PROBLEM SET 7
To be discussed: 6.12.2017

Instructions

This homework will not be collected or graded, but it is highly advisable to at least think through all of the problems before the next Wednesday lecture after they are distributed, as they will often serve as mental preparation for the material in that lecture. We will discuss the solutions in the Übung beforehand.

1. Assume $(I, \leqslant)$ is a directed set and $\{X_\alpha\}_{\alpha \in I}$ is a direct system of topological spaces with associated continuous maps $\varphi_{\beta\alpha} : X_\alpha \to X_\beta$ for each $\alpha, \beta \in I$ with $\beta \geqslant \alpha$, and corresponding maps

$$\varphi_\alpha : X_\alpha \to \varinjlim\{X_\beta\}$$

for each $\alpha \in I$. For any coefficient group G, the singular homology functor $H_*(\,\cdot\,; G)$ transforms this into a direct system of $\mathbb{Z}$-graded abelian groups $\{H_*(X_\alpha; G)\}_{\alpha \in I}$ with associated homomorphisms $\Phi_{\beta\alpha} := (\varphi_{\beta\alpha})_* : H_*(X_\alpha; G) \to H_*(X_\beta; G)$. Denote by

$$\Phi_\alpha : H_*(X_\alpha; G) \to \varinjlim\{H_*(X_\beta; G)\}$$

for each $\alpha \in I$ the natural homomorphism to the direct limit of the homology groups. Notice that the homomorphisms

$$\Psi_\alpha := (\varphi_\alpha)_* : H_*(X_\alpha; G) \to H_*(\varinjlim\{X_\beta\}; G)$$

satisfy $\Psi_\beta \circ \Phi_{\beta\alpha} = \Psi_\alpha$ since $\varphi_\beta \circ \varphi_{\beta\alpha} = \varphi_\alpha$ for all $\beta \geqslant \alpha$, so by the universal property of the direct limit (see Problem Set 6 #2), there exists a unique homomorphism

$$\Psi_\infty : \varinjlim\{H_*(X_\alpha; G)\} \to H_*(\varinjlim\{X_\alpha\}; G)$$

such that $\Psi_\infty \circ \Phi_\alpha = \Psi_\alpha$ for all $\alpha \in I$.

   (a) Show that if every compact subset $K \subset \varinjlim\{X_\beta\}$ is in $\varphi_\alpha(X_\alpha)$ for some $\alpha \in I$, then Ψ_∞ is an isomorphism, i.e. the direct limit of the homology groups matches the homology of the direct limit of spaces.
   Hint: The functor that takes spaces to their singular homology groups is actually the composition of two functors, one that takes a space to its singular homology chain complex, and another that transforms chain complexes to their homologies. We proved in lecture that the second functor commutes with the functor assigning to a direct system its direct limit. You need to prove the corresponding statement for the first functor, that is, show that the direct limit of singular chain complexes matches the singular chain complex of the direct limit of spaces.

   (b) Show that without the extra assumption on compact sets, the result of part (a) is not true. Find an explicit counterexample.
   Hint: What was the strangest direct limit of spaces you saw on Problem Set 6?

   (c) Since singular cohomology $H^*(\,\cdot\,; G)$ is a contravariant functor, it transforms the direct system $\{X_\alpha\}_{\alpha \in I}$ into an inverse system $\{H^*(X_\alpha; G)\}_{\alpha \in I}$. Do you think the analogue of part (a) is likely to be true for cohomology, i.e. will the cohomology of the direct limit be isomorphic to the inverse limit of the cohomologies?

2. Let $\mathbb{R}^\infty$ denote the direct limit of the system of topological spaces $\{\mathbb{R}^n\}_{n \in \mathbb{N}}$ with the associated maps $\mathbb{R}^m \to \mathbb{R}^n$ defined for each $n \geqslant m$ as the inclusion of $\mathbb{R}^m \oplus \{0\} \subset \mathbb{R}^n$. As a set, $\mathbb{R}^\infty$ can be identified naturally with the direct sum $\bigoplus_{n=1}^\infty \mathbb{R}$; note that it is not a direct product, as every element in $\mathbb{R}^\infty$ can be regarded as living in $\mathbb{R}^n$ for some $n \in \mathbb{N}$ and thus has only finitely many nonzero coordinates.

   (a) Explain concretely what it means for a sequence in $\mathbb{R}^\infty$ to converge.

(b) Prove that $\mathbb{R}^\infty$ satisfies the hypothesis of Problem 1(a), i.e. every compact subset of $\mathbb{R}^\infty$ is contained in $\mathbb{R}^n \subset \mathbb{R}^\infty$ for some $n \in \mathbb{N}$.

3. The goal of this problem is to understand why Čech homology $\check{H}_*(X; G)$, in spite of its failure in general to satisfy the exactness axiom of Eilenberg-Steenrod, behaves nicely whenever X is a triangulable space. The answer, in a nutshell, is that for any triangulation of X, there is a natural isomorphism between $\check{H}_*(X; G)$ and the *simplicial* homology (with coefficients in G) of the simplicial complex triangulating X. A simplicial complex is also a CW-complex, and its simplicial homology is the same as its cellular homology, which is therefore isomorphic to the singular homology of X (or any other homology theory that satisfies all the axioms).[1]

For a k-simplex $\sigma \subset X$ in the triangulation, we define the **open star** of σ as the open subset

$$\operatorname{st}\sigma \subset X$$

formed by the union of the interiors of all simplices in the triangulation that have σ as a face.

(a) Let $\mathscr{U}$ denote the open cover of X consisting of the sets $\operatorname{st}\sigma$ for all 0-simplices σ in the triangulation. Show that for any finite collection of 0-simplicies $\sigma_0, \ldots, \sigma_k$, $\operatorname{st}\sigma_0 \cap \ldots \cap \operatorname{st}\sigma_k = \operatorname{st}\tau$ if the triangulation contains a (necessarily unique) k-simplex τ whose vertices are $\sigma_0, \ldots, \sigma_k$, and if not, then this intersection is empty.

(b) Part (a) gives a one-to-one correspondence between the degree k generators of the Čech chain complex $\check{C}_k(\mathscr{U}; G)$ for the cover $\mathscr{U}$ and the set of all k-simplices in the triangulation. Show that this correspondence defines a chain map from $\check{C}_*(\mathscr{U}; G)$ to the simplicial chain complex defined by the triangulation.

(c) Show that if the triangulation of X is replaced by its barycentric subdivision and $\mathscr{U}'$ denotes the open cover defined via open stars of vertices in this subdivided complex, then the natural map $\check{H}_*(\mathscr{U}'; G) \to \check{H}_*(\mathscr{U}; G)$ is an isomorphism.

(d) Use the subdivisions in part (c) to compute the inverse limit and conclude $\check{H}_*(X; G) \cong H_*(X; G)$.

(e) If M is a closed and connected topological n-manifold, we saw in Problem Set 2 #4 that an oriented triangulation of M determines a distinguished element $[M] \in H_n(M; \mathbb{Z})$, the **fundamental class**. Choose any point $x \in M$ that is in the interior of an n-simplex in the triangulation: then if we represent a generator $[M]_x \in H_n(M, M\backslash\{x\}; \mathbb{Z})$ by a single embedded n-simplex that has x in its interior (cf. Problem Set 5 #2), it is not hard to see that the map $H_n(M; \mathbb{Z}) \to H_n(M, M\backslash\{x\}; \mathbb{Z})$ induced by the inclusion $(M, \varnothing) \hookrightarrow (M, M\backslash\{x\})$ sends $[M]$ to a generator $[M]_x$, also known as a **local orientation** at x.

Try to translate this whole discussion into Čech homology, i.e. describe what a fundamental class $[M] \in \check{H}_n(M; \mathbb{Z})$ and a local orientation $[M]_x \in \check{H}_n(M, M\backslash\{x\}; \mathbb{Z})$ might look like in terms of oriented triangulations, such that the natural map $\check{H}_n(M; \mathbb{Z}) \to \check{H}_n(M, M\backslash\{x\}; \mathbb{Z})$ sends $[M]$ to $[M]_x$.
Hint: To understand $\check{H}_n(M, M\backslash\{x\}; \mathbb{Z})$ in terms of specific choices of open covers, you may find it convenient to modify the assumption that x is in the interior of an n-simplex of the triangulation, and assume instead that it coincides with a vertex.

[1] For Problem 2, feel free to use without proof the fact that the simplicial or cellular homology of a simplicial/CW complex is isomorphic to the singular homology of the underlying space.