Humboldt-Universität zu Berlin
Institut für Mathematik
C. Wendl, S. Dwivedi, O. Müller

Differentialgeometrie I
WiSe 2021–22

Problem Set 4

To be discussed: 16.–17.11.2021

Problem 1
Suppose $(\mathcal{U}, x)$ and $(\widetilde{\mathcal{U}}, \widetilde{x})$ are two overlapping smooth charts on a manifold M, and denote the component functions of a tensor field $S \in \Gamma(T^k_\ell M)$ with respect to these charts by $S^{i_1 \dots i_k}{}_{j_1 \dots j_\ell}$ and $\widetilde{S}^{i_1 \dots i_k}{}_{j_1 \dots j_\ell}$ respectively. Prove that on $\mathcal{U} \cap \widetilde{\mathcal{U}}$,

$$\widetilde{S}^{i_1 \dots i_k}{}_{j_1 \dots j_\ell} = \frac{\partial \widetilde{x}^{i_1}}{\partial x^{a_1}} \dots \frac{\partial \widetilde{x}^{i_k}}{\partial x^{a_k}} S^{a_1 \dots a_k}{}_{b_1 \dots b_\ell} \frac{\partial x^{b_1}}{\partial \widetilde{x}^{j_1}} \dots \frac{\partial x^{b_\ell}}{\partial \widetilde{x}^{j_\ell}}.$$

Hint: The case $(k, \ell) = (1, 0)$ applies to vector fields, for which we did this last week in lecture. For the case $(k, \ell) = (0, 1)$, use the relation $df = \frac{\partial f}{\partial x^j} dx^j$, which holds (why?) for real-valued functions f, and in particular for the coordinates $\widetilde{x}^i$. Then deduce the general case via multilinearity.

Problem 2
In lecture, we defined the *smoothness* of a tensor field $S \in \Gamma(T^k_\ell M)$ via the condition that for all tuples of smooth 1-forms $\lambda^1, \dots, \lambda^k \in \Omega^1(M)$ and smooth vector fields $X_1, \dots, X_\ell \in \mathfrak{X}(M)$, the real-valued function $S(\lambda^1, \dots, \lambda^k, X_1, \dots, X_\ell)$ on M is smooth. Show that the following condition is equivalent: for every smooth chart $(\mathcal{U}, x)$, all of the component functions $S^{i_1 \dots i_k}{}_{j_1 \dots j_\ell} : \mathcal{U} \to \mathbb{R}$ for $i_1, \dots, i_k, j_1, \dots, j_\ell \in \{1, \dots, n\}$ are smooth.
Advice: We originally defined the smoothness of a vector field $X \in \mathfrak{X}(M)$ to mean that the map $X : M \to TM$ is smooth, but it is useful to observe that this is equivalent to the component functions $X^1, \dots, X^n : \mathcal{U} \to \mathbb{R}$ with respect to all smooth charts $(\mathcal{U}, x)$ being smooth. If you are not feeling wide awake enough to think about the smooth structure of the tangent bundle, feel free to skip this initial step and assume the result.

Problem 3
Show that in local coordinates, the components of two tensor fields $S \in \Gamma(T^k_\ell M)$, $T \in \Gamma(T^r_s M)$ and their tensor product $S \otimes T \in \Gamma(T^{k+r}_{\ell+s} M)$ are related by

$$(S \otimes T)^{i_1 \dots i_k a_1 \dots a_r}{}_{j_1 \dots j_\ell b_1 \dots b_s} = S^{i_1 \dots i_k}{}_{j_1 \dots j_\ell} T^{a_1 \dots a_r}{}_{b_1 \dots b_s}.$$

Problem 4
We proved in lecture that any multilinear map

$$S : \underbrace{\Omega^1(M) \times \dots \times \Omega^1(M)}_{k} \times \underbrace{\mathfrak{X}(M) \times \dots \times \mathfrak{X}(M)}_{\ell} \to C^\infty(M)$$

that is C^∞-linear in each of its arguments defines a smooth tensor field of type (k, ℓ). Use the usual canonical isomorphism to derive the following corollary: any multilinear map

$$S : \underbrace{\mathfrak{X}(M) \times \dots \times \mathfrak{X}(M)}_{k} \to \mathfrak{X}(M)$$

that is C^∞-linear in each of its arguments defines a smooth tensor field of type $(1, k)$.

Problem 5
Fix a 1-form $\lambda \in \Omega^1(M)$.

(a) Show that the bilinear map $\omega : \mathfrak{X}(M) \times \mathfrak{X}(M) \to C^\infty(M)$ defined by $\omega(X, Y) := \mathcal{L}_X(\lambda(Y)) - \mathcal{L}_Y(\lambda(X)) - \lambda([X, Y])$ defines a tensor field of type $(0, 2)$.

(b) Independently of part (a), use the coordinate transformation formula of Problem 1 to show that the local coordinate formula $\omega_{ij} := \partial_i \lambda_j - \partial_j \lambda_i$ also defines a tensor field on M (i.e. the formula is coordinate invariant). Then convince yourself that it is the same tensor field as in part (a).

(c) Convince yourself that the local coordinate definition $\omega_{ij} := \partial_i \lambda_j$ is *not* coordinate invariant, i.e. in general there does not exist any $\omega \in \Gamma(T^0_2 M)$ whose components are related to those of λ in this way for *every* choice of chart.

Problem 6

Suppose $J \in \Gamma(T^1_1 M)$ is a smooth almost complex structure, which we will regard as a smooth map $J : TM \to TM$ whose restriction to each tangent space $T_p M$ is a linear map $J_p : T_p M \to T_p M$ with $J_p^2 = -\mathbb{1}$. The *Nijenhuis tensor*[1] is defined from J via the map

$$N : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M), \qquad N(X, Y) := [JX, JY] - J[JX, Y] - J[X, JY] - [X, Y].$$

(a) Use Problem 4 to prove that this formula defines a tensor field of type $(1, 2)$.

(b) Show that in local coordinates, the components of N and J are related by

$$N^i{}_{jk} = J^\ell{}_j \partial_\ell J^i{}_k - J^\ell{}_k \partial_\ell J^i{}_j + J^i{}_\ell \left(\partial_k J^\ell{}_j - \partial_j J^\ell{}_k \right).$$

(c) Show that N vanishes identically if $\dim M = 2$.
Hint: Notice that $N(X, Y)$ is antisymmetric in X and Y. What is $N(X, JX)$?

(d) An almost complex structure J is called *integrable* if near every point $p \in M$ there exists a chart $(\mathcal{U}, x)$ in which the components $J^i{}_j$ become the entries of the constant matrix

$$\mathbf{J}_0 := \begin{pmatrix} 0 & -\mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \in \mathbb{R}^{2n \times 2n},$$

where each of the four blocks is an n-by-n matrix and $\dim M = 2n$. Show that if J is integrable, then $N \equiv 0$.
Advice: One can use the formula in part (b) for this, but an argument based directly on the definition of N via Lie brackets is also possible.

Remark: The matrix $\mathbf{J}_0$ represents the linear transformation $\mathbb{C}^n \to \mathbb{C}^n : \mathbf{z} \mapsto i\mathbf{z}$ if one identifies $\mathbb{C}^n$ with $\mathbb{R}^{2n}$ via the correspondence $\mathbb{C}^n \ni \mathbf{x} + i\mathbf{y} \leftrightarrow (\mathbf{x}, \mathbf{y}) \in \mathbb{R}^n \times \mathbb{R}^n = \mathbb{R}^{2n}$, thus an integrable almost complex structure makes M into a "complex manifold". By a deep theorem of Newlander and Nirenberg from 1957, the converse of part (d) is also true: if the Nijenhuis tensor vanishes, then J is integrable.

[1] Approximate pronounciation: "NIGH-en-house", where "nigh" rhymes with English "sigh".