Topology II — Humboldt-Universität zu Berlin
C. Wendl — Winter Semester 2017–18

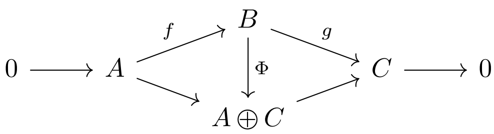
PROBLEM SET 3
To be discussed: 8.11.2017

Instructions

This homework will not be collected or graded, but it is highly advisable to at least think through all of the problems before the next Wednesday lecture after they are distributed, as they will often serve as mental preparation for the material in that lecture. We will discuss the solutions in the Übung beforehand.

***Reminder**: There will be no lecture on Friday, November 3.*

1. (a) Given a short exact sequence of abelian groups $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$, show that the following conditions are equivalent:
   (i) There exists a homomorphism $\pi : B \to A$ such that $\pi \circ f = \mathbb{1}_A$;
   (ii) There exists a homomorphism $i : C \to B$ such that $g \circ i = \mathbb{1}_C$;
   (iii) There exists an isomorphism $\Phi : B \to A \oplus C$ such that $\Phi \circ f(a) = (a, 0)$ and $g \circ \Phi^{-1}(a, c) = c$.

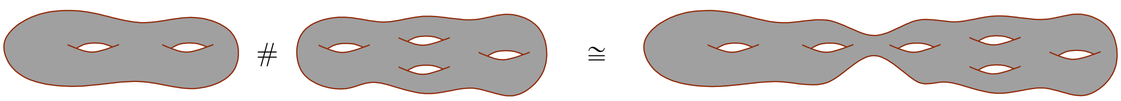


   If any of these conditions holds, we say that the sequence **splits**.
   (b) Show that if the groups in part (a) are all finite-dimensional vector spaces and the homomorphisms are linear maps, then the sequence always splits.
   (c) Find an example of a short exact sequence that does not split.

2. Given two spaces X and Y, use excision and the long exact sequences of the pairs $(X \amalg Y, X)$ and $(X \amalg Y, Y)$ to prove that for the natural inclusions $i^X : X \hookrightarrow X \amalg Y$ and $i^Y : Y \hookrightarrow X \amalg Y$, the map
$$H_*(X; G) \oplus H_*(Y; G) \to H_*(X \amalg Y; G) : (x, y) \mapsto i^X_* x + i^Y_* y$$
is an isomorphism. (Try to produce a proof that does not mention singular simplices at all!)

3. Show that for the singular homology of a space $X = \mathring{A} \cup \mathring{B}$ with coefficient group G, the lowest-degree connecting homomorphism $\partial_* : H_1(X; G) \to H_0(A \cap B; G)$ in the Mayer-Vietoris sequence is trivial whenever $A \cap B$ is path-connected.
Hint: You could prove this directly from the formula for ∂_* that we wrote down in lecture, but it might be quicker and easier to think in terms of reduced homology. What is $\widetilde{H}_0(X; G)$ when X is path-connected?

4. Recall that a **topological n-manifold** is a second-countable Hausdorff space X in which every point has a neighborhood homeomorphic to $\mathbb{R}^n$. Given two connected topological n-manifolds X and Y, their **connected sum** $X \# Y$ is defined by deleting an open n-disk $\mathring{\mathbb{D}}^n$ from each of X and Y and then gluing $X \backslash \mathring{\mathbb{D}}^n$ and $Y \backslash \mathring{\mathbb{D}}^n$ together along an identification of their boundary spheres:



More precisely, we can choose topological embeddings $\iota_X : \mathbb{D}^n \hookrightarrow X$, $\iota_Y : \mathbb{D}^n \hookrightarrow Y$ of the closed unit n-disk $\mathbb{D}^n \subset \mathbb{R}^n$ and then define
$$X \# Y := \left(X \backslash \iota_X(\mathring{\mathbb{D}}^n) \right) \cup_{S^{n-1}} \left(Y \backslash \iota_Y(\mathring{\mathbb{D}}^n) \right),$$

where the gluing identifies the boundaries of both pieces in the obvious way with $S^{n-1} = \partial\mathbb{D}^n$. There are one or two subtle issues about the extent to which $X\#Y$ is (up to homeomorphism) independent of choices, e.g. in general this need not be true without an extra condition involving orientations, but don't worry about this for now. Last semester (see Problem Set 6 #3) we used the Seifert-van Kampen theorem to show that $\pi_1(X\#Y) \cong \pi_1(X) * \pi_1(Y)$ whenever $n \geqslant 3$. We can now use the Mayer-Vietoris sequence to derive a similar formula for the homology of a connected sum.

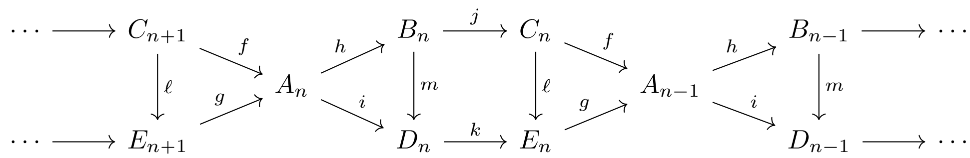
(a) Prove that for any $k = 1, \dots, n-2$ and any coefficient group G, $H_k(X\#Y; G) \cong H_k(X;G) \oplus H_k(Y;G)$.
Hint: There are two steps, as you first need to derive a relation between $H_k(X;G)$ and $H_k(X \backslash \mathring{\mathbb{D}}^n; G)$, and then see what happens when you glue $X\backslash\mathring{\mathbb{D}}^n$ and $Y\backslash\mathring{\mathbb{D}}^n$ together. The case $k = 1$ may need to be handled slightly differently from the others.

(b) It turns out that the formula $H_{n-1}(X\#Y;\mathbb{Z}) \cong H_{n-1}(X;\mathbb{Z}) \oplus H_{n-1}(Y;\mathbb{Z})$ also holds if X and Y are both closed orientable n-manifolds with $n \geqslant 2$, and without orientability we still have $H_{n-1}(X\#Y;\mathbb{Z}_2) \cong H_{n-1}(X;\mathbb{Z}_2) \oplus H_{n-1}(Y;\mathbb{Z}_2)$. Prove this under the additional assumption that $X\backslash\mathring{\mathbb{D}}^n$ and $Y\backslash\mathring{\mathbb{D}}^n$ both admit (possibly oriented) triangulations (cf. Problem Set 2 #4).

(c) Find a counterexample to the formula $H_1(X\#Y;\mathbb{Z}) \cong H_1(X;\mathbb{Z}) \oplus H_1(Y;\mathbb{Z})$ where X and Y are both closed (but not necessarily orientable) 2-manifolds.

5. In lecture we derived the Mayer-Vietoris sequence directly from a short exact sequence of singular chain complexes, but it is interesting to observe that it can also be deduced from the previous theorems we've proved about singular homology, namely its functoriality, the excision property and the long exact sequence of the pair. When we discuss the Eilenberg-Steenrod axioms for homology, this will imply that *any* homology theory satisfying the axioms (not just singular homology) has a Mayer-Vietoris sequence.

   (a) Here's your diagram-chase for this week. Suppose that the following diagram commutes and that the top and bottom rows (both including the A_* terms) are both exact:



   Prove that

$$\cdots \longrightarrow E_{n+1} \xrightarrow{h \circ g} B_n \xrightarrow{(j,-m)} C_n \oplus D_n \xrightarrow{\ell + k} E_n \xrightarrow{h \circ g} B_{n-1} \longrightarrow \cdots$$

   is then an exact sequence.

   (b) Prove that if $X = \mathring{A} \cup \mathring{B}$, then the inclusion of pairs $(B, A \cap B) \hookrightarrow (X, A)$ induces an isomorphism $H_*(B, A\cap B; G) \to H_*(X, A; G)$ via excision. With this in mind, write down an example of the diagram in part (a) where the top and bottom rows are the long exact sequences of the pairs $(B, A\cap B)$ and (X, A) respectively. What does this imply?