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Differentialgeometrie I

WiSe 2021–22

Problem Set 9: Solutions

Notation: *As in the lectures,* $\mathbb{F}$ *denotes either* $\mathbb{R}$ *or* $\mathbb{C}$*, and all vector spaces, vector bundles and linear maps are over* $\mathbb{F}$ *unless otherwise specified. The dual of a vector space* V *is* $V^* := \mathrm{Hom}(V, \mathbb{F})$*, and* $\mathrm{Hom}(V, W)$ *denotes the space of linear maps* $V \to W$*.*

Problem 1

Recall that in lecture we defined the tensor product $V \otimes W$ of two finite-dimensional vector spaces V, W as the space of bilinear maps $V^* \times W^* \to \mathbb{F}$. (In case you have seen a more general definition of the tensor product elsewhere, you can view this problem as an effort to convince you that that definition is equivalent to ours.)

(a) Verify that if $v_1, \ldots, v_m$ and $w_1, \ldots, w_n$ are bases of V and W respectively, then the mn elements $v_i \otimes w_j \in V \otimes W$ for $i = 1, \ldots, m$ and $j = 1, \ldots, n$ form a basis of $V \otimes W$.

Let $v_*^1, \ldots, v_*^m \in V^*$ and $w_*^1, \ldots, w_*^n \in W^*$ denote the dual bases to $v_1, \ldots, v_m$ and $w_1, \ldots, w_n$ respectively, and given any $A \in V \otimes W$, let

$$A^{ij} := A(v_*^i, w_*^j) \in \mathbb{F}$$

for $i = 1, \ldots, m$, $j = 1, \ldots, n$. Then the element[1] $A^{ij} v_i \otimes w_j \in V \otimes W$ evaluates the same as A on all pairs of the form $(v_*^k, w_*^\ell) \in V^* \times W^*$, so we conclude via bilinearity that $A = A^{ij} v_i \otimes w_j$, i.e. every element of $V \otimes W$ is a linear combination of the elements $v_i \otimes w_j$. These elements also are linearly independent since any linear combination $A := A^{ij} v_i \otimes w_j$ for which some coefficient $A^{k\ell}$ is nonzero satisfies $A(v_*^k, w_*^\ell) = A^{k\ell} \neq 0$.

(b) Show that for any vector space X, there is a canonical isomorphism between the space of linear maps $V \otimes W \to X$ and the space of *bilinear* maps $V \times W \to X$.

Denote the space of bilinear maps $V \times W \to X$ by $\mathrm{Hom}_2(V, W; X)$. An obvious way to define a linear map $\Phi : \mathrm{Hom}(V \otimes W, X) \to \mathrm{Hom}_2(V, W; X)$ is by

$$\Phi(A)(v, w) := A(v \otimes w),$$

which works due to the fact that $V \times W \to V \otimes W : (v, w) \mapsto v \otimes w$ is a bilinear map. (Note: bilinearity is actually the only fact one usually needs to know about tensor products—we do not even need to know the precise definition of the space $V \otimes W$ for this problem!) It is clear that Φ is linear and injective; the latter follows from part (a), because if $\Phi(A)(v, w) = 0$ for all $v \in V$ and $w \in W$, it implies that A must vanish on all elements of some basis of $V \otimes W$ and therefore is trivial. Conversely, given any bilinear map $B \in \mathrm{Hom}_2(V, W; X)$, there is clearly a unique $A \in \mathrm{Hom}(V \otimes W, X)$ that satisfies $A(v_i \otimes w_j) = B(v_i, w_j)$ for the basis elements $v_1, \ldots, v_m \in V$ and $w_1, \ldots, w_n \in W$ we considered in part (a). To show that $\Phi(A) = B$, we can write arbitrary elements of V and W as linear combinations $a^i v_i \in V$ and $b^j w_j \in W$ for some coefficients $a^i, b^j \in \mathbb{F}$, and then use bilinearity to compute

$$\Phi(A)(a^i v_i, b^j w_j) = a^i b^j \Phi(A)(v_i, w_j) = a^i b^j A(v_i \otimes w_j) = a^i b^j B(v_i, w_j) = B(a^i v_i, b^j w_j).$$

[1] summation convention in effect!

(c) Find a canonical isomorphism between $(V \otimes W) \otimes X$ and $V \otimes (W \otimes X)$ that identifies $(v \otimes w) \otimes x$ with $v \otimes (w \otimes x)$ for every $v \in V$, $w \in W$ and $x \in X$.
Hint: Identify both spaces with the space of all multilinear maps $V^* \times W^* \times X^* \to \mathbb{F}$. In the same manner, one can dispense with parentheses and identify any finite tensor product $V_1 \otimes \ldots \otimes V_k$ with the space of multilinear maps $V_1^* \times \ldots \times V_k^* \to \mathbb{F}$.

Let $\mathrm{Hom}_3(V^*, W^*, X^*; \mathbb{F})$ denote the space of multilinear maps $V^* \times W^* \times X^* \to \mathbb{F}$. We claim that there is an isomorphism $\Phi : (V \otimes W) \otimes X \to \mathrm{Hom}_3(V^*, W^*, X^*; \mathbb{F})$ uniquely determined by the condition that for every $v \in V$, $w \in W$ and $x \in X$, the multilinear map $\Phi((v \otimes w) \otimes x) : V^* \times W^* \times X^* \to \mathbb{F}$ is given by

$$\Phi((v \otimes w) \otimes x)(\lambda, \mu, \xi) = \lambda(v)\mu(w)\xi(x) \tag{1}$$

for $\lambda \in V^*$, $\mu \in W^*$, $\xi \in X^*$. Indeed, one can define Φ in the first place by choosing bases $v_1, \ldots, v_m \in V$, $w_1, \ldots, w_n \in W$ and $x_1, \ldots, x_\ell \in X$, and requiring (1) to hold whenever v, w, x are all basis elements; since the elements $(v_i \otimes w_j) \otimes x_\ell$ form a basis of $(V \otimes W) \otimes X$ according to part (a), this determines Φ uniquely via linearity. A general formula for Φ is then

$$\Phi(c^{ijk}(v_i \otimes w_j) \otimes x_k)(\lambda, \mu, \xi) = c^{ijk}\lambda(v_i)\mu(w_j)\xi(x_k)$$

for arbitrary coefficients $c^{ijk} \in \mathbb{F}$ with $i = 1, \ldots, m$, $j = 1, \ldots, n$ and $k = 1, \ldots, \ell$. Using the dual bases to write $\lambda = \lambda_i v_*^i \in V^*$, $\mu = \mu_j w_*^j \in W^*$ and $\xi = \xi_k x_*^k \in X^*$, this formula becomes

$$\Phi(c^{ijk}(v_i \otimes w_j) \otimes x_k)(\lambda, \mu, \xi) = c^{ijk}\lambda_i \mu_j \xi_k, \tag{2}$$

making it clear that Φ is an isomorphism. To check that (1) is *always* satisfied, one can now consider arbitrary linear combinations $v = a^i v_i \in V$, $w = b^j w_j \in W$ and $x = c^\ell x_\ell \in X$, and compute

$$\begin{aligned}
\Phi((v \otimes w) \otimes x)(\lambda, \mu, \xi) &= \Phi((a^i v_i \otimes b^j w_j) \otimes c^\ell x_\ell)(\lambda, \mu, \xi) \\
&= a^i b^j c^\ell \Phi((v_i \otimes w_j) \otimes x_\ell)(\lambda, \mu, \xi) \\
&= a^i b^j c^\ell \lambda(v_i)\mu(w_j)\xi(x_\ell) = \lambda(a^i v_i)\mu(b^j w_j)\xi(c^\ell x_\ell) \\
&= \lambda(v)\mu(w)\xi(x).
\end{aligned}$$

The claim about the isomorphism $(V \otimes W) \otimes X \to \mathrm{Hom}_3(V^*, W^*, X^*; \mathbb{F})$ is thus proven. In a precisely analogous manner, one can show that there is a unique isomorphism $\Psi : V \otimes (W \otimes X) \to \mathrm{Hom}_3(V^*, W^*, X^*; \mathbb{F})$ for which $\Psi(v \otimes (w \otimes x)) : V^* \times W^* \times X^* \to \mathbb{F}$ is the same multilinear map as $\Phi((v \otimes w) \otimes x)$. The canonical isomorphism $(V \otimes W) \otimes X \to V \otimes (W \otimes X)$ is then given by $\Psi^{-1} \circ \Phi$.

Addendum:
It is now straightforward to extend part (b) as follows. Suppose $V_1, \ldots, V_k$ are finite-dimensional vector spaces and $\mathrm{Hom}_k(V_1, \ldots, V_k; X)$ denotes the space of multilinear maps $V_1 \times \ldots \times V_k \to X$. Then there is a canonical isomorphism $\Phi : \mathrm{Hom}(V_1 \otimes \ldots \otimes V_k, X) \to \mathrm{Hom}_k(V_1, \ldots, V_k; X)$ such that

$$\Phi(A)(v_1, \ldots, v_k) = A(v_1 \otimes \ldots \otimes v_k)$$

for all $(v_1, \ldots, v_k) \in V_1 \times \ldots \times V_k$. The proof of this requires only two facts: (1) The map $V_1 \times \ldots \times V_k \to V_1 \otimes \ldots \otimes V_k : (v_1, \ldots, v_k) \mapsto v_1 \otimes \ldots \otimes v_k$ is multilinear, and (2) Given any bases of the spaces $V_1, \ldots, V_k$, the set of all tensor products of these basis elements forms a basis of $V_1 \otimes \ldots \otimes V_k$. If one has these properties, one never needs to know the actual definition of the space $V_1 \otimes \ldots \otimes V_k$.

Problem 2

(a) Prove that for two finite-dimensional vector spaces V and W, there is a canonical isomorphism $\Psi : V^* \otimes W \to \mathrm{Hom}(V, W)$ such that for all $\lambda \in V^*$ and $w \in W$, $\Psi(\lambda \otimes w)v = \lambda(v)w$.

We observe first that there is a bilinear map $\widehat{\Psi} : V^* \times W \to \mathrm{Hom}(V, W)$ given by $\widehat{\Psi}(\lambda, w)v := \lambda(v)w$. The result of Problem 1(b) implies that this corresponds under the canonical isomorphism

$$\mathrm{Hom}_2(V^*, W; \mathrm{Hom}(V, W)) \cong \mathrm{Hom}(V^* \otimes W, \mathrm{Hom}(V, W))$$

to a unique linear map $\Psi : V^* \otimes W \to \mathrm{Hom}(V, W)$ such that $\Psi(\lambda \otimes w)v = \lambda(v)w$. To see that it is an isomorphism, choose bases $e_1, \ldots, e_m \in V$, $f_1, \ldots, f_n \in W$ and denote the dual basis of V^* by $e_*^1, \ldots, e_*^m \in V^*$. Then arbitrary elements of $V^* \otimes W$ take the form $A_i^{\ j} e_*^i \otimes f_j$ for coefficients $A_i^{\ j} \in \mathbb{F}$, and for $v = v^k e_k \in V$,

$$\Psi(A_i^{\ j} e_*^i \otimes f_j)v = A_i^{\ j} \Psi(e_*^i \otimes f_j)v = A_i^{\ j} e_*^i(v) f_j = A_i^{\ j} v^i f_j.$$

This last expression says that the coefficients $A_i^{\ j}$ are the entries in the matrix representing the linear map $\Psi(A_i^{\ j} e_*^i \otimes f_j) : V \to W$ with respect to the bases $e_1, \ldots, e_m \in V$ and $f_1, \ldots, f_n \in W$. This linear map vanishes if and only if the matrix entries are all 0, thus Ψ is injective, and it similarly is surjective since any choice of the matrix entries $A_i^{\ j}$ gives rise to a corresponding element $A_i^{\ j} e_*^i \otimes f_j \in V^* \otimes W$.

(b) Given smooth vector bundles E and F of rank m and k respectively over the same manifold M, describe a collection of smoothly compatible local trivializations of

$$\mathrm{Hom}(E, F) := \bigcup_{p \in M} \mathrm{Hom}(E_p, F_p),$$

giving $\mathrm{Hom}(E, F)$ the structure of a smooth vector bundle of rank mk over M. *Hint: One can just as well describe local frames instead of trivializations.*

I will give two solutions.

Solution 1:
In light of part (a), it will be equivalent to describe smoothly compatible local trivializations on $E^* \otimes F$, so let's do that first. As preparation, we start by describing how to obtain a family of smoothly compatible trivializations of the dual bundle E^*. (This was described briefly in lecture, but the fact that they are smoothly compatible was not proved—let's prove it!)
Any point in M admits a neighborhood $\mathcal{U} \subset M$ on which there exists a smooth frame $e_1, \ldots, e_m$ for E, and this determines a dual frame $e_*^1, \ldots, e_*^m$ for E^* over $\mathcal{U}$, defined via the condition $e_*^i(e_j) \equiv \delta_j^i$. We need to check that any two frames for E^* that are constructed in this way are smoothly compatible on the region where they overlap, where "smoothly compatible" means that each consists of sections whose component functions with respect to the other frame are smooth (cf. Proposition 17.4 in the notes). To this end, suppose

$\widehat{\mathcal{U}} \subset M$ is another open set that intersects $\mathcal{U}$, and $\widehat{e}_1, \ldots, \widehat{e}_m$ is a smooth frame for E over $\widehat{\mathcal{U}}$. On $\mathcal{U} \cap \widehat{\mathcal{U}}$, we can write

$$\widehat{e}_i = g_i^{\ j} e_j$$

for unique smooth functions $g_i^{\ j} : \mathcal{U} \cap \widehat{\mathcal{U}} \to \mathbb{F}$. The corresponding dual frame $\widehat{e}_*^1, \ldots, \widehat{e}_*^m : \widehat{\mathcal{U}} \to E^*$ likewise satisfies

$$\widehat{e}_*^i = h^i_{\ j} e_*^j$$

on $\mathcal{U} \cap \widehat{\mathcal{U}}$ for uniquely determined functions $h^i_{\ j} : \mathcal{U} \cap \widehat{\mathcal{U}} \to \mathbb{F}$; we need to show that the latter functions are smooth. To see this, we observe

$$\begin{aligned}\delta^i_j = \widehat{e}_*^i(\widehat{e}_j) = h^i_{\ a} e_*^a \left(g_j^{\ b} e_b \right) = h^i_{\ a} g_j^{\ b} e_*^a(e_b) \\ = h^i_{\ a} g_j^{\ b} \delta^a_b = h^i_{\ a} g_j^{\ a}.\end{aligned}$$

This is a matrix relation: what it says is that if $\mathbf{g}, \mathbf{h} : \mathcal{U} \cap \widehat{\mathcal{U}} \to \mathbb{F}^{m \times m}$ denote the matrix-valued functions whose entries in row i and column j are $g_i^{\ j}$ and $h^i_{\ j}$ respectively, then $\mathbf{h}\mathbf{g}^T \equiv \mathbb{1}$, or equivalently $\mathbf{g}\mathbf{h}^T = \mathbb{1}$, hence $\mathbf{g}(p) \in \mathbb{F}^{m \times m}$ is invertible for every $p \in \mathcal{U} \cap \widehat{\mathcal{U}}$ and $\mathbf{h}(p)$ is the transpose of its inverse. The smoothness of $\mathbf{g}$ thus implies the smoothness of $\mathbf{h}$ and therefore of the individual functions $h^i_{\ j} : \mathcal{U} \cap \widehat{\mathcal{U}} \to \mathbb{F}$. One shows in the same way after reversing the roles of the two frames that each e_*^i has smooth component functions with respect to the frame $\widehat{e}_*^1, \ldots, \widehat{e}_*^m$, thus the two frames correspond to smoothly compatible local trivializations of E^*.

Now consider again $\mathrm{Hom}(E, F)$. Any point in M has a neighborhood $\mathcal{U} \subset M$ on which both E and F admit smooth frames $e_1, \ldots, e_m$ and $f_1, \ldots, f_k$ respectively, and we will again denote the dual frame for E^* by $e_*^1, \ldots, e_*^m$. By Problem 1(a), the sections

$$e_*^i \otimes f_j : \mathcal{U} \to E^* \otimes F, \qquad i = 1, \ldots, m, \; j = 1, \ldots, k$$

then define a frame for $E^* \otimes F$ over U. We can also use part (a) to interpret it as a frame for $\mathrm{Hom}(E, F)$, namely by identifying $e_*^i(p) \otimes f_j(p) \in E_p^* \otimes F_p$ for each $p \in \mathcal{U}$ with the linear map $E_p \to F_p : v \mapsto e_*^i(p)(v) f_j(p)$. With this understood, it remains only to check that any two local frames for $\mathrm{Hom}(E, F)$ constructed in this way are smoothly compatible on the region where they overlap. Suppose $\widehat{\mathcal{U}} \subset M$ is another open set that intersects $\mathcal{U}$ and $\widehat{e}_1, \ldots, \widehat{e}_m$ and $\widehat{f}_1, \ldots, \widehat{f}_k$ denote smooth frames for E and F respectively over $\widehat{\mathcal{U}}$, defining also the dual frame $\widehat{e}_*^1, \ldots, \widehat{e}_*^m$ for E^*. On $\mathcal{U} \cap \widehat{\mathcal{U}}$, we can write

$$\widehat{e}_*^i = h^i_{\ a} e_*^a, \qquad \widehat{f}_j = G_j^{\ b} f_b$$

for unique functions $h^i_{\ a}, G_j^{\ b} : \mathcal{U} \cap \widehat{\mathcal{U}} \to \mathbb{F}$, where the $G_j^{\ b}$ are smooth due to the assumption that both frames for F are smoothly compatible, and the $h^i_{\ a}$ are smooth by the result of the previous paragraph. We then have

$$\widehat{e}_*^i \otimes \widehat{f}_j = h^i_{\ a} e_*^a \otimes G_j^{\ b} f_b = h^i_{\ a} G_j^{\ b} e_*^a \otimes f_b,$$

showing that the components of each of the sections $\widehat{e}_*^i \otimes \widehat{f}_j$ with respect to the frame formed by the sections $e_*^a \otimes f_b$ on $\mathrm{Hom}(E, F)$ are all functions of the form $h^i_{\ a} G_j^{\ b}$, which are clearly smooth. This completes the proof that the corresponding local trivializations for $\mathrm{Hom}(E, F)$ arising from this construction are all smoothly compatible.

Solution 2:
Here is a more direct construction, without using local frames.

Suppose $\mathcal{U}_\alpha \subset M$ is an open set on which both E and F admit local trivializations $\Phi_\alpha : E|_{\mathcal{U}_\alpha} \to \mathcal{U}_\alpha \times \mathbb{F}^m$ and $\Psi_\alpha : F|_{\mathcal{U}_\alpha} \to \mathcal{U}_\alpha \times \mathbb{F}^k$. They can both be written in the form

$$\Phi_\alpha(v) = (p, \Phi_{\alpha,p} v), \qquad \Psi_\alpha(v) = (p, \Psi_{\alpha,p} v), \qquad \text{for} \qquad p \in \mathcal{U}_\alpha,\ v \in E_p,$$

where $\Phi_{\alpha,p} : E_p \to \mathbb{F}^m$ and $\Psi_{\alpha,p} : E_p \to \mathbb{F}^k$ are vector space isomorphisms defined for each $p \in \mathcal{U}_\alpha$. These give rise to a family of vector space isomorphisms

$$\Pi_{\alpha,p} : \mathrm{Hom}(E_p, F_p) \to \mathrm{Hom}(\mathbb{F}^m, \mathbb{F}^k) = \mathbb{F}^{k \times m}$$

defined by $\Pi_{\alpha,p}(A) := \Psi_{\alpha,p} \circ A \circ \Phi_{\alpha,p}^{-1}$, and these can be assembled into a bijection

$$\Pi_\alpha : \mathrm{Hom}(E, F)|_{\mathcal{U}_\alpha} \to \mathcal{U}_\alpha \times \mathbb{F}^{k \times m} : A \mapsto (p, \Pi_{\alpha,p}(A)) \quad \text{for} \quad p \in \mathcal{U}_\alpha,\ A \in \mathrm{Hom}(E_p, F_p),$$

which we shall interpret as a local trivialization (you only have to choose your favorite isomorphism of the vector space $\mathbb{F}^{k \times m}$ with $\mathbb{F}^{km}$). Clearly the entirety of $\mathrm{Hom}(E, F)$ can be covered by local trivializations that are constructed in this way, and we claim that any two of them are smoothly compatible. To see this, suppose $\mathcal{U}_\beta \subset M$ is another open set intersecting $\mathcal{U}_\alpha$, on which there is another pair of local trivializations $\Phi_\beta : E|_{\mathcal{U}_\beta} \to \mathcal{U}_\beta \times \mathbb{F}^m$ and $\Psi_\beta : F|_{\mathcal{U}_\beta} \to \mathcal{U}_\beta \times \mathbb{F}^k$ that are related to Φ_α and Ψ_α by

$$\Phi_\beta \circ \Phi_\alpha^{-1}(p, v) = (p, g_{\beta\alpha}(p)v), \qquad \Psi_\beta \circ \Psi_\alpha^{-1}(p, v) = (p, h_{\beta\alpha}(p)v),$$

thus defining transition functions $g_{\beta\alpha} : \mathcal{U}_\alpha \cap \mathcal{U}_\beta \to \mathrm{GL}(m, \mathbb{F})$ and $h_{\beta\alpha} : \mathcal{U}_\alpha \cap \mathcal{U}_\beta \to \mathrm{GL}(k, \mathbb{F})$. These can also be written as $g_{\beta\alpha}(p) = \Phi_{\beta,p} \circ \Phi_{\alpha,p}^{-1}$ and $h_{\beta\alpha}(p) = \Psi_{\beta,p} \circ \Psi_{\alpha,p}^{-1}$, thus the new local trivialization $\Pi_\beta : \mathrm{Hom}(E, F)|_{\mathcal{U}_\beta} \to \mathcal{U}_\beta \times \mathbb{F}^{k \times m}$ is related to Π_α by $\Pi_\beta \circ \Pi_\alpha^{-1}(p, A) = (p, G_{\beta\alpha}(p)A)$, where

$$\begin{aligned} G_{\beta\alpha}(p)A &= \Pi_{\beta,p} \circ \Pi_{\alpha,p}^{-1}(A) = \Psi_{\beta,p} \circ \Psi_{\alpha,p}^{-1} \circ A \circ \Phi_{\alpha,p} \circ \Phi_{\beta,p}^{-1} = h_{\beta\alpha}(p) A g_{\beta\alpha}(p)^{-1} \\ &= h_{\beta\alpha}(p) A g_{\alpha\beta}(p). \end{aligned}$$

This formula defines a smooth function $G_{\beta\alpha} : \mathcal{U}_\alpha \cap \mathcal{U}_\beta \to \mathrm{End}(\mathbb{F}^{k \times m}) := \mathrm{Hom}(\mathbb{F}^{k \times m}, \mathbb{F}^{k \times m})$, thus Π_α and Π_β are smoothly compatible.

Addendum:
You may be wondering: what relation is there between Solutions 1 and 2, i.e. is the local frame for $\mathrm{Hom}(E, F)$ constructed in Solution 1 over a subset $\mathcal{U} := \mathcal{U}_\alpha \subset M$ equivalent to the local trivialization $\Pi_\alpha : \mathrm{Hom}(E, F)|_{\mathcal{U}_\alpha} \to \mathcal{U}_\alpha \times \mathbb{F}^{k \times m}$ in Solution 2 if we use the same frames/trivializations on E and F for each? The answer is yes. To see it, note that our trivialization in Solution 2 identifies each fiber of $\mathrm{Hom}(E, F)|_{\mathcal{U}_\alpha}$ with $\mathrm{Hom}(\mathbb{F}^m, \mathbb{F}^k)$, so to understand what local frame corresponds to this, we must first decide what to call the "standard" basis of $\mathrm{Hom}(\mathbb{F}^m, \mathbb{F}^k)$. Using the isomorphism $\mathrm{Hom}(\mathbb{F}^m, \mathbb{F}^k) = (\mathbb{F}^m)^* \otimes \mathbb{F}^k$ from part (a), I would say the natural basis of $\mathrm{Hom}(\mathbb{F}^m, \mathbb{F}^k)$ consists of the elements

$$\mathbf{e}_*^i \otimes \mathbf{e}_j, \qquad i = 1, \ldots, m,\ j = 1, \ldots, k,$$

where we denote by $\mathbf{e}_i$ the standard basis vectors on $\mathbb{F}^m$ or $\mathbb{F}^k$ and write $\mathbf{e}_*^i$ for the corresponding dual vectors. In other words, $\mathbf{e}_*^i \otimes \mathbf{e}_j \in \mathrm{Hom}(\mathbb{F}^m, \mathbb{F}^k)$ denotes the unique linear map that sends $\mathbf{e}_i \mapsto \mathbf{e}_j$ and sends all other standard basis vectors of $\mathbb{F}^m$ to 0. Assuming $e_1, \ldots, e_m$ and $f_1, \ldots, f_k$ are the local frames of E and F respectively that are identified with the standard bases via our trivializations $\Phi_\alpha : E|_{\mathcal{U}_\alpha} \to \mathcal{U}_\alpha \times \mathbb{F}^m$ and $\Psi_\alpha : F|_{\mathcal{U}_\alpha} \to \mathcal{U}_\alpha \times \mathbb{F}^k$, you will find that $\Pi_\alpha : \mathrm{Hom}(E, F)|_{\mathcal{U}_\alpha} \to \mathcal{U}_\alpha \times \mathrm{Hom}(\mathbb{F}^m, \mathbb{F}^k)$ now identifies this standard basis of $\mathrm{Hom}(\mathbb{F}^m, \mathbb{F}^k)$ with the local frame we constructed in Solution 1.

(c) Assume $A : E \to F$ is a map whose restriction $A_p := A|_{E_p}$ to the fiber $E_p \subset E$ over each point $p \in M$ is a linear map to the corresponding fiber $F_p \subset F$; in other words, the map $p \mapsto A_p$ is a section of the bundle $\mathrm{Hom}(E, F)$. Show that it is a *smooth* section if and only if $A : E \to F$ is a smooth map between manifolds.
Remark: This shows that the notion of a "smooth linear bundle map $E \to F$" as we defined it in lecture is the same thing as a smooth section of $\mathrm{Hom}(E, F)$.

To talk about smooth maps $A : E \to F$, we need to recall how the smooth structures on the total spaces E and F are defined in terms of smooth local trivializations and charts. Assume $\mathcal{U} \subset M$ is an open set that is small enough so that there exists both a smooth chart $x : \mathcal{U} \to \mathbb{R}^n$ for M and smooth local trivializations $\Phi : E|_{\mathcal{U}} \to \mathcal{U} \times \mathbb{F}^m$ and $\Psi : F|_{\mathcal{U}} \to \mathcal{U} \times \mathbb{F}^k$. These choices give rise to charts on the open subsets $E|_{\mathcal{U}} \subset E$ and $F|_{\mathcal{U}} \subset F$ of the total spaces, in the form

$$\phi := (x \times \mathbb{1}) \circ \Phi : E|_{\mathcal{U}} \to \mathbb{R}^n \times \mathbb{F}^m, \qquad \psi := (x \times \mathbb{1}) \circ \Psi : F|_{\mathcal{U}} \to \mathbb{R}^n \times \mathbb{F}^k.$$

In other words, if we write $\Phi_p : E_p \to \mathbb{F}^m$ for the unique vector space isomorphism such that $\Phi(v) = (p, \Phi_p v)$ for each $p \in \mathcal{U}$ and $v \in E_p$, then ϕ maps $v \in E_p$ to $(x(p), \Phi_p v) \in \mathbb{R}^n \times \mathbb{F}^m$, and we use the obvious identification of $\mathbb{R}^n \times \mathbb{F}^m$ with $\mathbb{R}^{n+m}$ or (in the case $\mathbb{F} = \mathbb{C}$) $\mathbb{R}^{n+2m}$ in order to regard ϕ as a chart (and ψ similarly). Any charts of this form arising from different choices of the smooth chart x and smooth local trivializations Φ, Ψ are smoothly compatible, and the smooth structures of the total spaces E and F are defined as the unique maximal smooth atlases that contain all charts of this form. In practice, we are free to consider *only* charts of this form since E and F can be covered by open subsets of the form $E|_{\mathcal{U}}$ and $F|_{\mathcal{U}}$ for $\mathcal{U} \subset M$ open.
Now, using the charts ϕ and ψ defined above, the key question is this: under what conditions is the map

$$\psi \circ A \circ \phi^{-1} : x(\mathcal{U}) \times \mathbb{F}^m \to x(\mathcal{U}) \times \mathbb{F}^k$$

smooth? For any given point $p \in \mathcal{U}$, writing $q := x(p) \in \mathbb{R}^n$, ϕ^{-1} sends $\{q\} \times \mathbb{F}^m$ to the fiber E_p via the linear isomorphism Φ_p^{-1}, and by assumption A then sends it to F_p via the linear map $A_p : E_p \to F_p$, after which ψ sends it by another linear isomorphism Ψ_p to $\{q\} \times \mathbb{F}^k$. This shows that we can associate to each $p \in \mathcal{U}$ a matrix

$$\mathbf{B}(p) := \Psi_p \circ A_p \circ \Phi_p^{-1} \in \mathrm{Hom}(\mathbb{F}^m, \mathbb{F}^k) = \mathbb{F}^{k \times m},$$

such that

$$\psi \circ A \circ \phi^{-1}(q, v) = (q, \mathbf{B}(x^{-1}(q))v),$$

and since $x^{-1} : x(\mathcal{U}) \to \mathcal{U}$ is smooth, it follows that $\psi \circ A \circ \phi^{-1}$ is smooth if and only if $\mathbf{B} : \mathcal{U} \to \mathbb{F}^{k \times m}$ is a smooth function. Now look again at Solution 2 to part (b): if we use a local trivialization $\Pi : \mathrm{Hom}(E, F)|_{\mathcal{U}} \to \mathcal{U} \times \mathbb{F}^{k \times m}$ as constructed there out of the local trivializations Φ and Ψ for E and F respectively, then the function $\mathbf{B} : \mathcal{U} \to \mathbb{F}^{k \times m}$ is precisely the local representation of the section $M \to \mathrm{Hom}(E, F) : p \mapsto A_p$ with respect to that trivialization. We conclude that the section is smooth if and only if $A : E \to F$ is a smooth map.

(d) Show that if $F \subset E$ is a smooth subbundle of the vector bundle $E \xrightarrow{\pi} M$, then the natural map $E \to E/F$ that restricts to each fiber $E_p \subset E$ as the quotient projection $E_p \to E_p/F_p : v \mapsto [v]$ is a smooth linear bundle map.

The result of part (c) makes this very easy. Recall first how smooth local trivializations of the quotient bundle $E/F \to M$ are constructed: assuming E and F have ranks m and k respectively, one can cover E with local trivializations $\Phi_\alpha : E|_{\mathcal{U}_\alpha} \to \mathcal{U}_\alpha \times \mathbb{F}^m$ having the special property that $\Phi_\alpha(F|_{\mathcal{U}_\alpha}) = \mathcal{U}_\alpha \times \mathbb{F}^k$, where we identify $\mathbb{F}^k$ with the subspace $\mathbb{F}^k \times \{0\} \subset \mathbb{F}^m$. Writing $\Phi_\alpha(v) = (p, \Phi_{\alpha,p} v)$ for $p \in \mathcal{U}_\alpha$ and $v \in E_p$, the corresponding local trivialization of E/F over $\mathcal{U}_\alpha$ takes the form

$$\Psi_\alpha : (E/F)|_{\mathcal{U}_\alpha} \to \mathcal{U}_\alpha \times (\mathbb{F}^m/\mathbb{F}^k) : [v] \mapsto (p, [\Phi_{\alpha,p} v]) \qquad \text{for } p \in \mathcal{U}_\alpha,\ [v] \in E_p/F_p,$$

which is well defined because the isomorphism $\Phi_{\alpha,p} : E_p \to \mathbb{F}^m$ sends $F_p \subset E_p$ isomorphically to $\mathbb{F}^k \subset \mathbb{F}^m$ and thus descends to an isomorphism of quotient spaces $E_p/F_p \to \mathbb{F}^m/\mathbb{F}^k$. Using the trivializations Φ_α and Ψ_α for E and E/F respectively over $\mathcal{U}_\alpha$, the fiberwise projection map $P : E \to E/F$ looks like

$$\Psi_\alpha \circ P \circ \Phi_\alpha^{-1} : \mathcal{U}_\alpha \times \mathbb{F}^m \to \mathcal{U}_\alpha \times \mathbb{F}^m/\mathbb{F}^k : (q, v) \mapsto (q, [v]),$$

i.e. it is represented on every fiber over points in $\mathcal{U}_\alpha$ by the same linear map $\mathbb{F}^m \to \mathbb{F}^m/\mathbb{F}^k$, namely the natural quotient projection. In other words, P is a section of the bundle $\mathrm{Hom}(E, E/F)$ that looks *constant* when expressed in the local trivialization determined by Φ_α and Ψ_α; it is thus smooth since constant functions are smooth.

Final comment: The point of this problem was to make your life easier when working with smooth linear bundle maps in the future. Our initial definition in lecture for the notion of "smoothness" of a linear bundle map $A : E \to F$ was the quickest definition we could give at the time, as we had already proved that the total spaces E and F have natural smooth structures, and could thus talk about smooth maps $E \to F$. But it is usually not convenient in practice to think of bundle maps in this way, and actually checking smoothness in terms of smooth charts on E and F is a bit cumbersome—it is typically easier to think in terms of smooth sections of the bundle $\mathrm{Hom}(E, F)$*, as we did in part (d). From this perspective, the smoothness of many bundle maps that arise naturally in various situations becomes almost obvious.*

Problem 3

Assume (M, g) is a Riemannian n-manifold and $N \subset M$ is a smooth k-dimensional submanifold, so for every $p \in N$, $T_pN \subset T_pM$ is a linear subspace and has a well-defined orthogonal complement $(T_pN)^\perp \subset T_pM$ with respect to the inner product g_p. Prove:

(a) $TN^\perp := \bigcup_{p \in N} (T_pN)^\perp$ is a smooth subbundle of $TM|_N$.
Hint: Construct smooth local frames $X_1, \dots, X_n$ for $TM|_N$ such that $X_1, \dots, X_k$ are tangent to N and $X_{k+1}, \dots, X_n$ lie in $(TN)^\perp$.

We already know $TN \subset TM|_N$ is a smooth subbundle, thus near any point $p \in N$, one can find a neighborhood $\mathcal{U} \subset N$ of p and a smooth local frame $Y_1, \dots, Y_n$ for $TM|_N$ over $\mathcal{U}$ such that $Y_1, \dots, Y_k$ is a local frame for TN over $\mathcal{U}$. Now define a new frame $X_1, \dots, X_n$ for $TM|_N$ over the same region by applying the Gram-Schmidt algorithm to $Y_1, \dots, Y_n$. This ensures that at every point, the vectors $X_1, \dots, X_k$ have the same span as $Y_1, \dots, Y_k$, namely the tangent space to N, but the vectors $X_{k+1}, \dots, X_n$ are also orthogonal to all of these and thus belong to $TN^\perp$. The local trivialization corresponding to the frame $X_1, \dots, X_n$ thus defines isomorphisms $T_qM \to \mathbb{R}^n$ for every $q \in \mathcal{U}$ that identify $T_qN \subset T_qM$ with $\mathbb{R}^k \times \{0\} \subset \mathbb{R}^n$ and $(T_qN)^\perp \subset T_qM$ with $\{0\} \times \mathbb{R}^{n-k} \subset \mathbb{R}^n$, thus making $TN^\perp \subset TM|_N$ a smooth subbundle according to Proposition 17.2 in the notes.

(b) The composition of the inclusion $TN^\perp \hookrightarrow TM|_N$ with the fiberwise quotient projection $TM|_N \to (TM|_N)/TN =: \nu N$ from Problem 2(d) defines a bundle isomorphism $TN^\perp \to \nu N$.

Basic linear algebra implies that on each fiber, the map $(T_pN)^\perp \to (\nu N)_p$ is an isomorphism. That it also depends smoothly on p follows from the fact that the inclusion $TN^\perp \hookrightarrow TM|_N$ is a smooth linear bundle map (because $TN^\perp \subset TM|_N$ is a subbundle) and so is $TM|_N \to \nu N$ (by Problem 2(b)).

Problem 4
Prove:

(a) A (real or complex) line bundle is trivial if and only if it admits a section that is nowhere zero.

Recall, the term "line bundle" means the fibers are 1-dimensional, so a nowhere-zero section of $E \to M$ is in this case the same thing as a *global* frame for E, i.e. a frame that is defined on all of M. There is a natural bijective correspondence between global frames and global trivializations: concretely, we associate to any nowhere-zero section $s : M \to E$ the unique bundle isomorphism $\Phi : E \to M \times \mathbb{F}$ such that $\Phi^{-1}(p, \lambda) = \lambda s(p)$ for all $p \in M$ and $\lambda \in \mathbb{F}$.

(b) A real vector bundle of any rank is orientable if and only if it admits a volume form.

Assume $E \to M$ is a real bundle of rank m and $\mu \in \Lambda^m E^*$ is a volume form, meaning $\mu(p)$ is a nontrivial alternating m-form on E_p for every $p \in M$. Proposition 18.28 in the notes then implies that E must be orientable; this was proved as a corollary of the fact that $\mathrm{SL}(m, \mathbb{R})$ is a subgroup of $\mathrm{GL}_+(m, \mathbb{R})$. More concretely, one can define an orientation of E via the condition that for each $p \in M$, an ordered basis $v_1, \ldots, v_m$ of E_p is positively oriented if and only if $\mu(p)(v_1, \ldots, v_m) > 0$. Conversely, if E is orientable, then there are at least two ways to show that a volume form $\mu \in \Lambda^m E^*$ exists: (1) choose a positive bundle metric (these always exist by Theorem 18.18 in the notes) and appeal to the fact that every bundle metric on an oriented real vector bundle determines a canonical volume form (Proposition 18.29 in the notes); (2) Cover E with local trivializations $(\mathcal{U}_\alpha, \Phi_\alpha)$, choose a volume form $\mu_\alpha \in \Lambda^m E^*|_{\mathcal{U}_\alpha}$ that matches the standard volume form of $\mathbb{R}^m$ in each trivialization, then piece these local choices together using a partition of unity (cf. Problem Set 8 #2).

(c) A real line bundle is orientable if and only if it is trivial.

Every trivial real vector bundle is orientable. The converse is not true for bundles of arbitrary rank m, but it is true when $m = 1$, for the following reason: if $E \to M$ is an orientable line bundle, then by part (b), it admits a volume form, which in the case $m = 1$ means a nowhere-zero section of the dual bundle E^*. The latter is also a line bundle, so part (a) now implies that E^* is trivial, or equivalently, E^* admits a global frame. This frame is dual to a unique global frame of E, thus E is also trivial.

(d) A real vector bundle $E \to M$ of rank m is orientable if and only if the bundle $\Lambda^m E \to M$ is trivial.

The only tricky aspect of this problem is that it says $\Lambda^m E$ instead of $\Lambda^m E^*$. Let us first show that E is orientable if and only if $\Lambda^m E^*$ is trivial: by part (a), the latter is equivalent to $\Lambda^m E^*$ admitting a nowhere-zero section, which is the same thing as a volume form for E, whose existence is equivalent via part (b) to E being orientable. So far so good.
I can think of three methods for proving the same thing about $\Lambda^m E$ instead of $\Lambda^m E^*$.

Method 1: Here is a fairly direct argument. Suppose first that $\Lambda^m E$ is trivial; since $\operatorname{rank}(\Lambda^m E) = 1$, it then follows from part (a) that there exists a nowhere-zero section $\mu \in \Gamma(\Lambda^m E)$. For any $p \in M$ and any basis $v_1, \ldots, v_m$ of E_p, the wedge product $v_1 \wedge \ldots \wedge v_m \in \Lambda^m E_p$ is then nontrivial and must be a scalar multiple of $\mu(p)$ since the fibers of $\Lambda^m E$ are 1-dimensional, so

$$v_1 \wedge \ldots \wedge v_m = c\mu(p)$$

for some $c \in \mathbb{R}\backslash\{0\}$. We define an orientation on E via the condition that $v_1, \ldots, v_m$ is positively oriented if and only if $c > 0$. Conversely, if E is oriented, then we have a preferred class of local frames $e_1, \ldots, e_m$ for E over subsets $\mathcal{U} \subset M$, forming positively-oriented bases on the fibers at points in $\mathcal{U}$. Any such frame gives rise to a nontrivial section $e_1 \wedge \ldots \wedge e_m$ of $\Lambda^m E$ over $\mathcal{U}$, which can also be interpreted as a frame for $\Lambda^m E$ since the fibers of $\Lambda^m E$ are 1-dimensional. We claim that if $e_1', \ldots, e_m'$ is another positively-oriented frame for E on a region $\mathcal{U}' \subset M$ that intersects $\mathcal{U}$, then

$$e_1' \wedge \ldots \wedge e_m' = f\, e_1 \wedge \ldots \wedge e_m \qquad \text{on } \mathcal{U} \cap \mathcal{U}' \tag{3}$$

for a *positive* function $f : \mathcal{U} \cap \mathcal{U}' \to (0, \infty)$. Indeed, applying Proposition 9.10 from the notes to the vector space $V := E_p^*$ at any point $p \in \mathcal{U} \cap \mathcal{U}'$, we find

$$e_1' \wedge \ldots \wedge e_m' = \det \begin{pmatrix} e_*^1(e_1') & \cdots & e_*^m(e_1') \\ \vdots & \ddots & \vdots \\ e_*^1(e_m') & \cdots & e_*^m(e_m') \end{pmatrix} e_1 \wedge \ldots \wedge e_m, \tag{4}$$

where $e_*^1, \ldots, e_*^m$ denotes the dual frame of $e_1, \ldots, e_m$. To see that the determinant of this matrix is positive, note that since $e_1, \ldots, e_m$ and $e_1', \ldots, e_m'$ define the same orientation of E_p at any point $p \in \mathcal{U} \cap \mathcal{U}'$, one can deform the basis $e_1'(p), \ldots, e_m'(p)$ to $e_1(p), \ldots, e_m(p)$ through a continuous family of ordered bases of E_p. This deformation has the effect of deforming the matrix in (4) to the identity matrix through a continuous family of invertible matrices; since $\det(\mathbb{1}) > 0$, the determinant in (4) must therefore also be positive, proving the claim. If we now convert frames of the form $e_1 \wedge \ldots \wedge e_m$ into local trivializations of $\Lambda^m E$, the positivity of the function f in (3) implies that these trivializations are always compatible with each other via *positive* transition functions, i.e. the transition functions take values in $\mathrm{GL}_+(1, \mathbb{R}) = (0, \infty)$. In this way, we have endowed $\Lambda^m E$ with an orientation, so by part (c), $\Lambda^m E$ is therefore trivial.

The other two methods both derive the result from the observation above that E is orientable if and only if $\Lambda^m E^*$ is trivial.

Method 2: Convince yourself that E is orientable if and only if E^* is. In fact, any orientation on a vector space V canonically determines an orientation on its dual space V^* via the condition that an ordered basis of V is positively oriented if and only if its dual basis is a positively-oriented basis of V^*. In this way, any continuous family of orientations of the fibers of E gives rise to a continuous family of orientations of the fibers of E^*, and vice versa. Once this is established, you can view nowhere-zero sections of $\Lambda^m E$ as volume forms on E^* and then appeal to the previous observation: identifying E^{**} with E, E^* is orientable if and only if $\Lambda^m E^{**} = \Lambda^m E$ is trivial.

Method 3: There is a natural isomorphism of $\Lambda^m E^*$ with the dual bundle of $\Lambda^m E$, implying that the latter is trivial if and only if the former is. In fact, for any real m-dimensional vector space V, each top-dimensional alternating form $\mu \in \Lambda^m V^*$ defines a linear map $\Lambda^m V \to \mathbb{R}$ in the following way. As an m-fold multilinear map $\mu : V \times \ldots \times V \to \mathbb{R}$, the addendum to our solution of Problem 1 identifies μ with a linear map $\widehat{\mu} : V^{\otimes m} := \underbrace{V \otimes \ldots \otimes V}_{m} \to \mathbb{R}$ that is uniquely determined by the formula

$$\widehat{\mu}(v_1 \otimes \ldots \otimes v_m) = \mu(v_1, \ldots, v_m) \qquad \text{for all} \qquad v_1, \ldots, v_m \in V.$$

The restriction of $\widehat{\mu}$ to the subspace $\Lambda^m V \subset V^{\otimes m}$ thus defines a linear map $\Lambda^m V \to \mathbb{R}$, or in other words, an element of the dual space $(\Lambda^m V)^*$, so that we have in this way defined a linear map

$$\Lambda^m(V^*) \to (\Lambda^m V)^* : \mu \mapsto \widehat{\mu}|_{\Lambda^m V}. \tag{5}$$

To see that this map is injective, note that for any basis $v_1, \ldots, v_m$ of V, $\Lambda^m V$ is spanned by

$$v_1 \wedge \ldots \wedge v_m = \sum_{\sigma \in S_m} (-1)^{|\sigma|} v_{\sigma(1)} \otimes \ldots \otimes v_{\sigma(m)} \in \Lambda^m V \subset V^{\otimes m},$$

and

$$\begin{aligned}
\widehat{\mu}(v_1 \wedge \ldots \wedge v_m) &= \widehat{\mu}\left(\sum_{\sigma \in S_m} (-1)^{|\sigma|} v_{\sigma(1)} \otimes \ldots \otimes v_{\sigma(m)} \right) \\
&= \sum_{\sigma \in S_m} (-1)^{|\sigma|} \widehat{\mu}(v_{\sigma(1)} \otimes \ldots \otimes v_{\sigma(m)}) = \sum_{\sigma \in S_m} (-1)^{|\sigma|} \mu(v_{\sigma(1)}, \ldots, v_{\sigma(m)}) \\
&= m!\, \mu(v_1, \ldots, v_m),
\end{aligned}$$

where in the last line we have appealed to the fact that μ is antisymmetric in order to remove the permutations. Since μ is a top-dimensional form on V and $v_1, \ldots, v_m$ is a basis, this expression is nonzero if and only if $\mu \neq 0$, proving that the map (5) is injective. It is therefore also surjective, as $\dim \Lambda^m(V^*) = \dim(\Lambda^m V)^* = 1$. Applying this isomorphism to all fibers of the bundle $E \to M$ defines a bijection

$$\Psi : \Lambda^m(E^*) \to (\Lambda^m E)^*$$

that gives a vector space isomorphism $\Lambda^m(E_p^*) \to (\Lambda^m E_p)^*$ for every $p \in M$. We claim that this is a *smooth* linear bundle map, and therefore a bundle isomorphism. For this it suffices to show that on a neighborhood of any given point in M, Ψ maps some smooth frame for $\Lambda^m(E^*)$ to a smooth frame for $(\Lambda^m E)^*$. Indeed, suppose $e_1, \ldots, e_m$ is a smooth frame for E on some region $\mathcal{U} \subset M$, with dual frame $e^1_*, \ldots, e^m_*$, so $e^1_* \wedge \ldots \wedge e^m_*$ defines a smooth frame for $\Lambda^m(E^*)$ and $e_1 \wedge \ldots \wedge e_m$ a smooth frame for $\Lambda^m E$ over the same region $\mathcal{U}$. At each point in $\mathcal{U}$, evaluating $\Psi(e^1_* \wedge \ldots \wedge e^m_*)$ on $e_1 \wedge \ldots \wedge e_m$ then gives

$$\begin{aligned}\Psi(e^1_* \wedge \ldots \wedge e^m_*)(e_1 \wedge \ldots \wedge e_m) &= \Psi(e^1_* \wedge \ldots \wedge e^m_*)\left(\sum_{\sigma \in S_m} (-1)^{|\sigma|} e_{\sigma(1)} \otimes \ldots \otimes e_{\sigma(m)}\right) \\ &= \sum_{\sigma \in S_m} (-1)^{|\sigma|} \Psi(e^1_* \wedge \ldots \wedge e^m_*)(e_{\sigma(1)} \otimes \ldots \otimes e_{\sigma(m)}) \\ &= \sum_{\sigma \in S_m} (-1)^{|\sigma|} (e^1_* \wedge \ldots \wedge e^m_*)(e_{\sigma(1)}, \ldots, e_{\sigma(m)}) \\ &= m!\,(e^1_* \wedge \ldots \wedge e^m_*)(e_1, \ldots, e_m) = m!.\end{aligned}$$

This reveals that Ψ maps the smooth frame $\frac{1}{m!} e^1_* \wedge \ldots \wedge e^m_*$ to the dual frame of $e_1 \wedge \ldots \wedge e_m$, which is indeed a smooth frame for $(\Lambda^m E)^*$ over $\mathcal{U}$.

Problem 5

(a) Prove that every real vector bundle is isomorphic to its dual bundle.
Hint: For a finite-dimensional vector space V, an isomorphism $V \to V^*$ always exists but is not typically canonical. Your bundle isomorphism $E \to E^*$ will similarly need to depend on a non-canonical choice.

Recall that every vector bundle $E \to M$ admits a positive bundle metric $\langle\ ,\ \rangle$, so we can choose one and define a bundle isomorphism $E \to E^*$ by

$$v \mapsto \langle v, \cdot \rangle.$$

(b) Do you think every complex vector bundle is isomorphic to its dual? Just think about it—don't try to prove anything.

The answer is no: E and E^* are not always isomorphic if $E \to M$ is a complex vector bundle. The reason the construction from part (a) does not work in the complex case is that if $\langle\ ,\ \rangle$ is an inner product on a complex vector space V, then $v \mapsto \langle v, \cdot \rangle$ gives a bijection $V \to V^*$ but it is not a complex-linear map; it is in fact complex *antilinear* since $\langle iv, \cdot \rangle = -i\langle v, \cdot \rangle$. One could try to get around this problem by using the map $v \mapsto \langle \cdot, v \rangle$ instead of $\langle v, \cdot \rangle$, but now there is a different problem: $\langle \cdot, v \rangle : V \to \mathbb{R}$ is a complex-antilinear map, and is thus not an element of the dual space V^*. (One can more accurately regard it as an element of the *conjugate* dual space $\bar{V}^*$; see Exercise 17.21 in the notes.)

Commentary:
To actually show that no isomorphism $E \to E^*$ exists in certain examples requires topological tools that we have not discussed in this course, but I can summarize the simplest case as follows. If M is a closed oriented 2-manifold and $E \to M$ is a complex line bundle, then there is a numerical invariant associated to E, called its *first Chern number* $c_1(E) \in \mathbb{Z}$, with the property that for any section $s \in \Gamma(E)$ that vanishes at only finitely many points, counting the zeroes of s with suitable signs and weights always gives $c_1(E)$. (In complex analysis, you may have seen the notion of the *order* of a zero of a holomorphic function, which is a positive integer one can define in terms of a winding number. If one instead considers an antiholomorphic function, then a zero will instead have *negative* order. The order of a zero in this sense is what I meant by "suitable signs and weights" above.) Once you've convinced yourself that $c_1(E)$ is well defined, it is not hard to show that (1) $c_1(E) = c_1(F)$ whenever E and F are isomorphic complex line bundles over M, and (2)

$c_1(E^*) = -c_1(E)$. It follows that a complex line bundle over a closed oriented surface cannot be isomorphic to its dual bundle unless its first Chern number is 0, and it is easy to come up with examples for which $c_1(E)$ is nonzero. One such example is $TS^2 \to S^2$, if one chooses a complex structure in order to regard the tangent spaces T_pS^2 as complex 1-dimensional vector spaces (see Section 18.7 in the notes). One then has $c_1(TS^2) = 2$, implying that TS^2 is not complex-isomorphic to its dual bundle. (Incidentally, this also implies a famous result known colloquially as the "hairy sphere" theorem: there exists no continuous vector field on S^2 that is nowhere zero, i.e. "you can't comb the hair on a sphere". We will be able to prove this by the end of the semester.)

Problem 6
For subbundles $E^1, \dots, E^k \subset E$, we write $E = E^1 \oplus \dots \oplus E^k$ if the natural map

$$E^1 \oplus \dots \oplus E^k \to E : (v_1, \dots, v_k) \mapsto v_1 + \dots + v_k$$

is a bundle isomorphism. Suppose a splitting of this form exists, and write $m_i := \operatorname{rank}(E^i)$ for $i = 1, \dots, k$ and $m := \operatorname{rank}(E)$. What does the existence of this splitting tell you about the structure group of E, i.e. to what subgroup of $\operatorname{GL}(m, \mathbb{F})$ can it be reduced?

Define the subgroup $G \subset \operatorname{GL}(m, \mathbb{F})$ to consist of all block-diagonal matrices of the form

$$\mathbf{A} = \begin{pmatrix} \mathbf{A}_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \mathbf{A}_k \end{pmatrix}$$

where $\mathbf{A}_i \in \operatorname{GL}(m_i, \mathbb{R})$ for $i = 1, \dots, k$. We claim that splittings $E = E^1 \oplus \dots \oplus E^k$ are equivalent to G-structures on E for this particular group G. Indeed, if such a splitting is given, then on a sufficiently small neighborhood of any point in M one can find smooth frames for each of the subbundles $E^i \subset E$ and assemble them together into a frame $e_1, \dots, e_m$ for E such that $e_1, \dots, e_{m_1}$ have values in E^1, $e_{m_1+1}, \dots, e_{m_1+m_2}$ have values in E^2, and so forth. The transition functions relating any two trivializations that correspond to frames of this form take values in G. Conversely, if a G-structure on E is given, then one can consider the obvious splitting

$$\mathbb{R}^m = \mathbb{R}^{m_1} \oplus \dots \oplus \mathbb{R}^{m_k} \tag{6}$$

and define subbundles $E^i \subset E$ for each $i = 1, \dots, k$ by the condition that every G-compatible local trivialization $\Phi : E|_{\mathcal{U}} \to \mathcal{U} \times \mathbb{R}^m$ should identify $E^i|_{\mathcal{U}}$ with $\mathcal{U} \times \mathbb{R}^{m_i} \subset \mathcal{U} \times \mathbb{R}^m$. This is well defined because G is the group of all linear transformations $\mathbb{R}^m \to \mathbb{R}^m$ that preserve each of the summands in the splitting (6).

Problem 7
Prove: If $E \to M$ is a real vector bundle with an indefinite bundle metric $\langle\ ,\ \rangle$ of signature (k, ℓ), then $E = E^+ \oplus E^-$ for a pair of subbundles $E^+, E^- \subset E$ of ranks k and ℓ respectively such that $\langle\ ,\ \rangle$ is positive-definite on each fiber of E^+ and negative-definite on each fiber of E^-. Are the subbundles $E^\pm \subset E$ unique?

Choose a positive bundle metric $\langle\ ,\ \rangle_+$ on E; this is always possible by Theorem 18.18 in the notes. This determines a unique smooth linear bundle map $H : E \to E$ such that

$$\langle v, w \rangle = \langle v, Hw \rangle_+$$

for all $(v, w) \in E \oplus E$. On each fiber, $H_p := H|_{E_p}$ is then a linear map $E_p \to E_p$ that is invertible (due to the nondegeneracy of $\langle\ ,\ \rangle$) and symmetric with respect to the inner product $\langle\ ,\ \rangle_+$, so by the spectral theorem, it uniquely determines a splitting of E_p into two subspaces $E_p = E_p^+ \oplus E_p^-$, spanned by the eigenvectors of H_p with positive or negative eigenvalues respectively. These subspaces are mutually orthogonal with respect to $\langle\ ,\ \rangle_+$, and since they are each preserved by H_p, they are also mutually orthogonal with respect to $\langle\ ,\ \rangle$, and the latter is positive definite on E_p^+ and negative definite on E_p^-. By Corollary 18.23 in the notes, $E^\pm := \bigcup_{p \in M} E_p^\pm$ are smooth subbundles of E.
While the subspaces $E_p^\pm \subset E_p$ are uniquely determined by the linear map $H_p : E_p \to E_p$ for each $p \in M$, H_p itself depended on a choice, namely the positive bundle metric $\langle\ ,\ \rangle_+$. In fact, it is easy to see that E_p^+ and E_p^- are *not* uniquely determined by the condition that $\langle\ ,\ \rangle$ should be positive on one and negative on the other: these are open conditions, so they will continue to hold if E_p^+ and E_p^- are replaced by any sufficiently nearby perturbations of these subspaces. I realize in retrospect that I meant to state one more condition in the problem, namely that the subspaces E_p^+ and E_p^- are *orthogonal* to each other with respect to the indefinite bundle metric $\langle\ ,\ \rangle$. This is clearly satisfied for the construction of $E_p^\pm$ via eigenspaces given above, but even with this extra condition, the two subspaces are still not unique: one can allow any sufficiently small perturbation of E_p^+ and then use the Gram-Schmidt algorithm to obtain a similarly small perturbation of E_p^- that makes it orthogonal to the perturbed E_p^+.