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Differentialgeometrie I

WiSe 2021–22

Problem Set 6

To be discussed: 30.11–1.12.2021

Problem 1

Define a 1-form on $\mathbb{R}^2 \backslash \{0\}$ in the standard (x, y)-coordinates by $\lambda = \dfrac{1}{x^2 + y^2}(x\,dy - y\,dx)$, and let $i : S^1 \hookrightarrow \mathbb{R}^2$ denote the natural inclusion of the unit circle in $\mathbb{R}^2$.

(a) Using a finite covering by oriented charts and a subordinate partition of unity, compute $\int_{S^1} i^*\lambda = 2\pi$.

(b) Can you give the result of part (a) a nice interpretation in terms of polar coordinates?

(c) Without appealing to Stokes' theorem, show that for any smooth function $f : S^1 \to \mathbb{R}$, the 1-form $df \in \Omega^1(S^1)$ satisfies $\int_{S^1} df = 0$.

(d) Redo the computations of parts (a) and (c) without using a partition of unity: instead use a single chart whose domain covers all of S^1 except for a set of measure zero.

Problem 2

For every oriented n-manifold M with $n \geqslant 1$, there exists another oriented manifold $-M$ that is defined as the same manifold with the "reversed" orientation, meaning that one changes the orientation of every tangent space T_pM.

(a) Show that for every $\omega \in \Omega^n_c(M)$, $\displaystyle\int_{-M} \omega = -\int_M \omega$.
Hint: If you fix the reflection map $r(t^1, t^2, \dots, t^n) := (-t^1, t^2, \dots, t^n)$ on $\mathbb{R}^n$ and take any oriented chart $(\mathcal{U}, x)$ on M, then $(\mathcal{U}, r \circ x)$ will be an oriented chart on $-M$.

(b) What relation is there between $\int_M \varphi^*\omega$ and $\int_N \omega$ for $\omega \in \Omega^n_c(N)$ if $\varphi : M \to N$ is an orientation-*reversing* diffeomorphism?

Problem 3

In local coordinates with respect to an oriented chart $(\mathcal{U}, x)$ on an oriented n-manifold M, a Riemannian metric $g \in \Gamma(T^0_2 M)$ is described in terms of its components $g_{ij} := g(\partial_i, \partial_j)$, so that vectors $X, Y \in T_pM$ at points $p \in \mathcal{U}$ satisfy $g(X, Y) = g_{ij}X^iY^j$. The goal of this problem is to prove that the Riemannian volume form determined by g takes the form

$$d\mathrm{vol} = \sqrt{\det \mathbf{g}}\, dx^1 \wedge \ldots \wedge dx^n \qquad \text{on } \mathcal{U}, \tag{1}$$

where $\mathbf{g} : \mathcal{U} \to \mathbb{R}^{n \times n}$ denotes the matrix-valued function whose ith row and jth column is g_{ij}. Note that this matrix necessarily has positive determinant since g is everywhere positive definite. Fix a point $p \in \mathcal{U}$ and a positively-oriented orthonormal basis $(X_1, \dots, X_n)$ of T_pM, whose dual basis we will denote by $\lambda^1, \dots, \lambda^n \in T^*_pM$. In this case, it was shown in lecture that $d\mathrm{vol}_p = \lambda^1 \wedge \ldots \wedge \lambda^n$. Define matrices $\mathbf{X}, \boldsymbol{\lambda} \in \mathbb{R}^{n \times n}$ whose ith row and jth column are $dx^i(X_j)$ and $\lambda^i(\partial_j)$ respectively. By Proposition 9.10 in the lecture notes, $(\lambda^1 \wedge \ldots \wedge \lambda^n)(\partial_1, \dots, \partial_n) = \det \boldsymbol{\lambda}$.

(a) Prove $\boldsymbol{\lambda} = \mathbf{X}^{-1}$.

(b) Prove $\mathbf{X}^T \mathbf{g} \mathbf{X} = \mathbb{1}$.

(c) Deduce (1).

Problem 4
Using Cartesian coordinates (x, y, z) on $\mathbb{R}^3$, let $\omega := x\, dy \wedge dz + y\, dz \wedge dx + z\, dx \wedge dy \in \Omega^2(\mathbb{R}^3)$, and let $i : S^2 \hookrightarrow \mathbb{R}^3$ denote the inclusion of the unit sphere.

(a) Show that for an appropriate choice of orientation on S^2, $d\mathrm{vol}_{S^2} := i^*\omega \in \Omega^2(S^2)$ is the Riemannian volume form corresponding to the Riemannian metric on S^2 that is induced by the Euclidean inner product of $\mathbb{R}^3$.
Hint: Pick a good vector field $X \in \mathfrak{X}(\mathbb{R}^3)$ with which to write ω as $\iota_X(dx \wedge dy \wedge dz)$.

(b) Show that in the spherical coordinates (θ, ϕ) of Problem Set 1 #1, $d\mathrm{vol}_{S^2} = \cos\phi\, d\theta \wedge d\phi$.

(c) On the open upper hemisphere $\mathcal{U}_+ := \{z > 0\} \subset S^2 \subset \mathbb{R}^3$, one can define a chart $(x, y) : \mathcal{U}_+ \to \mathbb{R}^2$ by restricting to $\mathcal{U}_+$ the usual Cartesian coordinates x and y, which are then related to the z-coordinate on this set by $z = \sqrt{1 - x^2 - y^2}$. Show that $d\mathrm{vol}_{S^2} = \frac{1}{z}\, dx \wedge dy$ on $\mathcal{U}_+$.

(d) Compute the surface area of $S^2 \subset \mathbb{R}^3$ in two ways: once using the formula for $d\mathrm{vol}_{S^2}$ in part (b), and once using part (c) instead. In both cases, you should be able to express the answer in terms of a single Lebesgue integral[1] over a region in $\mathbb{R}^2$, and there will be no need for any partition of unity.

Problem 5
On an n-manifold M with a fixed volume form $d\mathrm{vol} \in \Omega^n(M)$, the *divergence* of a vector field $X \in \mathfrak{X}(M)$ with respect to $d\mathrm{vol}$ is the unique function $\mathrm{div}(X) : M \to \mathbb{R}$ such that the n-form $d(\iota_X d\mathrm{vol})$ matches $\mathrm{div}(X) \cdot d\mathrm{vol}$. Show that in local coordinates with respect to a chart $(\mathcal{U}, x)$ such that $d\mathrm{vol} = f\, dx^1 \wedge \ldots \wedge dx^n$ for a function $f : \mathcal{U} \to \mathbb{R}$, the divergence is given in terms of the components X^i of X by

$$\mathrm{div}(X) = \frac{1}{f}\partial_i(f X^i) \qquad \text{on } \mathcal{U}.$$

Problem 6
On an oriented Riemannian 3-manifold (M, g) with Riemannian volume form $d\mathrm{vol}$, the *curl* of a vector field $X \in \mathfrak{X}(M)$ is the unique vector field $\mathrm{curl}(X) \in \mathfrak{X}(M)$ satisfying $\iota_{\mathrm{curl}(X)} d\mathrm{vol} = d(X_\flat) \in \Omega^2(M)$, where $X_\flat := g(X, \cdot) \in \Omega^1(M)$.

(a) Show that in the special case of $M = \mathbb{R}^3$ with g chosen to be the standard Euclidean inner product,

$$\mathrm{curl}\begin{pmatrix} X^1 \\ X^2 \\ X^3 \end{pmatrix} = \begin{pmatrix} \partial_2 X^3 - \partial_3 X^2 \\ \partial_3 X^1 - \partial_1 X^3 \\ \partial_1 X^2 - \partial_2 X^1 \end{pmatrix} \in \mathfrak{X}(\mathbb{R}^3).$$

(b) The *gradient* $\nabla f \in \mathfrak{X}(M)$ of a function $f : M \to \mathbb{R}$ on a Riemannian manifold (M, g) is uniquely determined by the relation $g(\nabla f, \cdot) = df \in \Omega^1(M)$. Derive from $d \circ d = 0$ the following consequences for all smooth functions $f : M \to \mathbb{R}$ and vector fields $X \in \mathfrak{X}(M)$:

$$\mathrm{curl}(\nabla f) \equiv 0, \qquad \text{and} \qquad \mathrm{div}(\mathrm{curl}\, X) \equiv 0.$$

Problem 7
Prove the following version of *integration by parts*: if M is a compact oriented n-manifold with boundary, $\alpha \in \Omega^k(M)$ and $\beta \in \Omega^\ell(M)$ with $k + \ell = n - 1$, then

$$\int_M d\alpha \wedge \beta = \int_{\partial M} \alpha \wedge \beta - (-1)^k \int_M \alpha \wedge d\beta.$$

[1] You may assume that the upper and lower hemispheres have the same area.