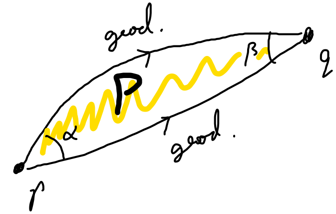
Curvature & geodesics

ex: On a surface (Σ, g)



Gauss-Bonnet $\Longrightarrow$

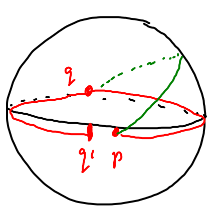
$$0 < \alpha + \beta = \underbrace{(N-2)}_{=0}\pi + \int_P K_G \, d\mathrm{vol}_\Sigma + \underbrace{\text{geod. curv.}}_{=0}$$

$$= \int_P K_G \, d\mathrm{vol}_\Sigma \implies \text{cannot happen if } K_G \leq 0.$$

Q: On a Riem. n-mfd (M, g), spse $\gamma : [a,b] \to M$ a geod. from p to q.

(1) $\exists$ other geods. $p \rightsquigarrow q$ near γ?

(2) $\exists$ shorter paths $p \rightsquigarrow q$ near γ?



Recall: γ geod. $\iff$ stationary for $E(\gamma) = \frac{1}{2}\int_a^b |\dot\gamma|^2 dt$

$\iff$ const. speed & stationary for

$$\ell(\gamma) = \int_a^b |\dot\gamma| \, dt.$$

γ might not be a local minimum.

idea: 2nd-deriv. test!

$E(\gamma) = \frac{1}{2}\int_a^b |\dot\gamma(t)|^2\, dt.$

$\mathcal{P} := \{ C^\infty\text{-maps } [a,b] \xrightarrow{\gamma} M \mid \gamma(a) = p,\ \gamma(b) = q \}$

$T_\gamma \mathcal{P} := \{ \eta \in \Gamma(\gamma^* TM) \mid \eta(a) = 0,\ \eta(b) = 0 \}.$

$E : \mathcal{P} \longrightarrow \mathbb{R}.$ For $\eta \in T_\gamma\mathcal{P}$, $\eta = \partial_s \gamma_s|_{s=0}$ for $\gamma_s \in \mathcal{P}$,

$$dE(\gamma)\eta := \frac{d}{ds} E(\gamma_s)\Big|_{s=0} = \int_a^b \langle \nabla_s \partial_t \gamma_s(t), \dot\gamma(t)\rangle\Big|_{s=0} dt$$

$$= \int_a^b \langle \nabla_t \eta(t), \dot\gamma(t)\rangle\, dt = -\int_a^b \langle \eta(t), \nabla_t \dot\gamma(t)\rangle\, dt$$

Defn. the "L^2-inner product" on $T_\gamma\mathcal{P}$:

$$\langle \eta, \xi\rangle_{L^2} := \int_a^b \langle \eta(t), \xi(t)\rangle\, dt$$

$\Rightarrow dE(\gamma)\eta = \langle -\nabla_t \dot\gamma, \eta\rangle_{L^2}$

Defn. "<u>L^2-gradient</u>" of E: $\nabla E(\gamma) := -\nabla_t \dot\gamma \in \Gamma(\gamma^* TM)$

γ is a geod. $\Leftrightarrow \nabla E(\gamma) = 0.$

<u>goal</u>: Compute "linearization" of ∇E at a pt. $\gamma \in (\nabla E)^{-1}(0)$.

Given path $\{\gamma_s \in \mathcal{P}\}_{s \in (-\epsilon,\epsilon)}$ w/ $\partial_s \gamma_s|_{s=0} = \eta \in T_\gamma \mathcal{P}$,

Defn: $\nabla_\eta \nabla E \in \Gamma(\gamma^* TM)$ by $(\nabla_\eta \nabla E)(t) := \nabla_s (\nabla E(\gamma_s)(t))\Big|_{s=0}$

$$= -\nabla_s(\nabla_t \dot\gamma_s(t))\Big|_{s=0} = -\nabla_s \nabla_t \partial_t \gamma_s(t)\Big|_{s=0}$$

$$= -\nabla_t \underbrace{\nabla_s \partial_t}_{\nabla_t \partial_s} \gamma_s(t)\Big|_{s=0} - R(\partial_s \gamma_s|_{s=0}, \partial_t \gamma_s|_{s=0})\, \partial_t \gamma_s(t)\Big|_{s=0}$$

$$= -\nabla_t^2 \eta(t) - R(\eta(t), \dot\gamma(t))\dot\gamma(t).$$

This defn. a linear operator $\nabla^2 E(\gamma) : \Gamma(\gamma^* TM) \to \Gamma(\gamma^* TM)$:

$\eta \longmapsto \nabla_\eta \nabla E = -\nabla_t^2 \eta - R(\eta, \dot\gamma)\dot\gamma$

<u>prop</u>: Spse $\gamma \in \mathcal{P}$ a geod. & $\gamma_{(\sigma,\tau)} \in \mathcal{P}$ a 2-param. family of paths w/ $\gamma_{(0,0)} = \gamma$, $\partial_\sigma \gamma_{(\sigma,\tau)}\Big|_{\sigma=\tau=0} =: \eta \in T_\gamma\mathcal{P}$, $\partial_\tau \gamma_{(\sigma,\tau)}\Big|_{\sigma=\tau=0} =: \xi$

Then $$\frac{\partial^2}{\partial\sigma\partial\tau} E(\gamma_{(\sigma,\tau)})\Big|_{\sigma=\tau=0} = \langle \underbrace{\nabla^2 E(\gamma)}_{\text{"second variation of } E\text{"}} \xi, \eta\rangle_{L^2}.$$

Pf: EX.

$\Rightarrow$ For a fam. $\gamma_s \in \mathcal{P}$ s.t. $\gamma_0 = \gamma$ is a geod.,

$$E(\gamma_s) = E(\gamma) + \frac{1}{2}s^2 \langle \nabla^2 E(\gamma)\eta, \eta\rangle_{L^2} + O(|s|^3)$$

<u>Q</u>: Under what circumstances can we say $\forall\ \eta \neq 0$,

$\langle \nabla^2 E(\gamma)\eta, \eta\rangle_{L^2} > 0$?

<u>Observation</u>: $$\langle \nabla^2 E(\gamma)\eta, \eta\rangle_{L^2} = \underbrace{-\langle \nabla_t^2 \eta, \eta\rangle_{L^2}}_{\langle \nabla_t \eta, \nabla_t \eta\rangle_{L^2} =: \|\nabla_t \eta\|_{L^2}^2 \geq 0,\ = 0 \text{ iff } \eta = 0.} - \langle \eta, R(\eta, \dot\gamma)\dot\gamma\rangle_{L^2}$$

<u>Q</u>: How to guarantee $-\mathrm{Riem}(\eta(t), \eta(t), \dot\gamma(t), \dot\gamma(t)) \geq 0$?

<u>Recall</u>: On a surface (Σ, g), $X, Y \in T_p\Sigma$ a basis

$$\Rightarrow K_G(p) = \frac{\mathrm{Riem}(X,X,Y,Y)}{|d\mathrm{vol}_\Sigma(X,Y)|^2} \Rightarrow$$

$$\mathrm{Riem}(X,X,Y,Y) = K_G(p)\cdot \mathrm{Area}(X,Y)^2 = K_G(p)\cdot \det\begin{pmatrix}\langle X,X\rangle & \langle X,Y\rangle \\ \langle Y,X\rangle & \langle Y,Y\rangle\end{pmatrix}$$

$$= K_G(p)\cdot\left(\langle X,X\rangle\cdot\langle Y,Y\rangle - \langle X,Y\rangle^2\right)$$

<u>Observe</u>: $\dfrac{\mathrm{Riem}(X,X,Y,Y)}{\det(\cdots)}$ is the same for <u>any</u> basis $X, Y \in T_p\Sigma$.

<u>Defn</u>: On a Riem. mfd (M, g), for $p \in M$ & a 2-dim. subspace $P \subseteq T_pM$, the <u>sectional curvature</u> along P is

$K_s(P) :=$ the Gaussian curv. at p of the submfd.

$\Sigma_p := \exp_p(\text{a suff. small nbhd of } 0 \text{ in } P \subseteq T_pM)$.

<u>Lemma</u>: The 2nd fund. form II of $\Sigma_p \subseteq M$ vanishes at p.

<u>pf</u>: $\exists$ a Riem. normal coord. system near p s.t.

$\Sigma_p = \{(x^1, x^2, 0, \ldots, 0)\}$. Any $X, Y \in T_p\Sigma_p$ extend to vec. flds. near p w/ const. cpnts, then $\nabla_{X(p)}Y = 0$

$\Rightarrow \mathrm{II}(X,Y) = (\nabla_{X(p)}Y)^\perp = 0$. ∎

<u>Cor</u>: For any basis $X, Y \in P \subseteq T_pM$,

$$K_s(P) = \frac{\mathrm{Riem}(X,X,Y,Y)}{\mathrm{Area}(X,Y)^2} = \frac{\mathrm{Riem}(X,X,Y,Y)}{\langle X,X\rangle\langle Y,Y\rangle - \langle X,Y\rangle^2}.$$ ∎

If $K_s(P) \leq 0 \quad \forall\ P, \implies$

$$\mathrm{Riem}(X, X, Y, Y) = K_s(\mathrm{span}(X, Y)) \cdot \mathrm{area}(X, Y)^2 \leq 0$$

$$\forall\ X, Y \in T_pM, \quad \forall\ p \in M.$$

$\implies \nabla^2 E(\gamma)$ always pos.-def.!

thm: Spse (M, g) a Riem. mfd w/ $K_s(P) \leq 0 \quad \forall\ P$.

Then if $\gamma \in \mathcal{P}$ a geod. & $\{\gamma_s \in \mathcal{P}\}_{s \in (-\varepsilon, \varepsilon)}$ s.t. $\gamma_0 = \gamma$; & $\partial_s \gamma_s|_{s=0} \neq 0$

for $\varepsilon > 0$ suff. small:

(1) No other path γ_s for $s \neq 0$ is a geodesic.

(2) $\ell(\gamma_s) > \ell(\gamma) \quad \forall\ s \neq 0$. $\boxed{?}$