$R(X,Y)Z = -K_G \, \mathrm{dvol}(X,Y) \, JZ$

for Σ a Riem. 2-mfd embedded $(\mathbb{R}^3, \pm dx^2 + dy^2 + dz^2)$.

<u>more generally</u>: (M,g) a ps-Riem mfd of dim $> n \geq 2$,

$\Sigma \subseteq M$ an n-dim. ps-Riem submfd.

$$\Rightarrow \begin{array}{ccccc} TM|_\Sigma & = & T\Sigma & \oplus & (T\Sigma)^\perp \\ \Psi & & \Psi & & \Psi \\ X & = & X^T & + & X^\perp \end{array}$$

<u>Recall</u>: ∇ the L.-C. conn. on (M,g), related to

$\hat{\nabla}$ the L.-C. conn. on Σ by $\hat{\nabla}_X Y = (\nabla_X Y)^T$.

<u>Lemma</u>: $\exists$ a symmetric bilinear bndl map $\mathrm{II} : T\Sigma \oplus T\Sigma \to (T\Sigma)^\perp$

(the <u>second fundamental form</u> of $\Sigma \subseteq M$) s.t. $\forall X, Y \in \mathfrak{X}(\Sigma)$,

$(\nabla_Y X)^\perp = \mathrm{II}(X,Y)$. $\quad (\Rightarrow \nabla_X Y = \hat{\nabla}_X Y + \mathrm{II}(X,Y))$.

<u>Pf</u>: Defn $\mathrm{II} : \mathfrak{X}(\Sigma) \times \mathfrak{X}(\Sigma) \to \Gamma(T\Sigma^\perp)$ by

$\mathrm{II}(X,Y) := (\nabla_Y X)^\perp$. Then

$\mathrm{II}(Y,X) - \mathrm{II}(X,Y) = (\nabla_X Y - \nabla_Y X)^\perp = [X,Y]^\perp = 0$.

II is C^∞-lin. w.r.t. Y $\Rightarrow$ also X due to symmetry. ∎

For any $\nu \in \Gamma(T\Sigma^\perp)$, defn. $\mathrm{II}_\nu \in \Gamma(T^0_2 \Sigma)$ by $\mathrm{II}_\nu(X,Y) :=$
$\langle \nu, \mathrm{II}(X,Y) \rangle$.

<u>defn</u>: The <u>Weingarten map</u> associated to $\nu \in \Gamma(T\Sigma^\perp)$ is

the self-adjoint lin. bndl map $W_\nu : T\Sigma \to T\Sigma$

determined by $\mathrm{II}_\nu(X,Y) = \langle X, W_\nu(Y) \rangle$.

For any $X \in \mathfrak{X}(\Sigma)$, $\langle X, \nu \rangle \equiv 0$, then for any $Y \in \mathfrak{X}(\Sigma)$,

$$0 = \partial_Y \langle X, \nu \rangle = \underbrace{\langle \nabla_Y X, \nu \rangle}_{= \langle \mathrm{II}(X,Y), \nu \rangle} + \underbrace{\langle X, \nabla_Y \nu \rangle}_{= \langle X, (\nabla_Y \nu)^T \rangle}$$

$$= \mathrm{II}_\nu(X,Y) + \langle X, (\nabla_Y \nu)^T \rangle$$

$$\Rightarrow \boxed{W_\nu(Y) = -(\nabla_Y \nu)^T.}$$

<u>special case</u>: $\Sigma \subseteq M$ a hypersurface w/ orientable normal bndl

$\Rightarrow \exists$ 2 canonical choices of $\nu \in \Gamma(T\Sigma^\perp)$ s.t. $\langle \nu, \nu \rangle = \pm 1$

$\Rightarrow \partial_Y \langle \nu, \nu \rangle = 0 = 2 \langle \nabla_Y \nu, \nu \rangle \Rightarrow \nabla_Y \nu = (\nabla_Y \nu)^T = -W_\nu(Y)$.

$\Rightarrow$ In case $\Sigma \subseteq \mathbb{R}^3$, $\Rightarrow K_G = \pm \det(W_\nu)$.

We call $\nabla \nu : T\Sigma \to T\Sigma$ the <u>shape operator</u> of Σ.

Let R, Riem = curvature tensors on (M, g)

$\hat{R}$, $\widehat{\mathrm{Riem}}$ = " Σ

<u>prop</u> (Gauss equation):

$$\widehat{\mathrm{Riem}}(V, X, Y, Z) = \mathrm{Riem}(V, X, Y, Z) + \langle \mathrm{II}(V, X), \mathrm{II}(Y, Z) \rangle - \langle \mathrm{II}(V, Y), \mathrm{II}(X, Z) \rangle.$$

<u>Pf</u>: If $V, X, Y, Z \in \mathfrak{X}(\Sigma)$,

$$0 = \partial_X \underbrace{\langle V, \mathrm{II}(Y, Z) \rangle}_{=0} = \langle \nabla_X V, \mathrm{II}(Y, Z) \rangle + \langle V, \nabla_X (\mathrm{II}(Y, Z)) \rangle$$

$$\Rightarrow \langle \mathrm{II}(X, V), \mathrm{II}(Y, Z) \rangle = -\langle V, \nabla_X (\mathrm{II}(Y, Z)) \rangle.$$

Now $\widehat{\mathrm{Riem}}(V, X, Y, Z) = \langle V, \hat{R}(X, Y) Z \rangle$

$$= \langle V, \hat{\nabla}_X \hat{\nabla}_Y Z - \hat{\nabla}_Y \hat{\nabla}_X Z - \hat{\nabla}_{[X, Y]} Z \rangle$$

$$= \langle V, \nabla_X (\nabla_Y Z - \mathrm{II}(Y, Z)) - \nabla_Y (\nabla_X Z - \mathrm{II}(X, Z)) - \nabla_{[X, Y]} Z \rangle$$

$$= \underbrace{\langle V, R(X, Y) Z \rangle}_{= \mathrm{Riem}(V, X, Y, Z)} + \langle \mathrm{II}(X, V), \mathrm{II}(Y, Z) \rangle - \langle \mathrm{II}(V, Y), \mathrm{II}(X, Z) \rangle.$$

$\boxtimes$

specialize: $\dim M = 3$, (M,g) loc. flat $(\Leftrightarrow R \equiv 0)$

$\dim \Sigma = 2$, $\langle\,,\,\rangle|_{T\Sigma}$ positive, $T\Sigma$ & $T\Sigma^\perp$ orientable

$\Rightarrow \exists$ (i) $d\mathrm{vol} \in \Omega^2(\Sigma)$

(ii) $J : T\Sigma \to T\Sigma$ $90°$ counterclockwise rotation

(iii) $\nu \in \Gamma(T\Sigma^\perp)$ s.t. $\langle \nu, \nu\rangle = \pm 1 = \begin{cases} +1 & \text{if } (M,g) \text{ has signature } (3,0) \\ -1 & \text{if sig. } (2,1) \end{cases}$

Now ν spans $T\Sigma^\perp \Rightarrow$

$$\mathrm{II}(X,Y) = \pm \mathrm{II}_\nu(X,Y)\,\nu = \pm \langle X, W_\nu(Y)\rangle \nu = \mp \langle X, \nabla_Y \nu\rangle \nu$$

$W_\nu = -\nabla\nu$ self-adjoint $\Rightarrow$ can choose O-N basis $X_1, X_2 \in T_p\Sigma$

s.t. $\nabla\nu(X_i) = \kappa_i X_i$, $\quad \kappa_1, \kappa_2 \in \mathbb{R}$.

WLOG $\quad X_2 = JX_1, \quad X_1 = -JX_2$.

Gauss eqn $\Rightarrow$

$$\langle V, \hat{R}(X_1,X_2)Z\rangle = \langle \mathrm{II}(V,X_1), \mathrm{II}(X_2,Z)\rangle - \langle \mathrm{II}(V,X_2), \mathrm{II}(X_1,Z)\rangle$$

$$= \langle \mp\langle V, \nabla_{X_1}\nu\rangle \nu, \mp \langle Z, \nabla_{X_2}\nu\rangle \nu\rangle - \langle \mp\langle V, \nabla_{X_2}\nu\rangle \nu, \mp\langle Z, \nabla_{X_1}\nu\rangle \nu\rangle$$

$$= \pm \kappa_1\kappa_2 \langle V, X_1\rangle \cdot \langle Z, X_2\rangle \mp \kappa_2\kappa_1 \langle V, X_2\rangle \cdot \langle Z, X_1\rangle$$

$$= \pm \kappa_1\kappa_2 \langle V, \langle Z, X_2\rangle X_1 - \langle Z, X_1\rangle X_2\rangle.$$

$$\langle Z, X_2\rangle X_1 - \langle Z, X_1\rangle X_2 = \langle JZ, -X_1\rangle X_1 - \langle JZ, X_2\rangle X_2$$

$$= -JZ. \qquad \underline{\text{Notice}}: \ d\mathrm{vol}(X_1, X_2) = 1$$

$$\Rightarrow \hat{R}(X_1,X_2)Z = \pm \kappa_1\kappa_2 \cdot d\mathrm{vol}(X_1,X_2) \cdot (-JZ)$$

$$= -K_G \cdot d\mathrm{vol}(X_1,X_2)\, JZ.$$

$\square$

local curvature 2-forms

$\pi: E \to M$ a VB w/ str. grp G, ∇ a G-compat. conn.,

any G-compat. loc. triv. $\Phi_\alpha : E|_{U_\alpha} \to U_\alpha \times \mathbb{F}^m$

$\rightsquigarrow$ connection 1-form $A_\alpha \in \Omega^1(U_\alpha, \mathfrak{g})$ (recall: $\mathfrak{g} := T_{\mathrm{id}} G$)

s.t. $(\nabla_X v)_\alpha = \partial_X v_\alpha + A_\alpha(X) v_\alpha$.

defn: The local curvature 2-form w.r.t. triv. Φ_α is def'd by

$$F_\alpha \in \Omega^2(U_\alpha, \mathbb{F}^{m\times m}), \quad F_\alpha(X,Y) v_\alpha = (\underbrace{\Omega_K(X,Y) v}_{R(X,Y)v})_\alpha .$$

$\Phi_\beta : E|_{U_\beta} \to U_\beta \times \mathbb{F}^m$ another triv w/ transition fn.

$g := g_{\beta\alpha} : U_\alpha \cap U_\beta \to G$, then $v_\beta = g v_\alpha$

$$\Rightarrow F_\beta(X,Y)(g v_\alpha) = (\Omega_K(X,Y)v)_\beta = g(\Omega_K(X,Y)v)_\alpha$$
$$= g F_\alpha(X,Y) v_\alpha$$

$$\Rightarrow \boxed{F_\beta(X,Y) = g F_\alpha(X,Y) g^{-1}.}$$

relation of F_α to A_α:

$$F_\alpha(X,Y) v_\alpha = (R(X,Y)v)_\alpha = (\nabla_X \nabla_Y v - \nabla_Y \nabla_X v - \nabla_{[X,Y]} v)_\alpha$$
$$= (\partial_X + A_\alpha(X))(\partial_Y + A_\alpha(Y)) v_\alpha - (\partial_Y + A_\alpha(Y))(\partial_X + A_\alpha(X)) v_\alpha$$
$$- (\partial_{[X,Y]} + A_\alpha([X,Y])) v_\alpha$$
$$= -\partial_Y(A_\alpha(X)) v_\alpha + \partial_X(A_\alpha(Y)) v_\alpha - A_\alpha([X,Y]) v_\alpha$$
$$+ (A_\alpha(X) A_\alpha(Y) - A_\alpha(Y) A_\alpha(X)) v_\alpha$$
$$= \Big(dA_\alpha(X,Y) + \underbrace{[A_\alpha(X), A_\alpha(Y)]}_{\text{matrix commutator}}\Big) v_\alpha$$

$$\Rightarrow \boxed{F_\alpha(X,Y) = dA_\alpha(X,Y) + [A_\alpha(X), A_\alpha(Y)]}$$

EX: If G abelian, then all $g \in G$ commute w/ all $B \in \mathfrak{g}$

& $[\ ,\ ]$ vanishes on $\mathfrak{g}$.

$\Rightarrow F_\alpha = dA_\alpha \in \Omega^2(U_\alpha, \mathfrak{g})$ & $F_\alpha = F_\beta$ on $U_\alpha \cap U_\beta$

$\Rightarrow \exists$ a global $F \in \Omega^2(M, \mathfrak{g})$ s.t. $F = F_\alpha$ on U_α

$\forall$ G-compat. trivs. Φ_α.

note: F is closed (but not exact)

Specialize: $E = T\Sigma$ for an oriented Riem. 2-mfd (M, g)

$$\Rightarrow \quad G = SO(2) = \left\{ \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \right\}$$

$$\cong U(1) = \{ e^{i\theta} \} \subseteq \mathbb{C}.$$

Replace $SO(2) \rightsquigarrow U(1)$

$\mathbb{R}^2 \rightsquigarrow \mathbb{C}$: $(x, y) \longmapsto x + iy$

triv. $\Phi_\alpha : T\Sigma|_{U_\alpha} \to U_\alpha \times \mathbb{R}^2 \rightsquigarrow \Phi_\alpha : T\Sigma|_{U_\alpha} \to U_\alpha \times \mathbb{C}$

trans. fns. have values in $U(1) \subseteq \mathbb{C}$

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \rightsquigarrow i$$

$\Rightarrow$ mult. by i on $T\Sigma$ is now $J : T\Sigma \to T\Sigma$.

$\mathfrak{u}(1) = i\mathbb{R} \Rightarrow F \in \Omega^2(\Sigma, i\mathbb{R})$.

$$F(X, Y) Z_\alpha := (R(X,Y)Z)_\alpha = -(K_G \, d\mathrm{vol}(X,Y) JZ)_\alpha$$

$$= -i K_G \, d\mathrm{vol}(X, Y) Z_\alpha$$

$$\Rightarrow F = -i K_G \, d\mathrm{vol}.$$

On any region $P \subseteq$ domain U_α of a local triv.,

$$F = dA_\alpha \Rightarrow \int_P K_G \, d\mathrm{vol} = \int_P i \, dA_\alpha = i \int_P dA_\alpha$$

$$= i \int_{\partial P} A_\alpha.$$