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Differentialgeometrie I
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Problem Set 2

To be discussed: 2.–3.11.2021

Problem 1

Identifying the complex plane $\mathbb{C}$ with $\mathbb{R}^2$ via the correspondence $\mathbb{C} \ni x + iy \leftrightarrow (x, y) \in \mathbb{R}^2$, we can regard S^1 as the unit circle in $\mathbb{C}$, which equivalently means the set of all complex numbers of the form $e^{i\theta}$ for $\theta \in \mathbb{R}$. The quotient group $\mathbb{R}^n/\mathbb{Z}^n$ then admits a natural bijection to the n-torus $\mathbb{T}^n = S^1 \times \ldots \times S^1$, given by

$$\mathbb{R}^n/\mathbb{Z}^n \to \mathbb{T}^n : [(\theta_1, \ldots, \theta_n)] \mapsto (e^{2\pi i\theta_1}, \ldots, e^{2\pi i\theta_n}).$$

(a) Use the same trick as in Problem Set 1 #3(b) to define a natural smooth structure on $\mathbb{R}^n/\mathbb{Z}^n$, making it a smooth n-manifold.

(b) Prove that the bijection $\mathbb{R}^n/\mathbb{Z}^n \to \mathbb{T}^n$ defined above is a diffeomorphism.

Problem 2

Show that for two smooth manifolds M, N and any two points $p \in M$ and $q \in N$, there is a canonical vector space isomorphism $T_{(p,q)}(M \times N) = T_pM \times T_qN$.

Problem 3

(a) Find a diffeomorphism from the tangent bundle TS^1 to the product manifold $S^1 \times \mathbb{R}$.

(b) More generally, find a diffeomorphism from $T\mathbb{T}^n$ to $\mathbb{T}^n \times \mathbb{R}^n$ for arbitrary $n \in \mathbb{N}$.
Hint: The diffeomorphism is somewhat easier to see if you take advantage of Problem 1 and replace $\mathbb{T}^n$ *with* $\mathbb{R}^n/\mathbb{Z}^n$.

Problem 4

For this problem, let us make use of Problem 1 and *redefine* $\mathbb{T}^n$ to mean $\mathbb{R}^n/\mathbb{Z}^n$; in the special case $n = 1$, this means $S^1 = \mathbb{R}/\mathbb{Z}$. We can then define a smooth map $f : \mathbb{T}^2 \to S^1$ by

$$f([(s, t)]) := [3s + \sin(2\pi t)].$$

(a) Using the diffeomorphisms of Problem 3 to identify $T\mathbb{T}^n$ with $\mathbb{T}^n \times \mathbb{R}^n$, write down an explicit formula for the tangent map $Tf : T\mathbb{T}^2 \to TS^1$ as a map $\mathbb{T}^2 \times \mathbb{R}^2 \to S^1 \times \mathbb{R}$.

(b) Show that f is a submersion. The implicit function theorem thus implies that for any $p \in S^1$, $f^{-1}(p)$ is a smooth submanifold of $\mathbb{T}^2$. Verify this for $f^{-1}([0])$ in particular, i.e. what precisely is this set? To which well-known manifold is it diffeomorphic?

Notational digression:

For the next two problems, we denote by $\mathbb{R}^{n\times n}$ the vector space of real n-by-n matrices. It contains the *general linear group*, $\mathrm{GL}(n, \mathbb{R}) \subset \mathbb{R}^{n\times n}$, which consists of all invertible matrices, and is an open subset since it is $\det^{-1}(\mathbb{R}\backslash\{0\})$, where $\det : \mathbb{R}^{n\times n} \to \mathbb{R}$ is the determinant map. The latter is a polynomial function of the matrix entries, and is therefore smooth.

Problem 5

We consider the n-dimensional *orthogonal group* $\mathrm{O}(n) = \left\{\mathbf{A} \in \mathbb{R}^{n\times n} \mid \mathbf{A}^T\mathbf{A} = \mathbb{1}\right\}$, where

$\mathbb{1}$ is the n-by-n identity matrix and $\mathbf{A}^T$ denotes the transpose of $\mathbf{A}$. This is precisely the set of all linear transformations $\mathbb{R}^n \to \mathbb{R}^n$ which preserve the Euclidean inner product, which means geometrically that they preserve lengths of vectors and angles between them. We will show in this problem that $\mathrm{O}(n)$ is a smooth submanifold of $\mathbb{R}^{n\times n}$.

(a) The set of all symmetric matrices

$$\Sigma(n) := \{\mathbf{A} \in \mathbb{R}^{n\times n} \mid \mathbf{A} = \mathbf{A}^T\} \subset \mathbb{R}^{n\times n}$$

forms a linear subspace of $\mathbb{R}^{n\times n}$. Defining a map $f : \mathbb{R}^{n\times n} \to \Sigma(n)$ by $f(\mathbf{A}) := \mathbf{A}^T\mathbf{A}$, the orthogonal group is the level set $\mathrm{O}(n) = f^{-1}(\mathbb{1})$. The entries of $f(\mathbf{A})$ are quadratic functions of the entries of $\mathbf{A}$, thus f is clearly a smooth map. Show that its derivative at any $\mathbf{A} \in \mathbb{R}^{n\times n}$ is the linear map

$$Df(\mathbf{A}) : \mathbb{R}^{n\times n} \to \Sigma(n) : \mathbf{H} \mapsto \mathbf{A}^T\mathbf{H} + \mathbf{H}^T\mathbf{A}.$$

Hint: In theory you can do this by computing all the partial derivatives of f with respect to the entries of $\mathbf{A}$, but it's much, much *easier to use the definition of the derivative, i.e. regarding $\mathbb{R}^{n\times n}$ and $\Sigma(n)$ simply as vector spaces, show that a "remainder" formula of the form*

$$f(\mathbf{A} + \mathbf{H}) = f(\mathbf{A}) + Df(\mathbf{A})\mathbf{H} + R(\mathbf{H}) \cdot |\mathbf{H}|$$

with $\lim_{\mathbf{H}\to 0} R(\mathbf{H}) = 0$ is satisfied. One useful thing you may want to assume: for a reasonable choice of norm on $\mathbb{R}^{n\times n}$, matrix products satisfy $|\mathbf{AB}| \leqslant |\mathbf{A}||\mathbf{B}|$.

(b) Show that $Df(\mathbf{A})$ is surjective if $\mathbf{A} \in \mathrm{O}(n)$. In fact, you won't even need to assume $\mathbf{A} \in \mathrm{O}(n)$, but it *is* useful to assume that $\mathbf{A}$ is invertible (which is automatically true for orthogonal matrices).

(c) It follows now from the implicit function theorem that $\mathrm{O}(n)$ is a smooth submanifold of $\mathbb{R}^{n\times n}$. What is its dimension? (For a sanity check I will tell you: $\dim \mathrm{O}(2) = 1$ and $\dim \mathrm{O}(3) = 3$.)

(d) Show that $T_{\mathbb{1}}\,\mathrm{O}(n) \subset T_{\mathbb{1}}\mathbb{R}^{n\times n} = \mathbb{R}^{n\times n}$ is the space of all *antisymmetric* matrices $\mathbf{H}$, i.e. those which satisfy $\mathbf{H}^T = -\mathbf{H}$.

Problem 6

The set $\mathrm{SL}(n,\mathbb{R}) = \{\mathbf{A} \in \mathbb{R}^{n\times n} \mid \det(\mathbf{A}) = 1\}$ is called the *special linear group.*

(a) Show that the derivative of $\det : \mathbb{R}^{n\times n} \to \mathbb{R}$ at $\mathbb{1}$ is given by the *trace*:

$$D(\det)(\mathbb{1})\mathbf{H} = \mathrm{tr}(\mathbf{H}).$$

Hint: Write $\mathbf{H} \in \mathbb{R}^{n\times n}$ in terms of n column vectors as $\begin{pmatrix}\mathbf{v}_1 & \cdots & \mathbf{v}_n\end{pmatrix}$, so

$$\det(\mathbb{1} + t\mathbf{H}) = \det\begin{pmatrix}\mathbf{e}_1 + t\mathbf{v}_1 & \cdots & \mathbf{e}_n + t\mathbf{v}_n\end{pmatrix},$$

where $\mathbf{e}_1, \ldots, \mathbf{e}_n$ denotes the standard basis of $\mathbb{R}^n$. Differentiate this expression with respect to t at $t = 0$, using the fact that the determinant of a matrix is linear in each of its columns.

(b) Use the relation $\det(\mathbf{AB}) = \det(\mathbf{A}) \cdot \det(\mathbf{B})$ to generalize the formula in part (a) to

$$D(\det)(\mathbf{A})\mathbf{H} = \det(\mathbf{A}) \cdot \mathrm{tr}(\mathbf{A}^{-1}\mathbf{H}) \qquad \text{for any} \qquad \mathbf{A} \in \mathrm{GL}(n,\mathbb{R}).$$

(c) Prove that $\mathrm{SL}(n,\mathbb{R})$ is a smooth submanifold of $\mathbb{R}^{n\times n}$, compute its dimension, and show $T_{\mathbb{1}}\,\mathrm{SL}(n,\mathbb{R}) = \{\mathbf{H} \in \mathbb{R}^{n\times n} \mid \mathrm{tr}(\mathbf{H}) = 0\}$.

(d) Is 0 a regular value of the map $\det : \mathbb{R}^{n\times n} \to \mathbb{R}$? Is $\det^{-1}(0)$ a submanifold of $\mathbb{R}^{n\times n}$? *Hint: Show that if $M := \det^{-1}(0)$ is a submanifold, then $T_0 M = \mathbb{R}^{n\times n}$.*