Topology II — Humboldt-Universität zu Berlin
C. Wendl — Winter Semester 2023–24

PROBLEM SET 2
To be discussed: 1.11.2023

Instructions

This homework will not be collected or graded, but it is highly advisable to at least think through all of the problems before the next week's lectures, as the problems will often serve as mental preparation for the lecture material. Solutions will be discussed in the Übung.

1. In lecture, we derived the long exact sequence of a pair (X, A) in singular homology by applying an algebraic result (essentially the functor $\mathsf{Short} \to \mathsf{Long}$ described in Problem 2(b) below) to an obvious short exact sequence of singular chain complexes. Work through enough of the diagram-chasing argument behind that result to show that the connecting homomorphisms $\partial_* : H_n(X, A) \to H_{n-1}(A)$ are given for each $n \in \mathbb{N}$ by the explicit formula
$$\partial_*[c] = [\partial c],$$
where $c \in C_n(X)$ is assumed to be a relative n-cylce in (X, A). Then use this formula to give a direct proof that the long exact sequence of (X, A) really is exact.

2. Consider the categories Short and Long, defined as follows. Objects in Short are short exact sequences of chain complexes $0 \to A_* \xrightarrow{f} B_* \xrightarrow{g} C_* \to 0$, with a morphism from this object to another object $0 \to A'_* \xrightarrow{f'} B'_* \xrightarrow{g'} C'_* \to 0$ defined as a triple of chain maps $A_* \xrightarrow{\alpha} A'_*$, $B_* \xrightarrow{\beta} B'_*$ and $C_* \xrightarrow{\gamma} C'_*$ such that the following diagram commutes:
$$\begin{tikzcd} 0 \arrow[r] & A_* \arrow[r, "f"] \arrow[d, "\alpha"] & B_* \arrow[r, "g"] \arrow[d, "\beta"] & C_* \arrow[r] \arrow[d, "\gamma"] & 0 \\ 0 \arrow[r] & A'_* \arrow[r, "f'"] & B'_* \arrow[r, "g'"] & C_* \arrow[r] & 0 \end{tikzcd}$$
The objects in Long are long exact sequences of $\mathbb{Z}$-graded abelian groups $\ldots \to C_{n+1} \xrightarrow{\delta} A_n \xrightarrow{F} B_n \xrightarrow{G} C_n \xrightarrow{\delta} A_{n-1} \to \ldots$, with morphisms from this to another object $\ldots \to C'_{n+1} \xrightarrow{\delta'} A'_n \xrightarrow{F'} B'_n \xrightarrow{G'} C'_n \xrightarrow{\delta'} A'_{n-1} \to \ldots$ defined as triples of homomorphisms $A_* \xrightarrow{\alpha} A'_*$, $B_* \xrightarrow{\beta} B'_*$ and $C_* \xrightarrow{\gamma} C'_*$ that preserve the $\mathbb{Z}$-gradings and make the following diagram commute:
$$\begin{tikzcd} \ldots \arrow[r] & C_{n+1} \arrow[r, "\delta"] \arrow[d, "\gamma"] & A_n \arrow[r, "F"] \arrow[d, "\alpha"] & B_n \arrow[r, "G"] \arrow[d, "\beta"] & C_n \arrow[r, "\delta"] \arrow[d, "\gamma"] & A_{n-1} \arrow[r] \arrow[d, "\alpha"] & \ldots \\ \ldots \arrow[r] & C'_{n+1} \arrow[r, "\delta'"] & A'_n \arrow[r, "F'"] & B'_n \arrow[r, "G'"] & C'_n \arrow[r, "\delta'"] & A'_{n-1} \arrow[r] & \ldots \end{tikzcd}$$
Recall also the category $\mathsf{Top}_{\mathrm{rel}}$, whose objects are pairs (X, A) of topological spaces X with subsets A, with a morphism $(X, A) \to (Y, B)$ being a continuous map of pairs.
   (a) Show that there is a covariant functor $\mathsf{Top}_{\mathrm{rel}} \to \mathsf{Short}$ assigning to each pair (X, A) its short exact sequence of singular chain complexes.
   (b) Show that there is also a covariant functor $\mathsf{Short} \to \mathsf{Long}$ assigning to each short exact sequence of chain complexes the corresponding long exact sequence of their homology groups. (Note that this can be composed with the functor in part (a) to define a functor $\mathsf{Top}_{\mathrm{rel}} \to \mathsf{Long}$.)
   (c) Let $\mathsf{Top}^h_{\mathrm{rel}}$ and Short^h denote categories with the same objects as in $\mathsf{Top}_{\mathrm{rel}}$ and Short respectively, but with morphisms of $\mathsf{Top}^h_{\mathrm{rel}}$ consisting of homotopy classes of maps of pairs, and morphisms of Short^h consisting of triples of chain homotopy classes of chain maps. Show that the functors in parts (a) and (b) also define functors $\mathsf{Top}^h_{\mathrm{rel}} \to \mathsf{Short}^h$ and $\mathsf{Short}^h \to \mathsf{Long}$, which then compose to define a functor $\mathsf{Top}^h_{\mathrm{rel}} \to \mathsf{Long}$.

3. In lecture we defined isomorphisms $S_* : \widetilde{H}_n(X) \to \widetilde{H}_{n+1}(SX)$ for every space X and $n \in \mathbb{Z}$, where $SX = (X \times [-1,1]) \big/ \sim$ is the suspension of X, defined via the equivalence relation with $(x,1) \sim (y,1)$ and $(x,-1) \sim (y,-1)$ for all $x, y \in X$.

   (a) Show that for any continuous map $f : X \to Y$, the map $Sf : SX \to SY : [(x,t)] \mapsto [(f(x),t)]$ is well defined and continuous, and moreover, that $S(\mathrm{Id}_X) = \mathrm{Id}_{SX}$ and $S(f \circ g) = Sf \circ Sg$ whenever f and g can be composed. In other words, show that S defines a functor $\mathsf{Top} \to \mathsf{Top}$.

   (b) Denote by $\widetilde{H}^S_{n+1} : \mathsf{Top} \to \mathsf{Ab}$ the composition of the functor $S : \mathsf{Top} \to \mathsf{Top}$ in part (a) with the functor $\widetilde{H}_{n+1} : \mathsf{Top} \to \mathsf{Ab}$ which sends X to $\widetilde{H}_{n+1}(X)$. Show that there exists a natural transformation from $\widetilde{H}_n$ to $\widetilde{H}^S_{n+1}$ which associates to each space X the isomorphism $S_* : \widetilde{H}_n(X) \to \widetilde{H}_{n+1}(SX)$.

4. (a) Given a short exact sequence of abelian groups $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$, show that the following conditions are equivalent:

   (i) There exists a homomorphism $\pi : B \to A$ such that $\pi \circ f = \mathbb{1}_A$;
   (ii) There exists a homomorphism $i : C \to B$ such that $g \circ i = \mathbb{1}_C$;
   (iii) There exists an isomorphism $\Phi : B \to A \oplus C$ such that $\Phi \circ f(a) = (a,0)$ and $g \circ \Phi^{-1}(a,c) = c$.

$$\begin{tikzcd}
 & & B \arrow[dd, "\Phi"] \arrow[dr, "g"] & & \\
0 \arrow[r] & A \arrow[ur, "f"] \arrow[dr] & & C \arrow[r] & 0 \\
 & & A \oplus C \arrow[ur] & &
\end{tikzcd}$$

   If any of these conditions holds, we say that the sequence **splits**.

   (b) Show that if the groups in part (a) are all finite-dimensional vector spaces and the homomorphisms are linear maps, then the sequence always splits.

5. The following exercise is needed for the start of the inductive computation of $\widetilde{H}_n(S^n; G) \cong G$ for every $n \geqslant 0$ and coefficient group G.

   (a) For any two spaces X and Y with maps $\epsilon^X : X \to \{\mathrm{pt}\}$ and $\epsilon^Y : Y \to \{\mathrm{pt}\}$, show that the natural isomorphism $H_*(X \amalg Y) \cong H_*(X) \oplus H_*(Y)$ identifies $\widetilde{H}_*(X \amalg Y)$ with the kernel of the map

$$\epsilon^X_* \oplus \epsilon^Y_* : H_*(X) \oplus H_*(Y) \to H_*(\{\mathrm{pt}\}).$$

   Hint: For the inclusions $i^X : X \hookrightarrow X \amalg Y$ and $i^Y : Y \hookrightarrow X \amalg Y$, the unique map $\epsilon : X \amalg Y \to \{\mathrm{pt}\}$ satisfies $\epsilon \circ i^X = \epsilon^X$ and $\epsilon \circ i^Y = \epsilon^Y$.

   (b) Apply part (a) in the case $X = Y = \{\mathrm{pt}\}$ to identify $\widetilde{H}_0(S^0) \cong \widetilde{H}_0(\{\mathrm{pt}\} \amalg \{\mathrm{pt}\})$ with the kernel of the map

$$\mathbb{1} \oplus \mathbb{1} : G \oplus G \to G : (g,h) \mapsto g + h,$$

   and show that the latter is isomorphic to the coefficient group G.

6. For a space X and abelian group G with singular chain complex $\ldots \to C_2(X;G) \xrightarrow{\partial_2} C_1(X;G) \xrightarrow{\partial_1} C_0(X;G)$, define $\epsilon_* : C_0(X;G) \to G$ by $\epsilon_*(\sum_i g_i \sigma_i) = \sum_i g_i$ for finite sums with $g_i \in G$ and $\sigma_i : \Delta^0 \to X$. Show that $\epsilon_* \circ \partial_1 = 0$, so that

$$\ldots \to C_2(X;G) \xrightarrow{\partial_2} C_1(X;G) \xrightarrow{\partial_1} C_0(X;G) \xrightarrow{\epsilon_*} G$$

   also forms a chain complex, and that the homology of this complex is the reduced homology $\widetilde{H}_*(X;G)$. *Hint: At the end of the section on reduced homology in the lecture notes, you will find a remark explaining where this definition of $\epsilon_* : C_0(X;G) \to G$ comes from.*

7. If you haven't encountered tensor products of abelian groups before this semester, then work through Exercises 28.3, 28.5 and 28.6 in the lecture notes.