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Differentialgeometrie I

WiSe 2021–22

Problem Set 3

To be discussed: 9.–10.11.2021

Problem 1

We consider the manifold $M := \mathbb{R}$ with its standard smooth structure and the global chart $(\mathbb{R}, x)$ where $x : \mathbb{R} \to \mathbb{R}$ is the identity map. Under the canonical isomorphism $T_p\mathbb{R} = \mathbb{R}$ for all $p \in M$, the coordinate vector field $\partial_x := \frac{\partial}{\partial x}$ then has value $1 \in \mathbb{R}$ at every point.

(a) Compute the flow $\mathbb{R} \times M \supset \mathcal{O} \to M : (t, p) \mapsto \varphi_X^t(p)$ for the vector field $X(p) := p^2\,\partial_x$, including a precise description of its domain $\mathcal{O} \subset \mathbb{R} \times \mathbb{R}$.

(b) For the continuous but nondifferentiable vector field $Y(p) := \sqrt{|p|}\,\partial_x$ on M, find two distinct solutions $\gamma : \mathbb{R} \to M$ to the initial value problem $\dot\gamma(t) = Y(\gamma(t))$ with $\gamma(0) = 0$. *(This is one of a few reasons why we generally restrict our attention to vector fields that are smooth.)*

Problem 2

The following algebraic notions generalize some objects that appeared in lecture this week:

- An *algebra* is a vector space $\mathcal{A}$ equipped with a bilinear "product" operation $\mathcal{A} \times \mathcal{A} \to \mathcal{A} : (x, y) \mapsto xy$ that is also associative, meaning $(xy)z = x(yz)$. A linear map $L : \mathcal{A} \to \mathcal{A}$ is then called a *derivation* on $\mathcal{A}$ if it satisfies the "Leibniz rule" $L(xy) = (Lx)y + x(Ly)$ for all $x, y \in \mathcal{A}$.
- A *Lie algebra* is a vector space V equipped with a bilinear operation $V \times V \to V : (v, w) \mapsto [v, w]$ satisfying the following two conditions:
  - antisymmetry: $[v, w] = -[w, v]$;
  - the Jacobi identity: $[u, [v, w]] + [v, [w, u]] + [w, [u, v]] = 0$.

Show that on any algebra $\mathcal{A}$, the set $\mathcal{D}$ of all derivations on $\mathcal{A}$ has a natural vector space structure, and becomes a Lie algebra if one defines $[L_1, L_2] := L_1 \circ L_2 - L_2 \circ L_1$.
Remark: In light of the natural bijection between the space $\mathfrak{X}(M)$ of smooth vector fields on a manifold M and the set of derivations on $C^\infty(M)$, it follows from this exercise that $\mathfrak{X}(M)$ with the Lie bracket $[\ ,\ ]$ forms a Lie algebra.

Problem 3

Assume $\psi : M \to N$ is a diffeomorphism between two smooth manifolds. Prove:

(a) $\mathcal{L}_{\psi_* X}(\psi_* f) = \psi_*(\mathcal{L}_X f)$ for all $X \in \mathfrak{X}(M)$ and $f \in C^\infty(M)$.

(b) $\psi_*[X, Y] = [\psi_* X, \psi_* Y]$ for all $X, Y \in \mathfrak{X}(M)$.

Problem 4

In this problem, we fix a chart $(\mathcal{U}, x)$ on a manifold M and express vector fields $Z \in \mathfrak{X}(M)$ over the domain $\mathcal{U}$ in terms of their components $Z^1, \dots, Z^n \in C^\infty(\mathcal{U})$ as $Z = Z^i \partial_i$. Here, $\partial_1, \dots, \partial_n \in \mathfrak{X}(\mathcal{U})$ denote the coordinate vector fields determined by the chart, and the Einstein summation convention is in effect so that the symbol "$\sum_{i=1}^n$" has been omitted.

(a) Prove the formula $[X, Y]^i = X^j \partial_j Y^i - Y^j \partial_j X^i$ for the Lie bracket of $X, Y \in \mathfrak{X}(M)$. *Achtung (Einstein alert!): "$\sum_{j=1}^n$" is implied on the right hand side.*

(b) Without using any knowledge of the Lie bracket, prove that for any $X, Y \in \mathfrak{X}(M)$, there exists a vector field $Z \in \mathfrak{X}(M)$ whose components in *any* chart on M are related to those of X and Y by $Z^i = X^j \partial_j Y^i - Y^j \partial_j X^i$. In other words, show that this formula for Z is "coordinate invariant".

(c) In contrast to part (b), show that the formula $Z^i = X^j \partial_j Y^i$ is *not* coordinate invariant, i.e. outside of trivial cases like $X \equiv Y \equiv 0$, there is no vector field $Z \in \mathfrak{X}(M)$ whose components are related to those of two given vector fields $X, Y \in \mathfrak{X}(M)$ in this way for *every* chart.

Comment: Physics textbooks sometimes define the Lie bracket by the coordinate formula in part (a), without mentioning derivations. The price to pay for this approach is that you have to do part (b) before you can be sure that your definition of $[X, Y]$ makes sense.

Problem 5

Prove that for any $X, Y \in \mathfrak{X}(M)$ and $f \in C^\infty(M)$ on a manifold M,

$$[fX, Y] = f[X, Y] - (\mathcal{L}_Y f)X, \qquad \text{and} \qquad [X, fY] = f[X, Y] + (\mathcal{L}_X f)Y.$$

Give two proofs: one in terms of derivations, and another using the local coordinate formula in Problem 4.

Problem 6

On $M := \mathbb{R}^2$, the standard Cartesian coordinates (x, y) define a global chart $(\mathbb{R}^2, (x, y))$. Assume $\mathcal{U} \subset \mathbb{R}^2 \backslash \{0\}$ is any open set such that $(\mathcal{U}, (r, \theta))$ also defines a chart, where (r, θ) are standard polar coordinates, related to (x, y) by $x = r \cos\theta$ and $y = r \sin\theta$.

(a) Prove the relation $\frac{\partial}{\partial r} = \frac{1}{\sqrt{x^2+y^2}}\left(x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}\right)$, and find a similar formula for $\frac{\partial}{\partial \theta}$ in terms of the coordinates x and y.
Remark: Strictly speaking, these relations are valid only on the domain $\mathcal{U} \subset \mathbb{R}^2 \backslash \{0\}$ where both charts are defined, but $\mathcal{U}$ can be chosen to contain any given point in $\mathbb{R}^2 \backslash \{0\}$, and the relations are independent of this choice. It is therefore sensible to define $\frac{\partial}{\partial r}$ and $\frac{\partial}{\partial \theta}$ by these formulas as vector fields on $\mathbb{R}^2 \backslash \{0\}$, though in this case, it is no longer entirely accurate to call them "coordinate" vector fields.

(b) Use the expressions for $\frac{\partial}{\partial r}$ and $\frac{\partial}{\partial \theta}$ in (x, y)-coordinates together with the formula in Problem 4 to prove $\left[\frac{\partial}{\partial r}, \frac{\partial}{\partial \theta}\right] = 0$. Then explain why this fact was already practically obvious.

Problem 7

Using the standard Cartesian coordinates (x, y) on $M := \mathbb{R}^2$, consider vector fields $X, Y \in \mathfrak{X}(\mathbb{R}^2)$ defined by $X(x, y) := \frac{\partial}{\partial x}$ and $Y(x, y) := \sin(2\pi x)\frac{\partial}{\partial y}$.

(a) Compute $[X, Y]$. (It is not zero.)

(b) Show that both X and Y admit global flows, and compute the diffeomorphisms $\varphi_X^t, \varphi_Y^t : \mathbb{R}^2 \to \mathbb{R}^2$ explicitly for all $t \in \mathbb{R}$.

(c) Fix a point $(x, y) \in \mathbb{R}^2$ and consider the "parallelogram map"

$$\alpha(s, t) := \varphi_Y^{-t} \circ \varphi_X^{-s} \circ \varphi_Y^t \circ \varphi_X^s(x, y) \in \mathbb{R}^2$$

for $s, t \in \mathbb{R}$. Show that the lowest-order nontrivial derivative of α at $(s, t) = (0, 0)$ is

$$\partial_s \partial_t \alpha(0, 0) = [X, Y](x, y).$$