Humboldt-Universität zu Berlin
Institut für Mathematik
C. Wendl, S. Dwivedi, O. Müller

Differentialgeometrie I
WiSe 2021–22

Problem Set 5

To be discussed: 23.–24.11.2021

Problem 1

Suppose M is a 3-manifold and $\alpha \in \Omega^1(M)$ is nowhere zero, so for every $p \in M$, there is a well-defined 2-dimensional subspace $\xi_p := \ker \alpha_p \subset T_pM$. The set $\xi := \bigcup_{p \in M} \xi_p \subset TM$ in this situation is called a *smooth 2-plane field* in M. We say that ξ is *integrable* if its defining 1-form α satisfies the condition $\alpha \wedge d\alpha \equiv 0$.

(a) Show that the integrability condition depends only on ξ and not on α, i.e. for any $\beta \in \Omega^1(M)$ that is also nowhere zero and satisfies $\ker \beta_p = \xi_p$ for all $p \in M$, $\alpha \wedge d\alpha \equiv 0$ if and only if $\beta \wedge d\beta \equiv 0$.
Hint: If $\ker \alpha_p = \ker \beta_p$, how are the two cotangent vectors $\alpha_p, \beta_p \in T_p^*M$ related?

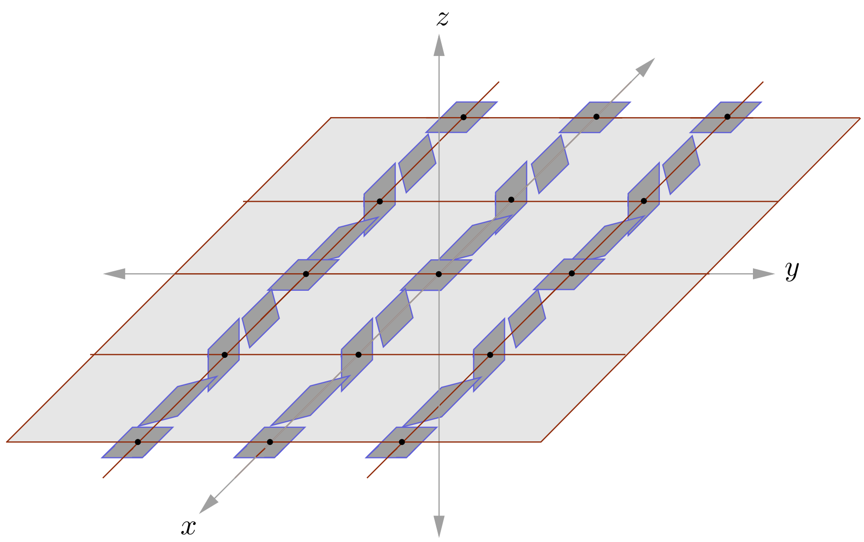
(b) Prove that the following conditions are each equivalent to integrability:

(i) $(d\alpha)_p|_{\xi_p} \in \Lambda^2 \xi_p^*$ vanishes for every $p \in M$.
Hint: Evaluate $(\alpha \wedge d\alpha)_p$ on a basis of T_pM that includes two vectors in ξ_p.

(ii) For every pair of vector fields $X, Y \in \mathfrak{X}(M)$ with $X(p), Y(p) \in \xi_p$ for all $p \in M$, $[X, Y] \in \mathfrak{X}(M)$ also satisfies $[X, Y](p) \in \xi_p$ for all $p \in M$.
Hint: Use our original definition of the exterior derivative, via C^∞-linearity.

(c) Using Cartesian coordinates (x, y, z) on $M := \mathbb{R}^3$, suppose $\alpha = f(x)\, dy + g(x)\, dz$ for smooth functions $f, g : \mathbb{R} \to \mathbb{R}$. Under what conditions on f and g is ξ integrable? Show that if these conditions hold, then for every point $p \in \mathbb{R}^3$ there exists a 2-dimensional submanifold $\Sigma \subset \mathbb{R}^3$ such that $p \in \Sigma$ and $T_q\Sigma = \xi_q$ for all $q \in \Sigma$.

Remark: The result of part (c) is a special case of the Frobenius integrability theorem, which we will prove later in this course. In case you're curious, the following picture gives an example of what $\xi \subset T\mathbb{R}^3$ might look like if it is **not** *integrable. Can you picture a 2-dimensional submanifold that is everywhere tangent to ξ? (I didn't think so.)*



Problem 2
Prove: On an n-dimensional vector space V, a set of dual vectors $\lambda^1, \dots, \lambda^k \in V^* = \Lambda^1 V^*$ is linearly independent if and only if $\lambda^1 \wedge \dots \wedge \lambda^k \in \Lambda^k V^*$ is nonzero.
Hint: Consider products of the form $\left(\sum_{i=1}^k c_i \lambda^i\right) \wedge \lambda^2 \wedge \dots \wedge \lambda^k$.

Problem 3

(a) Find explicit oriented atlases for S^1 and S^2.

(b) Use the oriented atlases in part (a) to show that the antipodal map $S^n \to S^n : p \mapsto -p$ is orientation preserving for $n = 1$, but orientation reversing for $n = 2$.

(c) Without talking about atlases, prove that S^n is orientable for every $n \in \mathbb{N}$ by defining a continuous family of orientations of the tangent spaces $\{T_p S^n \mid p \in S^n\}$.
Hint: Any $p \in S^n$ *together with a basis of* $T_p S^n$ *forms a basis of* $\mathbb{R}^{n+1}$.

(d) Show that the antipodal map $S^n \to S^n$ is orientation preserving for every odd n and orientation reversing for every even n.

Problem 4
Recall that a diffeomorphism $\mathbb{R}^n \supset \mathcal{U} \xrightarrow{\psi} \mathcal{V} \subset \mathbb{R}^n$ is called orientation preserving if $\det D\psi(p) > 0$ for all $p \in \mathcal{U}$, and orientation reversing if $\det D\psi(p) < 0$ for all $p \in \mathcal{U}$. The fact that ψ is a diffeomorphism implies that for any fixed p, one of these conditions must hold, but it need not hold everywhere, i.e. not every diffeomorphism is either orientation preserving or orientation reversing.

(a) Show that if M is an oriented manifold, then every chart $(\mathcal{U}, x)$ whose domain $\mathcal{U} \subset M$ is connected is either orientation preserving or orientation reversing.

(b) In Problem Set 1 #3, we defined the Klein bottle as $K^2 := \mathbb{R}^2/\sim$, where $(s,t) \sim (s, t+1)$ and $(s,t) \sim (s+1, -t)$ for all $(s,t) \in \mathbb{R}^2$. Find a pair of charts $(\mathcal{U}_1, x_1)$ and $(\mathcal{U}_2, x_2)$ on K^2 such that the subsets $\mathcal{U}_1$ and $\mathcal{U}_2$ are both connected but $\mathcal{U}_1 \cap \mathcal{U}_2$ has two connected components, and the transition map $x_1 \circ x_2^{-1}$ is neither orientation preserving nor orientation reversing.

(c) Explain why part (b) implies K^2 is not orientable.

(d) Find a continuous path $\gamma : [0,1] \to K^2$ with $\gamma(0) = \gamma(1) =: p$ and a continuous family of ordered bases $(X_1(t), X_2(t))$ of $T_{\gamma(t)} K^2$ such that $(X_1(0), X_2(0))$ and $(X_1(1), X_2(1))$ determine distinct orientations of the vector space $T_p K^2$.

Problem 5
Define a 1-form on $\mathbb{R}^2 \backslash \{0\}$ in the standard (x,y)-coordinates by

$$\lambda = \frac{1}{x^2 + y^2}(x\,dy - y\,dx),$$

and let $i : S^1 \hookrightarrow \mathbb{R}^2$ denote the natural inclusion of the unit circle in $\mathbb{R}^2$.

(a) Using a finite covering by oriented charts and a subordinate partition of unity, compute $\int_{S^1} i^*\lambda = 2\pi$.

(b) Can you give the result of part (a) a nice interpretation in terms of polar coordinates?

(c) Show that for any smooth function $f : S^1 \to \mathbb{R}$, the 1-form $df \in \Omega^1(S^1)$ satisfies $\int_{S^1} df = 0$.