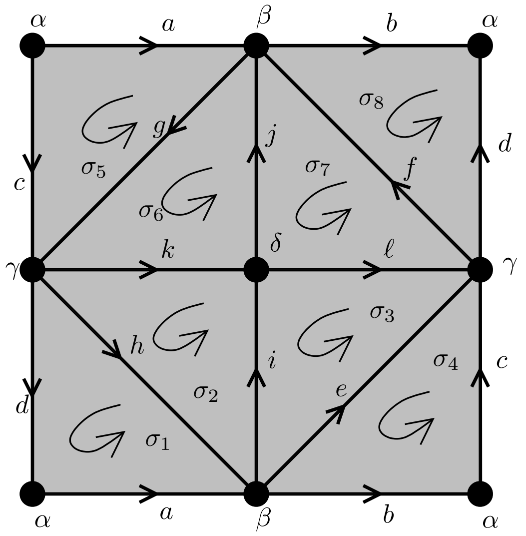
Topology II — Humboldt-Universität zu Berlin
C. Wendl — Winter Semester 2023–24

PROBLEM SET 3
To be discussed: 8.11.2023

Instructions

This homework will not be collected or graded, but it is highly advisable to at least think through all of the problems before the next week's lectures, as the problems will often serve as mental preparation for the lecture material. Solutions will be discussed in the Übung.

1. The following picture shows a simplicial complex $K = (V, S)$ whose associated polyhedron $|K|$ is homeomorphic to the Klein bottle.



There are four vertices $V = \{\alpha, \beta, \gamma, \delta\}$, twelve 1-simplices labeled by letters $a, \ldots, \ell$, and eight 2-simplices labeled σ_i for $i = 1, \ldots, 8$. The picture also shows a choice of orientation for each of the 2-simplices[1] (circular arrows represent a cyclic ordering of the vertices) and 1-simplices (arrows point from the first vertex to the last). If we additionally endow each 0-simplex with the positive orientation, every letter in the picture can be regarded as representing an oriented simplex, and thus a generator of the oriented simplicial chain complex $C_*^\Delta(K;\mathbb{Z})$.

   (a) Write down $\partial\sigma_i \in C_1^\Delta(K;\mathbb{Z})$ explicitly for each $i = 1, \ldots, 8$.

   (b) Prove that $H_2^\Delta(K;\mathbb{Z}_2) \cong \mathbb{Z}_2$, and write down a specific cycle in $C_2^\Delta(K;\mathbb{Z}_2)$ that generates it.

   (c) Prove that $H_2^\Delta(K;\mathbb{Z}) = 0$.

   (d) Show that the 1-cycle $c + d$ represents a nontrivial homology class $[c + d]$ in both $H_1^\Delta(K;\mathbb{Z})$ and $H_1^\Delta(K;\mathbb{Z}_2)$, but satisfies $2[c + d] = 0 \in H_1^\Delta(K;\mathbb{Z})$ and $[c + d] = 0 \in H_1^\Delta(K;\mathbb{Q})$.

2. Suppose M is a compact and connected n-manifold with a triangulation $(M, \partial M) \cong (|K|, |K'|)$ given by a simplicial pair (K, K').

   (a) Prove $H_n^\Delta(K, K';\mathbb{Z}_2) \cong \mathbb{Z}_2$.

   (b) Prove that if the triangulation admits an orientation, then $H_n^\Delta(K, K'; G) \cong G$ for every choice of coefficient group G.

[1] Notice however that this does not define an oriented triangulation, as the chosen orientations of neighboring 2-simplices are not always compatible with each other. The Klein bottle does not admit an oriented triangulation.

(c) Aside from the special case in part (a), under what assumptions on the group G can you prove $H_n^\Delta(K, K'; G) \cong G$ without assuming the triangulation is orientable?

(d) Show that if the triangulation is *not* orientable, then $H_n^\Delta(K, K'; \mathbb{Z}) = 0$.

3. Suppose the following diagram commutes and that both of its rows are exact, meaning $\operatorname{im} f = \ker g$, $\operatorname{im} g' = \ker h'$ and so forth:

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "\alpha"] & B \arrow[r, "g"] \arrow[d, "\beta"] & C \arrow[r, "h"] \arrow[d, "\gamma"] & D \arrow[r, "i"] \arrow[d, "\delta"] & E \arrow[d, "\varepsilon"] \\
A' \arrow[r, "f'"] & B' \arrow[r, "g'"] & C' \arrow[r, "h'"] & D' \arrow[r, "i'"] & E'
\end{tikzcd}$$

(a) Prove that if α, β, δ and ε are all isomorphisms, then so is γ.
This result is known as the ***five-lemma***.

(b) Here is an application: given a map of pairs $f : (X, A) \to (Y, B)$, show that if any two of the induced maps $H_k(X) \xrightarrow{f_*} H_k(Y)$, $H_k(A) \xrightarrow{f_*} H_k(B)$ and $H_k(X, A) \xrightarrow{f_*} H_k(Y, B)$ are isomorphisms for every k, then so is the third.

4. Show that for any oriented triangulation of S^n for $n \geqslant 1$, the suspension isomorphism $S_* : H_n(S^n; \mathbb{Z}) \to H_{n+1}(S^{n+1}; \mathbb{Z})$ sends a fundamental class $[S^n]$ to a fundamental class $[S^{n+1}]$ defined via a related oriented triangulation of $SS^n \cong S^{n+1}$.

5. In this problem I'm going to sketch an algorithm for triangulating the product of simplices $\Delta^k \times \Delta^\ell$ for each pair of integers $k, \ell \geqslant 0$ and defining an associated fundamental cycle. Your task is to convince yourself that everything I'm saying is true.

Write $n := k + \ell$, endow the set $\{0, \dots, k\} \times \{0, \dots, \ell\}$ with the total order such that $(i, j) \leqslant (i', j')$ if and only if both $i \leqslant i'$ and $j \leqslant j'$, and let $\mathbf{S}_m$ for $m = 0, \dots, n$ denote the set of all strictly increasing sequences $(i_0, j_0) < \ldots < (i_m, j_m)$ of $m + 1$ elements of that set. Let $e_0, \dots, e_k$ and $f_0, \dots, f_\ell$ denote the vertices of the standard simplices Δ^k and Δ^ℓ respectively. There is a triangulation of $\Delta^k \times \Delta^\ell$ such that for each $m = 0, \dots, n$, the m-simplices of the triangulation are the convex hulls of sets

$$\{(e_{i_0}, f_{j_0}), \dots, (e_{i_m}, f_{j_m})\} \subset \Delta^k \times \Delta^\ell \tag{1}$$

corresponding to elements $(i_0, j_0) < \ldots < (i_m, j_m)$ in $\mathbf{S}_m$. Verify that this really defines a triangulation. Let K denote the associated n-dimensional simplicial complex and $K' \subset K$ the subcomplex consisting of simplices whose sets of vertices have convex hulls contained in $\partial(\Delta^k \times \Delta^\ell)$. How can you describe the $(n-1)$-simplices of K? Which ones belong to K', which ones do not, and of which n-simplices can they be boundary faces?

For each $m = 0, \dots, n$ and $\mathbf{s} \in \mathbf{S}_m$, let $\mathbf{v_s} \in C_m^o(K; \mathbb{Z})$ denote the generator of the ordered simplicial chain complex given by the vertices of the associated m-simplex, arranged in the same order as in (1). Let $\mathbf{s}_0 \in \mathbf{S}_n$ denote the specific sequence $(0, 0) < (1, 0) < \ldots < (k, 0) < (k, 1) < \ldots < (k, \ell)$, and for any other $\mathbf{s} \in \mathbf{S}_n$, let $|\mathbf{s}| \in \mathbb{Z}_2$ denote the number (modulo 2) of modifications of the form

$$\ldots < (i-1, j) < (i, j) < (i, j+1) < \ldots \quad \mapsto \quad \ldots < (i-1, j) < (i-1, j+1) < (i, j+1) < \ldots$$

that are required in order to turn $\mathbf{s}_0$ into $\mathbf{s}$. Verify that this number is well defined, and then show that

$$c_{\Delta^k \times \Delta^\ell} := \sum_{\mathbf{s} \in \mathbf{S}_n} (-1)^{|\mathbf{s}|} \mathbf{v_s} \in C_n^o(K; \mathbb{Z})$$

defines a relative n-cycle in $C_*^o(K, K'; \mathbb{Z})$, and therefore determines an orientation of our triangulation of $\Delta^k \times \Delta^\ell$. Prove the formula

$$\partial c_{\Delta^k \times \Delta^\ell} = \sum_{i=0}^k (-1)^i c_{\partial_{(i)} \Delta^k \times \Delta^\ell} + (-1)^k \sum_{j=0}^\ell (-1)^j c_{\Delta^k \times \partial_{(j)} \Delta^\ell}.$$

The case $k = 1$ of this formula is useful for proving the homotopy invariance property of singular homology (how?). We will later discuss how the general case leads to product structures in singular homology and cohomology.