TOPOLOGY II
C. WENDL

HUMBOLDT-UNIVERSITÄT ZU BERLIN
WINTER SEMESTER 2023–24

TAKE-HOME MIDTERM
Due: 2.02.2023

Instructions

The purpose of this assignment is two-fold:

- It gives the instructors a chance to gauge your understanding more directly than usual, and give feedback.
- It provides an opportunity to improve your final grade in the course.

To receive feedback and/or credit, you must submit your written solutions by the start of the lecture (9:15am) on **Friday, February 2**. Submissions can be on paper or electronic via the moodle.

You are free to use any resources at your disposal and to discuss the problems with your comrades, but **you must write up your solutions alone**. Solutions may be written up in German or English, this is up to you.

A score of 75 points or better will boost your final exam grade according to the formula that was indicated in the course syllabus.

If a problem asks you to prove something, then unless it says otherwise, a **complete argument** is typically expected, not just a sketch of the idea. Partial credit may sometimes be given for incomplete arguments if you can demonstrate that you have the right idea, but for this it is important to write as clearly as possible. Less complete arguments can sometimes be sufficient, e.g. a clear and convincing picture is often a better way to prove that two spaces are homotopy equivalent than by writing down explicit maps and homotopies (use your best judgement). Unless stated otherwise, you are free to make use of all results that have appeared in the lecture notes or in problem sets, without reproving them. When using a result from a problem set or the lecture notes, say explicitly which one.

If you get stuck on one part of a problem, it may often still be possible to move on and do the next part. You are free to ask for clarification or hints via e-mail/moodle or in office hours or Übungen; of course we reserve the right not to answer such questions.

Problems

1. [30 pts total] Consider a knot $K \subset \mathbb{R}^3$, i.e. the image of a topological embedding[1] $S^1 \hookrightarrow \mathbb{R}^3$. For technical reasons, it is conventional in knot theory to assume that K is not too "wild," for instance it is good enough to assume that the embedding $S^1 \hookrightarrow \mathbb{R}^3$ is smooth (meaning C^∞).

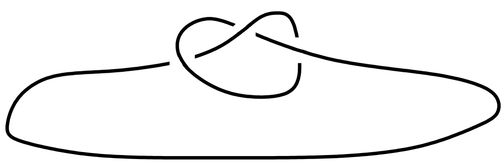

Figure 1: A smooth knot.

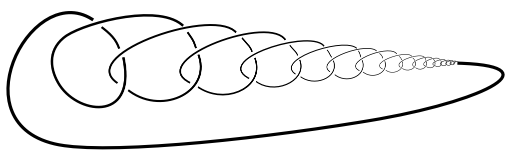

Figure 2: A "wild" knot, which is continuous, but not smooth. We will not consider these.

[1] Recall that a map $f : X \to Y$ between two topological spaces is called a **topological embedding** if it is continuous and injective and the inverse $f^{-1} : f(X) \to X$ is also continuous with respect to the subspace topology on $f(X) \subset Y$.

The smoothness condition has the following advantage: if K is the image of $f : S^1 \hookrightarrow \mathbb{R}^3$, we can always assume there exists an extension of f to a topological embedding $S^1 \times \mathbb{D}^2 \hookrightarrow \mathbb{R}^3$ that matches f along $S^1 \times \{0\}$. (Take a moment to convince yourself that no such extension exists for the knot in Figure 2.) We shall denote the image of this extension by $N \subset \mathbb{R}^3$, so

$$K \subset N \subset \mathbb{R}^3 \quad \text{where} \quad N \cong S^1 \times \mathbb{D}^2.$$

One way to distinguish topologically between two knots is via their **knot groups**, meaning the group $\pi_1(\mathbb{R}^3 \setminus K)$. As you might recall from *Topology I*, one can equivalently extend $\mathbb{R}^3$ to its one-point compactification S^3 and replace $\pi_1(\mathbb{R}^3 \setminus K)$ with $\pi_1(S^3 \setminus K)$, as it is easy to show via the Seifert-van Kampen theorem that these two groups are isomorphic. With this in mind, we lose nothing by regarding all knots as subsets of S^3.

(a) [20 pts] Prove that for every knot K, the abelianization of its knot group is isomorphic to $\mathbb{Z}$.

(b) [10 pts] Draw a picture of the knot K in Figure 1 together with a loop in $S^3 \setminus K$ representing a generator of the abelianization of $\pi_1(S^3 \setminus K)$.
Advice: For the purposes of this problem, you should imagine S^3 as $\mathbb{R}^3$ with an extra "point at infinity" that cannot be be shown in the picture.

Remark: Note that since the result of Problem 1(a) does not depend on the knot K, it is bad news if your goal is to distinguish inequivalent knots—you cannot do so by distinguishing the abelianizations of their knot groups. One has to find cleverer algebraic tricks for distinguishing two non-isomorphic knot groups.

2. [70 pts total] This problem concerns the cellular homology of mapping cones. As preparation, here is a quick definition of the *reduced* cellular homology groups of a CW-complex X:

$$\widetilde{H}_*^{\mathrm{CW}}(X;G) := \ker\left(H_*^{\mathrm{CW}}(X;G) \xrightarrow{\epsilon_*} H_*^{\mathrm{CW}}(\{\mathrm{pt}\};G)\right),$$

where the one-point space $\{\mathrm{pt}\}$ is regarded as a CW-complex containing only one cell, making the unique map $\epsilon : X \to \{\mathrm{pt}\}$ into a cellular map. Before continuing, you will want to convince yourself of the following facts:

- $H_k^{\mathrm{CW}}(X;G) \cong \widetilde{H}_k^{\mathrm{CW}}(X;G)$ for all $k \neq 0$, while $H_0^{\mathrm{CW}}(X;G) \cong \widetilde{H}_0^{\mathrm{CW}}(X;G) \oplus G$.
- $\widetilde{H}_*^{\mathrm{CW}}$ is a functor on the category of CW-complexes, i.e. cellular maps induce maps between the corresponding reduced cellular homology groups.
- There is a natural isomorphism $\widetilde{H}_*^{\mathrm{CW}}(X;G) \cong \widetilde{H}_*(X;G)$.
- $\widetilde{H}_*^{\mathrm{CW}}(X;\mathbb{Z})$ is also the homology of an *augmented* cellular chain complex $\widetilde{C}_*^{\mathrm{CW}}(X;\mathbb{Z})$ that takes the form

$$\ldots \longrightarrow C_2^{\mathrm{CW}}(X;\mathbb{Z}) \xrightarrow{\partial} C_1^{\mathrm{CW}}(X;\mathbb{Z}) \xrightarrow{\partial} C_0^{\mathrm{CW}}(X;\mathbb{Z}) \xrightarrow{\epsilon} \widetilde{C}_{-1}^{\mathrm{CW}}(X;\mathbb{Z}) := \mathbb{Z} \longrightarrow 0 \longrightarrow 0 \longrightarrow \ldots,$$

  where the **augmentation** $\epsilon : C_0^{\mathrm{CW}}(X;\mathbb{Z}) \to \mathbb{Z}$ is the unique homomorphism that sends each 0-cell (regarded as a generator of $C_0^{\mathrm{CW}}(X;\mathbb{Z})$) to $1 \in \mathbb{Z}$.
- Cellular maps $X \to Y$ naturally induce chain maps $\widetilde{C}_*^{\mathrm{CW}}(X;\mathbb{Z}) \to \widetilde{C}_*^{\mathrm{CW}}(Y;\mathbb{Z})$ between augmented chain complexes. (How do they act in degree -1?)

As a bookkeeping device, it is sometimes helpful to think of $\widetilde{H}_*^{\mathrm{CW}}(X)$ as the cellular homology of a CW-complex that has all the same cells as X, plus one extra "formal" cell e^{-1} of dimension -1, corresponding to the canonical generator of $\widetilde{C}_{-1}^{\mathrm{CW}}(X;\mathbb{Z}) = \mathbb{Z}$. The augmentation is then determined by the formula $\epsilon(e_\alpha^0) := e^{-1}$ for every 0-cell $e_\alpha^0 \subset X$.

With that out of the way, let's do a little homological algebra. Suppose $(A_*, \partial_A), (B_*, \partial_B)$ are chain complexes of abelian groups, and $f : A_* \to B_*$ is a chain map. The (homological) **mapping cone** of f is then the chain complex (C_*^f, ∂) with

$$C_n^f := A_{n-1} \oplus B_n, \qquad \text{and} \qquad \partial := \begin{pmatrix} -\partial_A & 0 \\ -f & \partial_B \end{pmatrix}.$$

The reason for the terminology will become clearer in part (b) below.

(a) [20 pts] Show that the homomorphisms $H_n(A_*) \xrightarrow{f_*} H_n(B_*)$ induced by the chain map f fit into a long exact sequence of the form

$$\ldots \longrightarrow H_{n+1}(C^f_*) \longrightarrow H_n(A_*) \xrightarrow{f_*} H_n(B_*) \longrightarrow H_n(C^f_*) \longrightarrow H_{n-1}(A_*) \xrightarrow{f_*} H_{n-1}(B_*) \longrightarrow \ldots,$$

and describe the other two maps in this sequence explicitly.
Hint: You can derive this from a short exact sequence of chain maps, but do not try to include f in that sequence explicitly. See if you can make f_* appear as a connecting homomorphism.

The topological **mapping cone** of a continuous map $f : X \to Y$ is a space C_f defined by

$$C_f := \big(([0,1] \times X) \amalg Y\big)\big/\!\sim,$$

where the equivalence relation is the smallest such that for every $x, x' \in X$, $(0,x) \sim f(x)$ and $(1,x) \sim (1,x')$. In other words, C_f is the space $CX \cup_f Y$ formed by gluing the usual cone CX of X to Y along its boundary via the map $CX \supset \partial(CX) := \{0\} \times X = X \xrightarrow{f} Y$. The usual cone CX is the special case of C_f where $X = Y$ and $f : X \to X$ is the identity, and more generally, one can imagine C_f as an enlargement of Y in which extra stuff has been attached in order to make the map

$$X \xrightarrow{f} Y \hookrightarrow C_f$$

homotopic to a constant.

(b) [25 pts] Assuming X, Y are CW-complexes and $f : X \to Y$ is a cellular map, describe a cell decomposition of C_f for which the quotient projection

$$([0,1] \times X) \amalg Y \xrightarrow{q} C_f$$

is a cellular map and the augmented cellular chain complex $\widetilde{C}^{\mathrm{CW}}_*(C_f;\mathbb{Z})$ is isomorphic to the (homological) mapping cone of the chain map $f_* : \widetilde{C}^{\mathrm{CW}}_*(X;\mathbb{Z}) \to \widetilde{C}^{\mathrm{CW}}_*(Y;\mathbb{Z})$.

(c) [25 pts] Prove: For any cellular map $f : X \to Y$, the induced homomorphism $f_* : H_*(X;\mathbb{Z}) \to H_*(Y;\mathbb{Z})$ is an isomorphism if and only if $\widetilde{H}_*(C_f;\mathbb{Z}) = 0$.

Comments: The statement in part (c) still holds if $f : X \to Y$ is only assumed continuous instead of cellular, because by the cellular approximation theorem, every continuous map between CW-complexes is homotopic to a cellular map. (See if you can convince yourself that homotopic maps always have homotopy equivalent mapping cones!) If X and Y are also assumed simply connected, then one can combine this with some fundamental tools from homotopy theory—namely the theorems of Whitehead and Hurewicz on higher homotopy groups—to establish the following elegant improvement: the map $f : X \to Y$ is a homotopy equivalence if and only if its mapping cone is contractible.