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Differentialgeometrie I
WiSe 2021–22

Problem Set 9

To be discussed: 5.01.2022

Notation: *As in the lectures,* $\mathbb{F}$ *denotes either* $\mathbb{R}$ *or* $\mathbb{C}$, *and all vector spaces, vector bundles and linear maps are over* $\mathbb{F}$ *unless otherwise specified. The dual of a vector space* V *is* $V^* := \mathrm{Hom}(V, \mathbb{F})$, *and* $\mathrm{Hom}(V, W)$ *denotes the space of linear maps* $V \to W$.

Problem 1

Recall that in lecture we defined the tensor product $V \otimes W$ of two finite-dimensional vector spaces V, W as the space of bilinear maps $V^* \times W^* \to \mathbb{F}$. (In case you have seen a more general definition of the tensor product elsewhere, you can view this problem as an effort to convince you that that definition is equivalent to ours.)

(a) Verify that if $v_1, \ldots, v_m$ and $w_1, \ldots, w_n$ are bases of V and W respectively, then the mn elements $v_i \otimes w_j \in V \otimes W$ for $i = 1, \ldots, m$ and $j, \ldots, n$ form a basis of $V \otimes W$.

(b) Show that for any vector space X, there is a canonical isomorphism between the space of linear maps $V \otimes W \to X$ and the space of *bilinear* maps $V \times W \to X$.

(c) Find a canonical isomorphism between $(V \otimes W) \otimes X$ and $V \otimes (W \otimes X)$ that identifies $(v \otimes w) \otimes x$ with $v \otimes (w \otimes x)$ for every $v \in V$, $w \in W$ and $x \in X$.
Hint: Identify both spaces with the space of all multilinear maps $V^* \times W^* \times X^* \to \mathbb{F}$. *In the same manner, one can dispense with parentheses and identify any finite tensor product* $V_1 \otimes \ldots \otimes V_k$ *with the space of multilinear maps* $V_1^* \times \ldots \times V_k^* \to \mathbb{F}$.

Problem 2

(a) Prove that for two finite-dimensional vector spaces V and W, there is a canonical isomorphism $\Psi : V^* \otimes W \to \mathrm{Hom}(V, W)$ such that for all $\lambda \in V^*$ and $w \in W$, $\Psi(\lambda \otimes w)v = \lambda(v)w$.

(b) Given smooth vector bundles E and F of rank m and k respectively over the same manifold M, describe a collection of smoothly compatible local trivializations of

$$\mathrm{Hom}(E, F) := \bigcup_{p \in M} \mathrm{Hom}(E_p, F_p),$$

giving $\mathrm{Hom}(E, F)$ the structure of a smooth vector bundle of rank mk over M.
Hint: One can just as well describe local frames instead of trivializations.

(c) Assume $A : E \to F$ is a map whose restriction $A_p := A|_{E_p}$ to the fiber $E_p \subset E$ over each point $p \in M$ is a linear map to the corresponding fiber $F_p \subset F$; in other words, the map $p \mapsto A_p$ is a section of the bundle $\mathrm{Hom}(E, F)$. Show that it is a *smooth* section if and only if $A : E \to F$ is a smooth map between manifolds.
Remark: This shows that the notion of a "smooth linear bundle map $E \to F$*" as we defined it in lecture is the same thing as a smooth section of* $\mathrm{Hom}(E, F)$.

(d) Show that if $F \subset E$ is a smooth subbundle of the vector bundle $E \xrightarrow{\pi} M$, then the natural map $E \to E/F$ that restricts to each fiber $E_p \subset E$ as the quotient projection $E_p \to E_p/F_p : v \mapsto [v]$ is a smooth linear bundle map.

Problem 3
Assume (M, g) is a Riemannian n-manifold and $N \subset M$ is a smooth k-dimensional submanifold, so for every $p \in N$, $T_pN \subset T_pM$ is a linear subspace and has a well-defined orthogonal complement $(T_pN)^\perp \subset T_pM$ with respect to the inner product g_p. Prove:

(a) $TN^\perp := \bigcup_{p \in N}(T_pN)^\perp$ is a smooth subbundle of $TM|_N$.
Hint: Construct smooth local frames $X_1, \ldots, X_n$ for $TM|_N$ such that $X_1, \ldots, X_k$ are tangent to N and $X_{k+1}, \ldots, X_n$ lie in $(TN)^\perp$.

(b) The composition of the inclusion $TN^\perp \hookrightarrow TM|_N$ with the fiberwise quotient projection $TM|_N \to (TM|_N)/TN =: \nu N$ from Problem 2(d) defines a bundle isomorphism $TN^\perp \to \nu N$.

Problem 4
Prove:

(a) A (real or complex) line bundle is trivial if and only if it admits a section that is nowhere zero.

(b) A real vector bundle of any rank is orientable if and only if it admits a volume form.

(c) A real line bundle is orientable if and only if it is trivial.

(d) A real vector bundle $E \to M$ of rank m is orientable if and only if the bundle $\Lambda^m E \to M$ is trivial.

Problem 5

(a) Prove that every real vector bundle is isomorphic to its dual bundle.
Hint: For a finite-dimensional vector space V, an isomorphism $V \to V^*$ always exists but is not typically canonical. Your bundle isomorphism $E \to E^*$ will similarly need to depend on a non-canonical choice.

(b) Do you think every complex vector bundle is isomorphic to its dual? Just think about it—don't try to prove anything.

Problem 6
For subbundles $E^1, \ldots, E^k \subset E$, we write $E = E^1 \oplus \ldots \oplus E^k$ if the natural map

$$E^1 \oplus \ldots \oplus E^k \to E : (v_1, \ldots, v_k) \mapsto v_1 + \ldots + v_k$$

is a bundle isomorphism. Suppose a splitting of this form exists, and write $m_i := \mathrm{rank}(E^i)$ for $i = 1, \ldots, k$ and $m := \mathrm{rank}(E)$. What does the existence of this splitting tell you about the structure group of E, i.e. to what subgroup of $\mathrm{GL}(m, \mathbb{F})$ can it be reduced?

Problem 7
Prove: If $E \to M$ is a real vector bundle with an indefinite bundle metric $\langle\ ,\ \rangle$ of signature (k, ℓ), then $E = E^+ \oplus E^-$ for a pair of subbundles $E^+, E^- \subset E$ of ranks k and ℓ respectively such that $\langle\ ,\ \rangle$ is positive-definite on each fiber of E^+ and negative-definite on each fiber of E^-. Are the subbundles $E^\pm \subset E$ unique?