Topology II — Humboldt-Universität zu Berlin
C. Wendl — Winter Semester 2023–24

**PROBLEM SET 6**
**To be discussed: 29.11.2023**

## Instructions

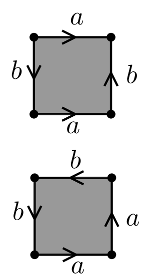
This homework will not be collected or graded, but it is highly advisable to at least think through all of the problems before the next week's lectures, as the problems will often serve as mental preparation for the lecture material. Solutions will be discussed in the Übung.

1. The set $\mathbb{R}^\infty := \bigoplus_{j\in\mathbb{N}} \mathbb{R}$ consists of all sequences of real numbers $(x_1, x_2, x_3, \dots)$ such that at most finitely many terms are nonzero. Identifying $\mathbb{R}^n$ for each $n \in \mathbb{N}$ with the subset
$$\left\{(x_1, x_2, x_3, \dots) \in \mathbb{R}^\infty \;\middle|\; x_j = 0 \text{ for all } j > n\right\},$$
we can define a topology on $\mathbb{R}^\infty$ such that a set $\mathcal{U} \subset \mathbb{R}^\infty$ is open if and only if $\mathcal{U} \cap \mathbb{R}^n$ is an open subset of $\mathbb{R}^n$ (with its standard topology) for all $n \in \mathbb{N}$.[1] Notice that every element $\mathbf{x} \in \mathbb{R}^\infty$ belongs to $\mathbb{R}^n$ for sufficiently large $n \in \mathbb{N}$. Prove that for any convergent sequence $\mathbf{x}^k \to \mathbf{x} \in \mathbb{R}^\infty$, there exists a (possibly larger) number $N \in \mathbb{N}$ such that $\mathbf{x}^k \in \mathbb{R}^N$ for all $k$. Deduce from this that every compact subset $K \subset \mathbb{R}^\infty$ is contained in $\mathbb{R}^N$ for some $N$ sufficiently large (depending on $K$).

2. Recall that $\mathbb{RP}^n$ has a cell decomposition with one $k$-cell for every $k = 0, \dots, n$, derived by starting from the cell decomposition of $S^n$ with two cells in each dimension and dividing the whole thing by a $\mathbb{Z}_2$-action. Use this to compute $H_*^{\mathrm{CW}}(\mathbb{RP}^n; \mathbb{Z})$, $H_*^{\mathrm{CW}}(\mathbb{RP}^n; \mathbb{Z}_2)$ and $H_*^{\mathrm{CW}}(\mathbb{RP}^n; \mathbb{Q})$.

3. For integers $g \geqslant 0$ and $m \geqslant 1$, let $\Sigma_{g,m}$ denote the compact surface with boundary obtained by deleting $m$ open disks from the closed oriented surface $\Sigma_g$ of genus $g$. Assuming the isomorphism between singular and cellular homology, compute $H_*(\Sigma_{g,m}; G)$ with $G$ an arbitrary coefficient group. *Hint: Since singular homology is homotopy invariant, you are free to replace $\Sigma_{g,m}$ by a CW-complex that is homotopy equivalent to it.*

4. The picture at the right shows two spaces that you may recall from *Topologie I* are both homeomorphic to the Klein bottle. Each also defines a cell complex $X = X^0 \cup X^1 \cup X^2$ consisting of one 0-cell, two 1-cells (labeled $a$ and $b$) and one 2-cell.




   (a) Compute $H_*^{\mathrm{CW}}(X; \mathbb{Z})$, $H_*^{\mathrm{CW}}(X; \mathbb{Z}_2)$ and $H_*^{\mathrm{CW}}(X; \mathbb{Q})$ for both complexes. (You'll know you've done something wrong if the answers you get from the two complexes are not isomorphic!)

   (b) Recall that the **rank** (*Rang*) of a finitely generated abelian group $G$ is the unique integer $k \geqslant 0$ such that $G \cong \mathbb{Z}^k \oplus T$ for some finite group $T$. Verify for both cell decompositions of the Klein bottle above that
$$\sum_k (-1)^k \operatorname{rank} H_k^{\mathrm{CW}}(X; \mathbb{Z}) = \sum_k (-1)^k \dim_{\mathbb{Z}_2} H_k^{\mathrm{CW}}(X; \mathbb{Z}_2) = \sum_k (-1)^k \dim_{\mathbb{Q}} H_k^{\mathrm{CW}}(X; \mathbb{Q}) = 0.$$
   *(Congratulations, you've just computed the* ***Euler characteristic*** *of the Klein bottle!)*

5. The **complex projective $n$-space** $\mathbb{CP}^n$ is a compact $2n$-manifold defined as the set of all complex lines through the origin in $\mathbb{C}^{n+1}$, or equivalently,
$$\mathbb{CP}^n = (\mathbb{C}^{n+1} \backslash \{0\}) / \sim$$

[1] This is the right topology so that the subspace topology on the set of unit vectors $S^\infty \subset \mathbb{R}^\infty$ matches the topology defined on $S^\infty$ via the infinite-dimensional cell decomposition we discussed in lecture.

where two points $z, z' \in \mathbb{C}^{n+1}\backslash\{0\}$ are equivalent if and only if $z' = \lambda z$ for some $\lambda \in \mathbb{C}$. It is conventional to write elements of $\mathbb{CP}^n$ in so-called *homogeneous coordinates*, meaning the equivalence class represented by $(z_0, \ldots, z_n) \in \mathbb{C}^{n+1}$ is written as $[z_0 : \ldots : z_n]$. Notice that $\mathbb{CP}^n$ can be partitioned into two disjoint subsets

$$\mathbb{C}^n \cong \{[1 : z_1 : \ldots : z_n] \in \mathbb{CP}^n\} \quad \text{and} \quad \mathbb{CP}^{n-1} \cong \{[0 : z_1 : \ldots : z_n] \in \mathbb{CP}^n\}.$$

(a) Show that the partition $\mathbb{CP}^n = \mathbb{C}^n \cup \mathbb{CP}^{n-1}$ gives rise to a cell decomposition of $\mathbb{CP}^n$ with one $2k$-cell for every $k = 0, \ldots, n$.

(b) Using the isomorphism between singular and cellular homology, compute $H_*(\mathbb{CP}^n; G)$ for an arbitrary coefficient group $G$. *Hint: This is easy.*

6. Adapt the proof of $H_*^{\mathrm{CW}}(X; G) \cong h_*(X)$ we saw in lecture to prove the relative version of this statement: for any axiomatic homology theory $h_*$ with coefficient group $h_0(\{\mathrm{pt}\}) \cong G$ and any finite-dimensional[2] CW-pair $(X, A)$, i.e. any CW-complex $X$ with a subcomplex $A \subset X$,

$$H_*^{\mathrm{CW}}(X, A; G) \cong h_*(X, A).$$

*Hint: Start by showing that $C_n^{\mathrm{CW}}(X, A; G)$ is canonically isomorphic to $h_n(X^n \cup A, X^{n-1} \cup A)$, and instead of the long exact sequence of the pair $(X^n, X^{n-1})$, consider the long exact sequence of the triple $(X^n \cup A, X^{n-1} \cup A, A)$.*
*Comment: This exercise is a bit lengthy, but it is not fundamentally difficult—every step is simply a minor generalization of something that we discussed in lecture. Working through it is one of the best ways to achieve a deeper understanding of the isomorphism $H_*^{\mathrm{CW}}(X; G) \cong h_*(X)$.*

[2] The assumption that $(X, A)$ is finite dimensional is not actually necessary. The ideas needed in order to lift this assumption in the case where $h_*$ is singular homology will be discussed next week.