(X, p)$ represented by the loop $f : (S^1, t_0) \to (X, p)$.

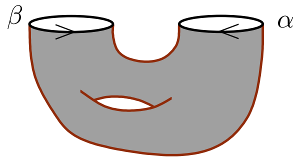
(g) Show that if W is a compact $(k+1)$-manifold with boundary $M = \partial W$ and an oriented triangulation, and $f = F|_M : M \to X$ for some continuous map $F : W \to X$, then $[f] = 0 \in H_k(X)$. Show that the same is true with $\mathbb{Z}_2$-coefficients if the orientation condition is dropped. Finally, if M is instead defined to be a compact subset of ∂W that is both a subcomplex and a k-dimensional submanifold with boundary such that $F(\overline{\partial W \backslash M}) \subset A \subset X$, show that $[f] = 0 \in H_k(X, A)$.

(h) Given two closed k-manifolds with oriented triangulations and continuous maps $f_0 : M_0 \to X$, $f_1 : M_1 \to X$, let $f : M_0 \amalg M_1 \to X$ denote the map that restricts to M_i as f_i for $i = 0, 1$. Show that $[f] = [f_0] + [f_1] \in H_k(X)$.

(i) Show that if the triangulation of M is oriented, then reversing the orientations of all its simplices (i.e. reordering their vertices by odd permutations) changes $[f] \in H_k(X)$ by a sign.
Hint: Extend $f : M \to X$ to a map $[0, 1] \times M \to X$ and use part (g) with a suitable choice of triangulation for $[0, 1] \times M$.

(j) Two maps $f_0 : M_0 \to X$ and $f_1 : M_1 \to X$ of closed triangulated k-manifolds are said to be **homologous** if $[f_0] = [f_1] \in H_k(X)$. Show that if $M_0 = M_1 = M$ with a fixed choice of oriented triangulation, then any two homotopic maps $M \to X$ are homologous.

(k) Let $\Sigma_{1,2}$ denote the 2-torus with two holes cut out, and suppose $\alpha, \beta : S^1 \hookrightarrow \partial\Sigma_{1,2}$ are loops parametrizing its two boundary components, with α following the boundary orientation of $\partial\Sigma_{1,2}$ and β following the opposite orientation. Show that α and β are homologous but not homotopic.



Hint: At this point, showing that they are homologous should be easy, but showing that they are not homotopic is a bit trickier. I can think of two approaches: one uses the fact that $\pi_1(\Sigma_{1,2})$ is a free group (why?) and the homotopy classes of maps $S^1 \to X$ correspond in general to conjugacy classes *of elements in $\pi_1(X)$ (see Exercise 6 on page 38 of Hatcher, or Problem Set 5 #3 from last semester's Topologie I course). Alternatively, if you know how to define the degree of a map between two closed manifolds of the same dimension, you can prove that a homotopy between α and β would imply the existence of a map $S^2 \to \mathbb{T}^2$ with degree 1, and then use the lifting theorem to show that no such map exists.*

Historical remarks: In the early decades of homology theory, it seems to have been generally assumed that the right way to prove statements like "$[f] \in H_k(X)$ is independent of the triangulation" would be via subdivision. In particular, our remarks following part (c) imply that this result would follow immediately from the so-called ***Hauptvermutung****, a conjecture formulated in 1908 stating that any two triangulations of the same space have a common subdivision. Unfortunately, the Hauptvermutung was shown in the 1960's to be false. Even worse, while it has been known since the 1940's that all smooth manifolds admit triangulations, counterexamples are known among topological manifolds of dimensions greater than three (for dimensions 5 and upward, this result is only a few years old). For a brief synopsis of this rather long and complicated story, the survey paper* `http://www.math.ucla.edu/~cm/tm.pdf` *makes interesting reading. The moral of the story is that while triangulations are sometimes helpful for intuition, they are not really the right way to go about proving general theorems in algebraic topology. We will see later in the course that a fundamental class $[M]$ can be defined for every closed oriented topological k-manifold M, without any need for triangulations (though defining the term "oriented" without them requires some cleverness), and it is a generator of $H_k(M) \cong \mathbb{Z}$. The class $[f] \in H_k(X)$ determined by a map $f : M \to X$ is then simply $f_*[M]$.*

[3] See Problem Set 1 #3.