PROBLEM SET 5
To be discussed: 22.11.2023

Instructions

This homework will not be collected or graded, but it is highly advisable to at least think through all of the problems before the next week's lectures, as the problems will often serve as mental preparation for the lecture material. Solutions will be discussed in the Übung.

1. If $f : X \to X$ is a homeomorphism, then the mapping torus $X_f = (X \times [0,1])\big/(x,0) \sim (f(x),1)$ admits an alternative definition as
$$X_f = (X \times \mathbb{R}) \Big/ (x,t) \sim (f(x), t+1)$$
where the equivalence is defined for every $x \in X$ and $t \in \mathbb{R}$. Take a moment to convince yourself that these two quotients are homeomorphic. The second perspective has the advantage that one can view $\widetilde{X} := X \times \mathbb{R}$ as a covering space for X_f, with the quotient projection defining a covering map $\widetilde{X} \to X_f$ of infinite degree. Writing $S^1 := \mathbb{R}/\mathbb{Z}$, we also see a natural continuous surjective map $\pi : X_f \to S^1 : [(x,t)] \mapsto [t]$, whose **fibers** $\pi^{-1}(t)$ are homeomorphic to X for all $t \in S^1$. We shall denote by $i : X \hookrightarrow X_f$ the inclusion of the fiber $\pi^{-1}([0])$.

   In lecture, we proved the existence of a long exact sequence
$$\ldots \longrightarrow h_{k+1}(X_f) \xrightarrow{\Phi} h_k(X) \xrightarrow{\mathbb{1}_* - f_*} h_k(X) \xrightarrow{i_*} h_k(X_f) \xrightarrow{\Phi} h_{k-1}(X) \longrightarrow \ldots$$
for any axiomatic homology theory h_*. The goal of this problem to gain a more concrete picture of the connecting homomorphism $\Phi : H_1(X_f;\mathbb{Z}) \to H_0(X;\mathbb{Z})$ for the special case of singular homology with integer coefficients, under the assumption that X is path-connected and f is a homeomorphism.

   Assuming X is path-connected, there is a natural isomorphism $H_0(X;\mathbb{Z}) \cong \mathbb{Z}$, and X_f is then also path-connected. Since $H_1(X_f;\mathbb{Z})$ is isomorphic to the abelianization of $\pi_1(X_f, x)$ for any choice of base point $x \in X_f$, we can identify X with $\pi^{-1}([0]) \subset X_f$, fix a base point $x \in X \subset X_f$ and represent any class in $H_1(X_f;\mathbb{Z})$ by a loop $\gamma : [0,1] \to X_f$ with $\gamma(0) = \gamma(1) = x$. Now let $\tilde{\gamma} : [0,1] \to \widetilde{X}$ denote the unique lift of γ to the cover $\widetilde{X} = X \times \mathbb{R}$ such that $\tilde{\gamma}(0) = (x,0)$. Since γ is a loop, it follows that $\tilde{\gamma}(1) = (f^m(x), m)$ for some $m \in \mathbb{Z}$, where f^m denotes the mth iterate of f if $m > 0$, the $(-m)$th iterate of f^{-1} if $m < 0$, and the identity map if $m = 0$.

   (a) Prove that under the natural identification of $H_0(X;\mathbb{Z})$ with $\mathbb{Z}$, the connecting homomorphism $\Phi : H_1(X_f;\mathbb{Z}) \to \mathbb{Z}$ can be chosen[1] such that
$$\Phi([\gamma]) = m,$$
so in particular, $[\gamma] \in \ker \Phi$ if and only if the lift of γ to the cover $\widetilde{X}$ is a loop.

   (b) Prove directly from the characterization in part (a) that $\Phi : H_1(X_f;\mathbb{Z}) \to H_0(X;\mathbb{Z})$ is surjective. *Remark: Of course this can also be deduced less directly from the exact sequence.*

2. The Klein bottle K^2 can be presented as the mapping torus of $f : S^1 \to S^1 : e^{i\theta} \mapsto e^{-i\theta}$. Use the exact sequence of the mapping torus to compute $H_*(K^2;\mathbb{Z})$ and $H_*(K^2;\mathbb{Z}_2)$.

3. Viewing S^1 as the unit circle in $\mathbb{C}$, fix a generator of $[S^1] \in H_1(S^1;\mathbb{Z})$ and use it to determine local orientations $[\mathbb{C}]_z \in H_n(\mathbb{C}, \mathbb{C}\backslash\{z\};\mathbb{Z})$ for every point $z \in \mathbb{C}$ via the natural isomorphisms $H_2(\mathbb{C}, \mathbb{C}\backslash\{z\};\mathbb{Z}) \cong H_2(\mathbb{D}_z, \partial\mathbb{D}_z;\mathbb{Z}) \cong H_1(\partial\mathbb{D}_z;\mathbb{Z})$, where $\mathbb{D}_z \subset \mathbb{C}$ denotes the closed unit disk centered at z, whose boundary is canonically identified with S^1. This choice will be used in the following for the definition of

[1] There is a bit of freedom allowed in the definition of Φ, e.g. we could replace it with $-\Phi$ and the sequence would still be exact since $\ker \Phi$ and $\operatorname{im} \Phi$ would not change.

local degrees of maps $f : \mathcal{U} \to \mathbb{C}$ defined on open subsets $\mathcal{U} \subset \mathbb{C}$; note that changing the generator $[S^1] \in H_1(S^1;\mathbb{Z})$ does not change the definition of $\deg(f;z)$ since it changes *both* $[\mathbb{C}]_z$ and $[\mathbb{C}]_{f(z)}$ by a sign.

(a) Show that if $f : \mathbb{C} \to \mathbb{C}$ is of the form $f(z) = (z - z_0)^k g(z)$ for some $z_0 \in \mathbb{C}$, $k \in \mathbb{N}$ and g a continuous map with $g(z_0) \neq 0$, then $\deg(f;z_0) = k$.

(b) Can you modify the example in part (a) to produce one with $\deg(f;z_0) = -k$ for $k \in \mathbb{N}$?

(c) Suppose $\mathcal{U} \subset \mathbb{C}$ is open and $f : \mathcal{U} \to \mathbb{C}$ is continuous with $f(z_0) = w_0$ and $\deg(f;z_0) \neq 0$ for some $z_0 \in \mathcal{U}$, $w_0 \in \mathbb{C}$. Prove that for any neighborhood $\mathcal{V} \subset \mathcal{U}$ of z_0, there exists an $\epsilon > 0$ such that every continuous map $\hat{f} : \mathcal{U} \to \mathbb{C}$ satisfying $|\hat{f} - f| < \epsilon$ maps some point in $\mathcal{V}$ to w_0.
Remark: I have stated this problem for maps on $\mathbb{C}$ just for convenience, but one can do something similar with maps on open subsets of $\mathbb{R}^n$ for any $n \in \mathbb{N}$.

(d) Find an example of a continuous map $f : \mathbb{C} \to \mathbb{C}$ that has an isolated zero at the origin with $\deg(f;0) = 0$ and admits arbitrarily small continuous perturbations that are nowhere zero.
Hint: You should probably not think in complex terms, but instead identify $\mathbb{C}$ with $\mathbb{R}^2$.

(e) Let $f : S^2 \to S^2$ denote the natural continuous extension to $S^2 := \mathbb{C} \cup \{\infty\}$ of a complex polynomial $\mathbb{C} \to \mathbb{C}$ of degree n. What is $\deg(f)$?

(f) Pick a constant $t_0 \in S^1$ and let $A \cong S^1 \vee S^1$ denote the subset $\{(x,y) \mid x = t_0 \text{ or } y = t_0\} \subset S^1 \times S^1 = \mathbb{T}^2$. Show that $\mathbb{T}^2/A \cong S^2$, and that the quotient map $\mathbb{T}^2 \to \mathbb{T}^2/A$ has degree ± 1 (depending on choices of generators for $H_2(\mathbb{T}^2;\mathbb{Z})$ and $H_2(S^2;\mathbb{Z})$).

4. Prove that for any axiomatic homology theory h_* and each $n \in \mathbb{N}$ and $x \in S^n$, the map $h_n(S^n) \to h_n(S^n, S^n \backslash \{x\})$ induced by the obvious inclusion of pairs is an isomorphism.
Hint: You can choose a Euclidean neighborhood $\mathcal{U} \subset S^n$ of x and use $h_n(S^n, S^n \backslash \mathcal{U})$ as a substitute for $h_n(S^n, S^n \backslash \{x\})$ (why?). What kind of space is $S^n \backslash \mathcal{U}$?

5. Suppose $f : S^n \to S^n$ is any continuous map, and $p_+ \in SS^n = C_+S^n \cup_{S^n} C_-S^n$ is the vertex of the top cone in the suspension $SS^n \cong S^{n+1}$. What is $\deg(Sf;p_+)$? Use this to give a new proof (different from the one we saw in lecture) that $\deg(Sf) = \deg(f)$.

6. (a) Show that every map $S^n \to \mathbb{T}^n$ has degree 0 if $n \geqslant 2$.
Hint: Lift $S^n \to \mathbb{T}^n$ to the universal cover of $\mathbb{T}^n$.

(b) Show that for every $d \in \mathbb{Z}$ and every $\mathbb{Z}$-admissible[2] n-dimensional manifold M with $n \geqslant 1$, there exists a map $M \to S^n$ of degree d.
Hint: Try a map that is interesting only on some n-ball in M and constant everywhere else.

7. For these problems you need to use the mod 2 degree, since $\mathbb{RP}^2$ and the Klein bottle are $\mathbb{Z}_2$-admissible but not $\mathbb{Z}$-admissible (in other words, they are closed and connected but not orientable).

(a) Find an example of a map $\mathbb{RP}^2 \to S^2$ that cannot be homotopic to a constant.

(b) Same problem but with $\mathbb{RP}^2$ replaced by the Klein bottle.
Hint: Problem 3(f) might provide some useful inspiration.

8. (a) Prove that for every positive even integer n, every continuous map $f : S^n \to S^n$ has at least one point $x \in S^n$ where either $f(x) = x$ or $f(x) = -x$. Deduce that every continuous map $\mathbb{RP}^n \to \mathbb{RP}^n$ has a fixed point if n is even.

(b) Construct counterexamples to the statement in part (a) for every odd n.
Hint: Consider linear transformations with no real eigenvalues.

[2]The terms "h_*-admissible" and "G-admissible" were not mentioned in the lecture, but are defined in the lecture notes. If you are happy to accept the "fact" about topological n-manifolds that was stated in the lecture, just assume M is closed, connected, and orientable: the relevant condition is that the natural map $H_n(M;\mathbb{Z}) \to H_n(M, M\backslash\{x\};\mathbb{Z})$ should be an isomorphism for every point $x \in M$, so that integer-valued mapping degrees can be defined.