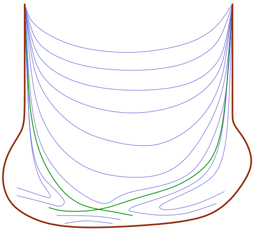
On Symplectic Manifolds with Contact Boundary

or "when is a Stein manifold merely symplectic?"

Chris Wendl

Humboldt-Universität zu Berlin

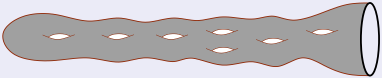
February 9, 2018



What do these questions have in common?

1. Topology

Given a closed manifold M, is it the boundary of a compact manifold?



2. Global analysis

Given two (almost) complex manifolds W and W', what is the structure of the space of **holomorphic maps** $W \to W'$?

Is it smooth? Is it compact? Is its topology interesting?

3. Hamiltonian dynamics

Given $H(q_1, p_1, \ldots, q_n, p_n)$, does $H^{-1}(c)$ contain periodic orbits of

$$\dot{q}_j = \frac{\partial H}{\partial p_j}, \qquad \dot{p}_j = -\frac{\partial H}{\partial q_j} ?$$

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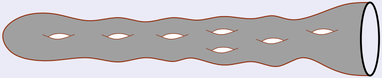

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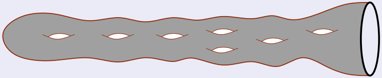

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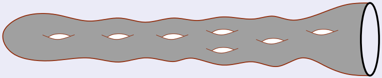

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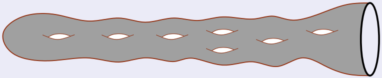

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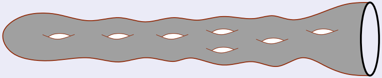

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- **Contact structures** $(\dim M = 2n - 1)$

The following answer to Question 3 may serve as motivation:

Theorem (Rabinowitz-Weinstein '78)

Every star-shaped hypersurface in $\mathbb{R}^{2n}$ has a **periodic** *Hamiltonian orbit.*

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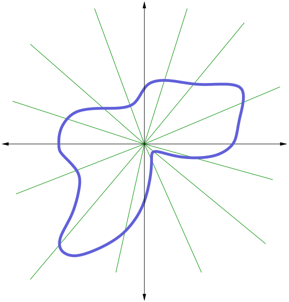

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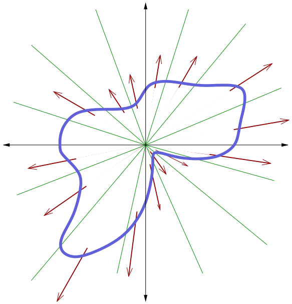

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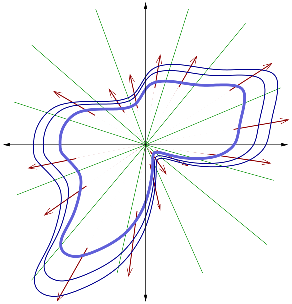

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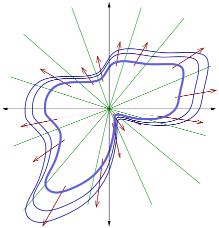

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$$\omega = dp_1 \wedge dq_1 + \ldots + dp_n \wedge dq_n.$$

The boundary ∂W is **convex** if it is transverse to a vector field that dilates the symplectic form: $\mathcal{L}_V \omega = \omega$.

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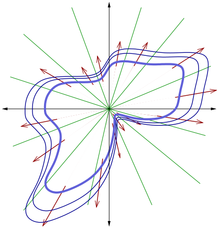

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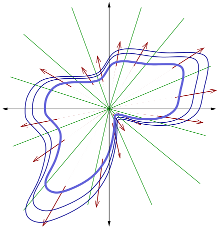

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An example from complex geometry

A **Stein manifold** is a complex manifold (W, J) with a proper holomorphic embedding

$$(W, J) \hookrightarrow (\mathbb{C}^N, i).$$

$\overset{\text{(Grauert)}}{\Leftrightarrow}$ (W, J) admits an exhausting **plurisubharmonic** function $f : W \to \mathbb{R}$, meaning

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Then ω_J is dilated by ∇f, so $W_c := f^{-1}((-\infty, c])$ is **symplectic with convex boundary** $M_c := f^{-1}(c)$.

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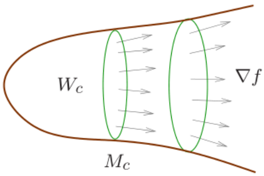

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The maximal complex subbundle

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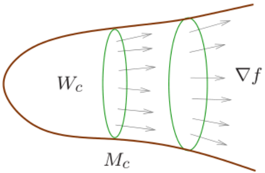
is then a **contact structure** on M_c, i.e. it is maximally nonintegrable.



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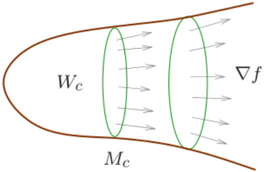

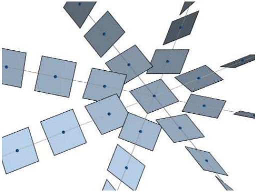

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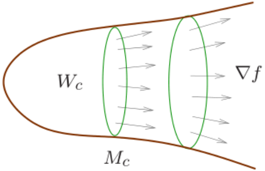

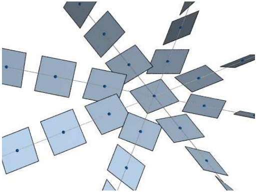

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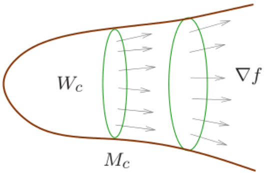

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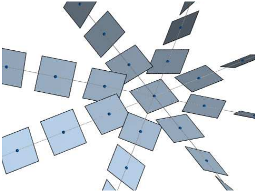

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Do Hamiltonian flows on compact contact hypersurfaces always have periodic orbits?

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Which (M', ξ') admit symplectic cobordisms to (M, ξ), i.e. "$(M', \xi') \prec (M, \xi)$"?

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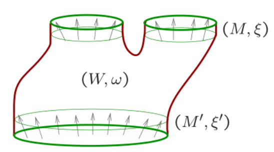
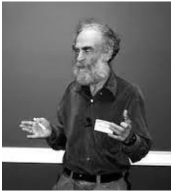


Rigidity and flexibility

Gromov (ICM 1986): "soft" vs. "hard" symplectic geometry



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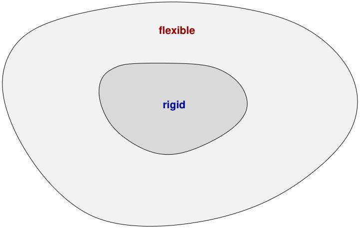

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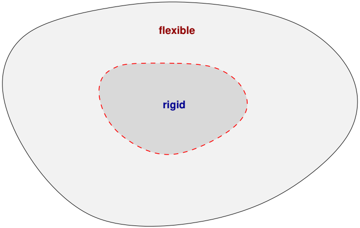

Insight: The interesting questions are on the borderline.

Flexibility

Flexibility ("soft") comes from the **h-principle**,

Examples of symplectic flexibility

- **Existence** of symplectic structures on **open** manifolds [Gromov 1969]:

$$\frac{\{\text{sympl. forms}\}}{\text{deformation}} \overset{1:1}{\longleftrightarrow} \frac{\{\text{almost } \mathbb{C}\text{-structures}\}}{\text{homotopy}}$$

- There is a flexible class of Stein structures: two such structures are **Stein homotopic** $\Leftrightarrow$ **homotopic** as almost complex structures.

  [Cieliebak-Eliashberg 2012]

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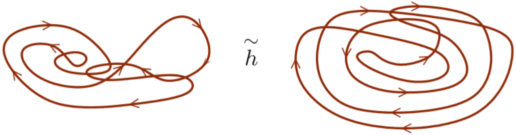

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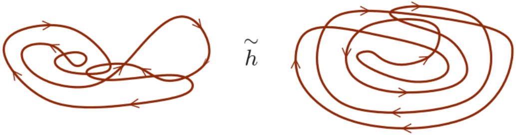

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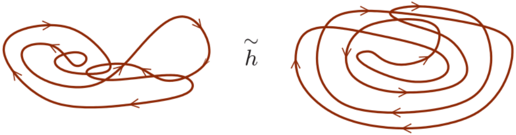

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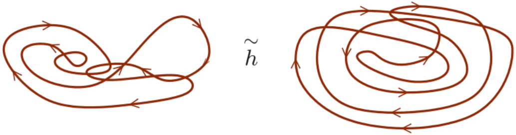

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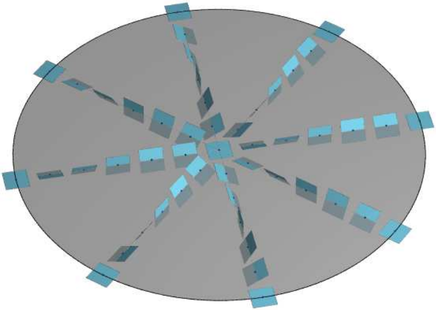
Flexibility

- Two "overtwisted" contact structures ξ_1, ξ_2 are **isotopic** $\Leftrightarrow$ they are **homotopic**. [Eliashberg 1989] + [Borman-Eliashberg-Murphy 2014]



Rigidity

Rigidity ("hard") comes from **invariants**: *Gromov-Witten, Floer homology, symplectic field theory (SFT), Seiberg-Witten...*

Examples of symplectic rigidity

- $(S^3, \xi_{\mathrm{std}})$ has a **unique Stein filling** up to deformation. [Gromov 1985], [Eliashberg 1989]
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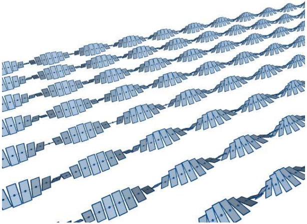
- The 3-torus admits an infinite sequence of contact structures that are **homotopic** but not isotopic. [Giroux 1994]
  Only the first is **fillable** [Eliashberg 1996], and its filling is **unique**. [W. 2010]



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Stein is generally **more rigid** than *symplectic*, e.g. Ghiggini '05 proved

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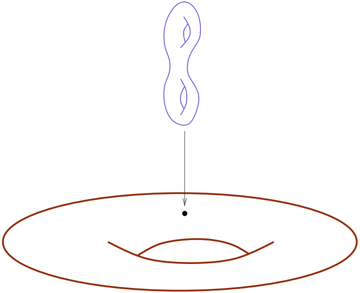

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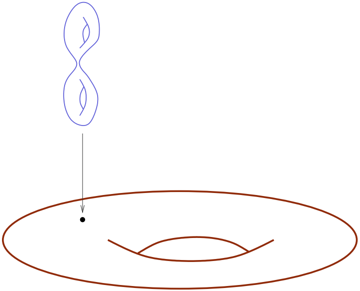

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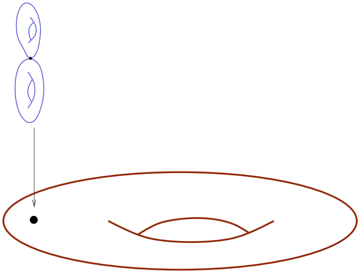

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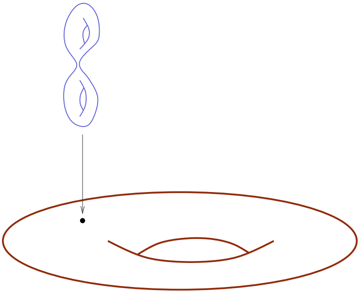

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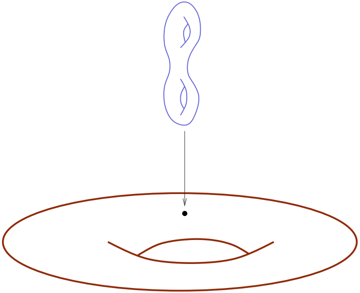

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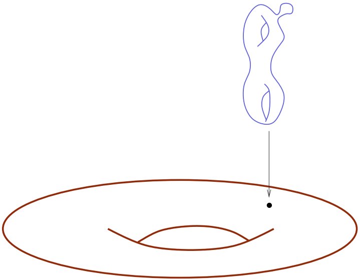

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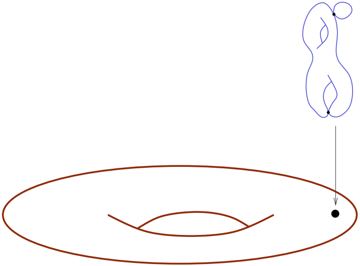

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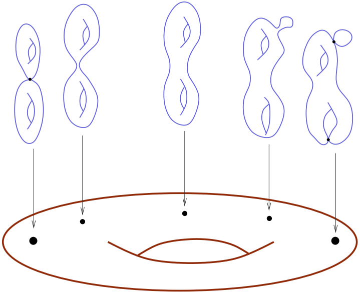

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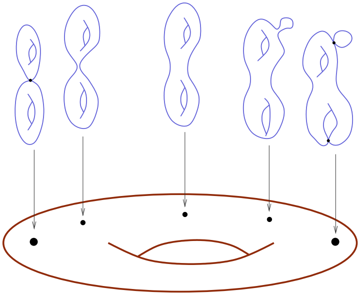
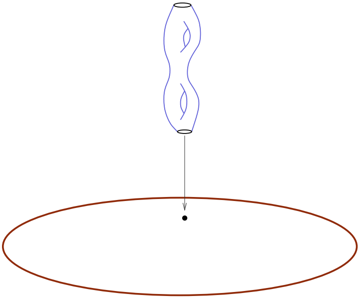


Theorem (Thurston, Gompf)

If [fiber] $\neq 0 \in H_2(W; \mathbb{Q})$, then W admits a canonical deformation class of symplectic forms with $\omega|_{\textit{fibers}} > 0$.

Lefschetz fibrations with boundary (over $\mathbb{D}^2$)



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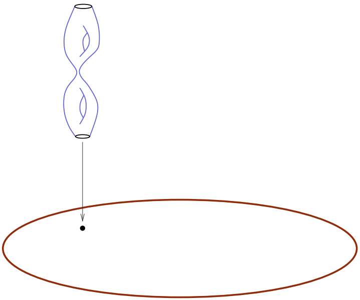

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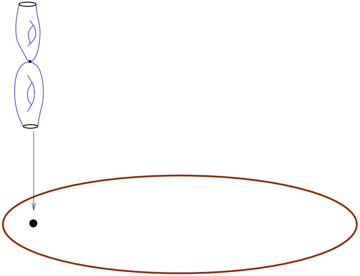

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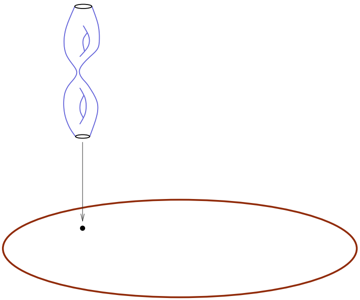

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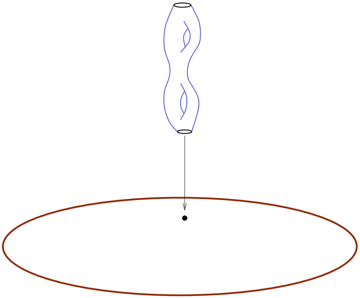

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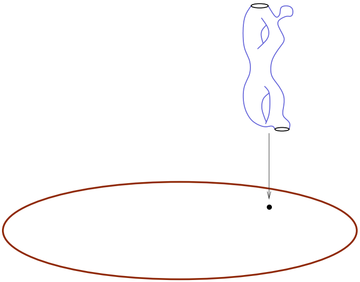

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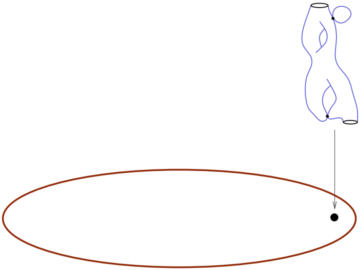

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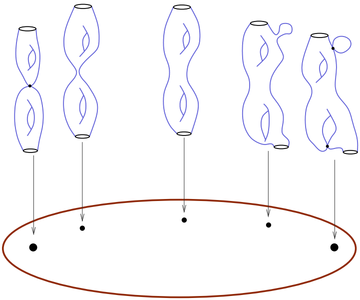

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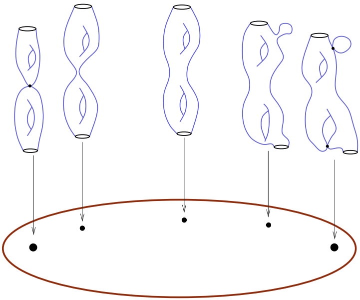

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$$\partial_v W := \pi^{-1}(\partial \mathbb{D}^2) \xrightarrow{\text{fibration}} \partial \mathbb{D}^2 = S^1,$$

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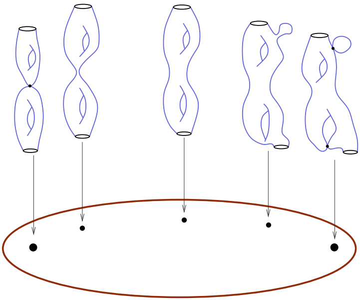

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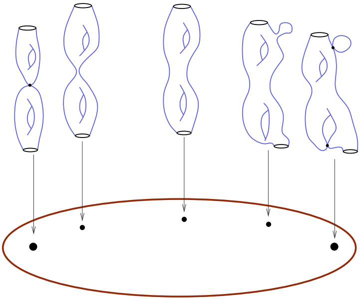
Thurston-Gompf for Stein structures



Lemma (Lisi, Van Horn-Morris, W.)

Suppose $\pi : W \to \mathbb{D}^2$ has **no closed components** *in its singular fibers (i.e. π is "allowable"). Then W admits a* **canonical deformation class of Stein structures** *such that the* **fibers are holomorphic curves***, and the contact structure on ∂W is* **supported** *(in the sense of Giroux) by the induced open book decomposition.*

Heavy artillery

Fundamental lemma of symplectic topology (Gromov '85)

On every symplectic manifold (W, ω), there is a **contractible** space of "tamed" almost complex structures

$$\left\{ J : TW \to TW \;\middle|\; J^2 = -\mathbb{1} \text{ and } \omega(X, JX) > 0 \text{ for all } X \neq 0 \right\}.$$

Given a Riemann surface (Σ, j), a map $u : \Sigma \to W$ is called J**-holomorphic** if it satisfies the **nonlinear Cauchy-Riemann equation**:

$$Tu \circ j = J \circ Tu$$

$\Leftrightarrow$ in local coordinates $s + it$,

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This is a **first-order elliptic PDE**.

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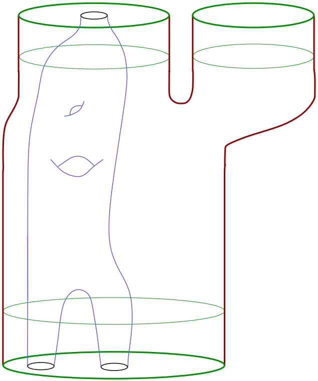

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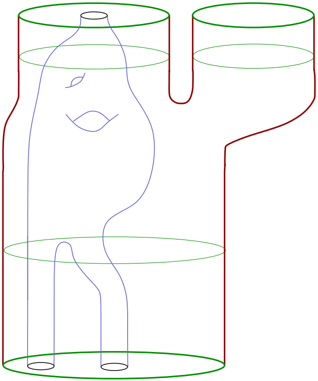

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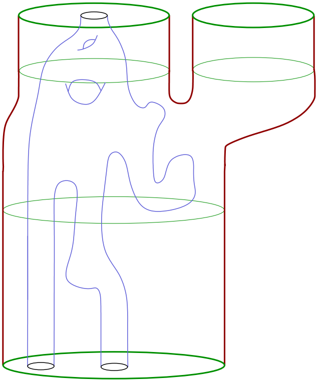

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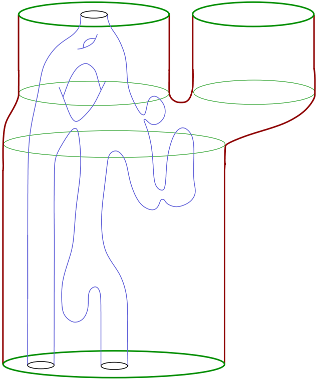

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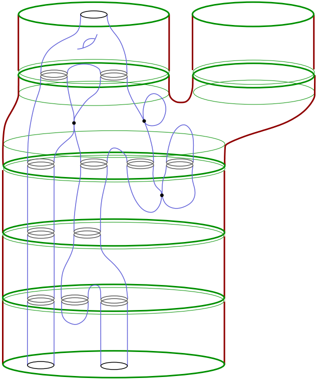

Switching on the machine...

Lemma (W. '10)

Suppose (W^4, ω_τ) is a 1-parameter family of symplectic fillings of (M^3, ξ), where ξ is supported by a planar open book (i.e. its fibers have genus zero).

Choose a generic family J_τ of ω_τ-tame **almost complex structures** *on the symplectic completion $(\widehat{W}, \widehat{\omega}_\tau)$.*

Then the open book extends to a smooth family of **Lefschetz fibrations**

$$W \xrightarrow{\pi_\tau} \mathbb{D}^2$$

*with J_τ-***holomorphic fibers***, and they are allowable if ω_τ is exact for any τ.*

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Switching on the machine...

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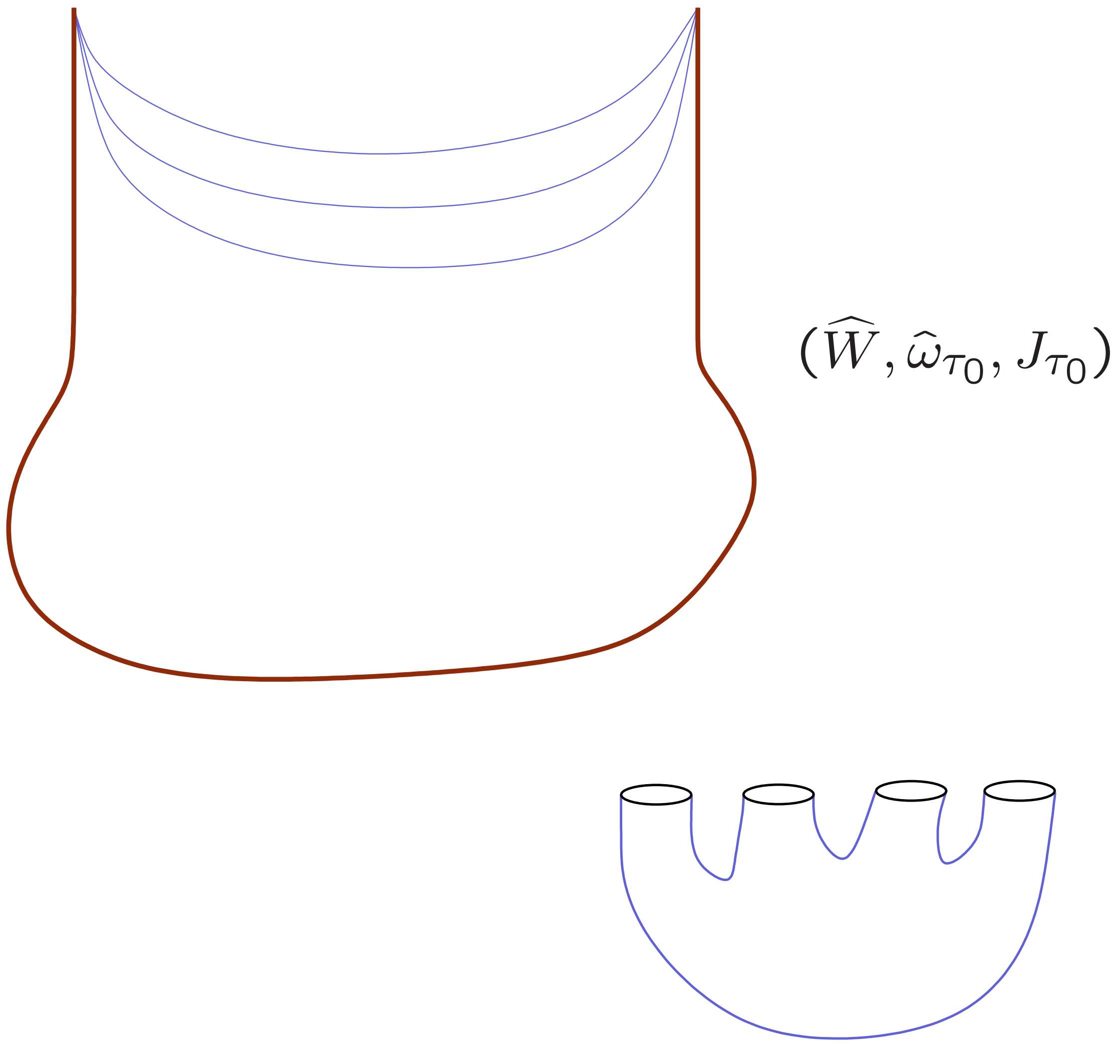
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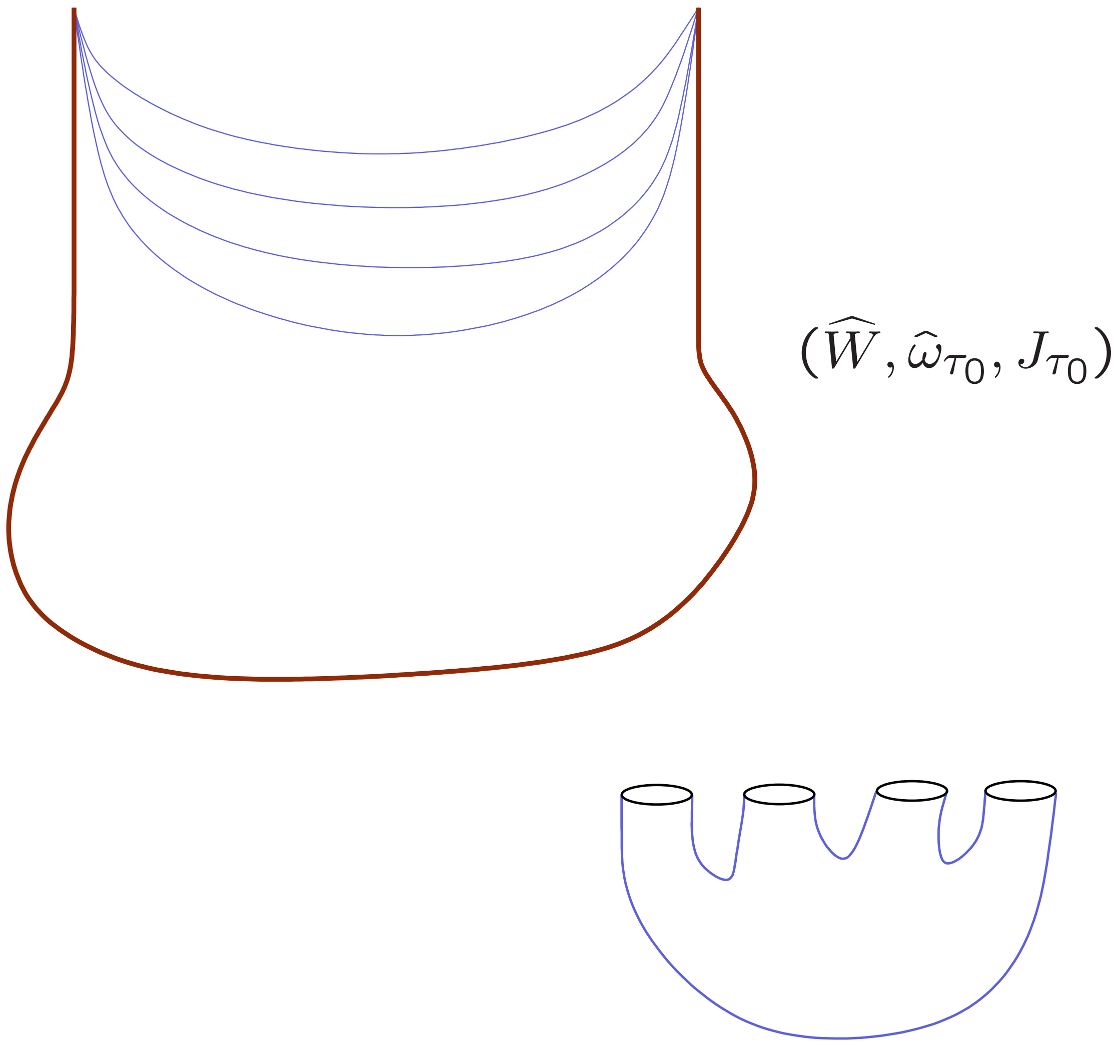
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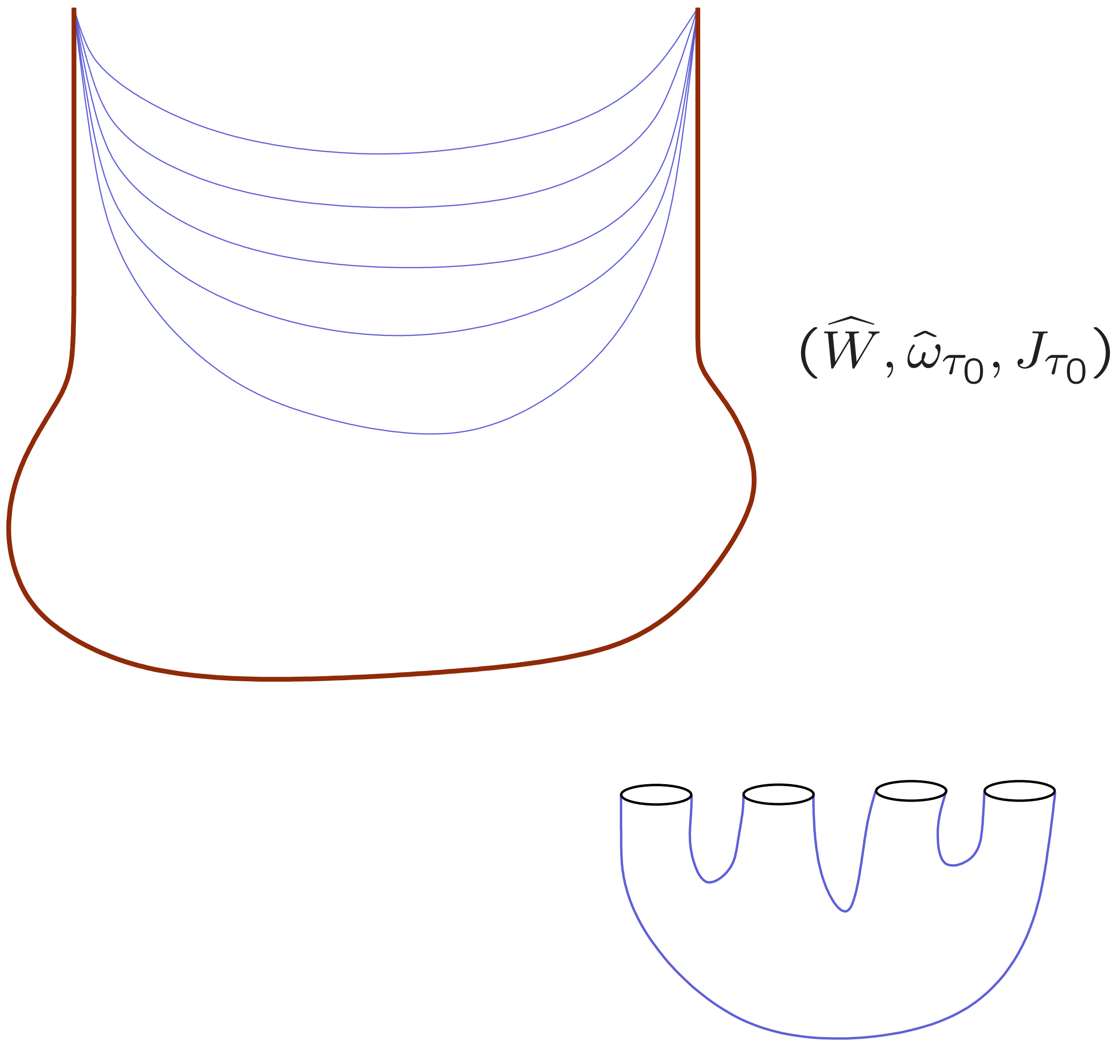
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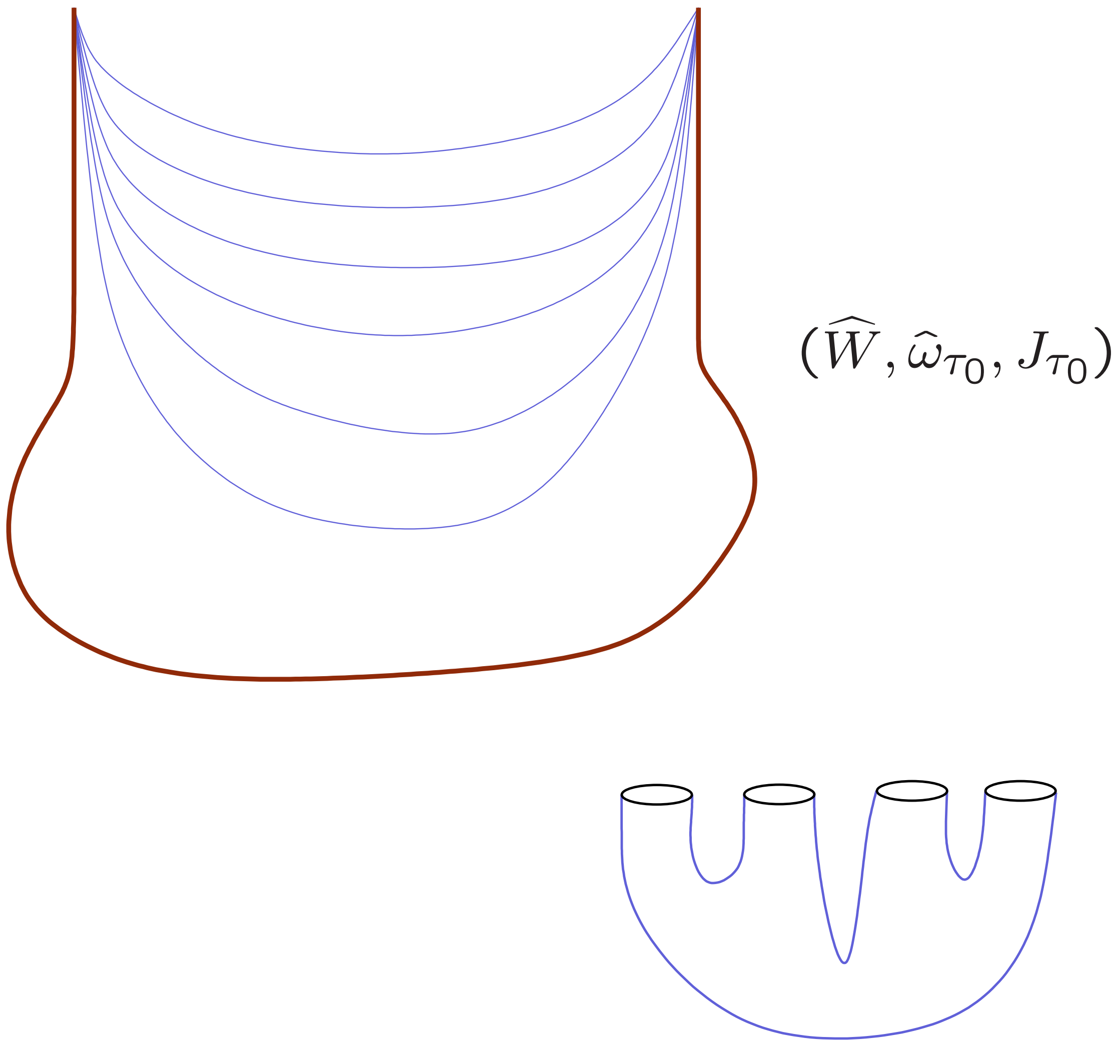
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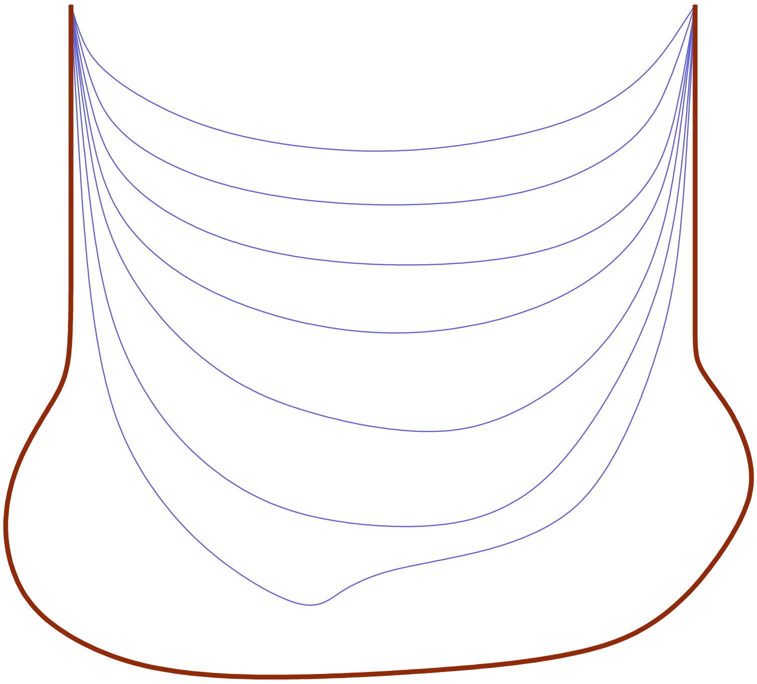

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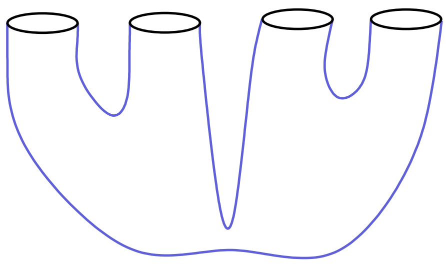

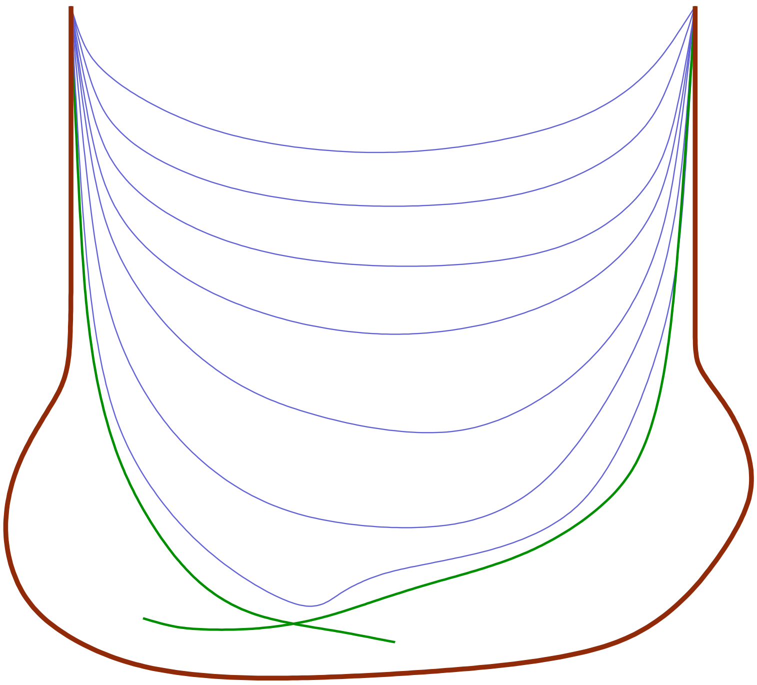
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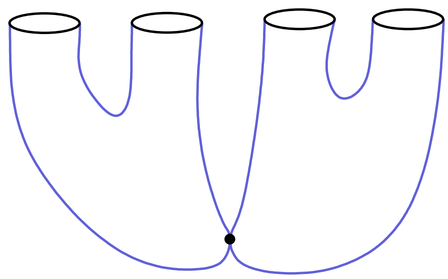

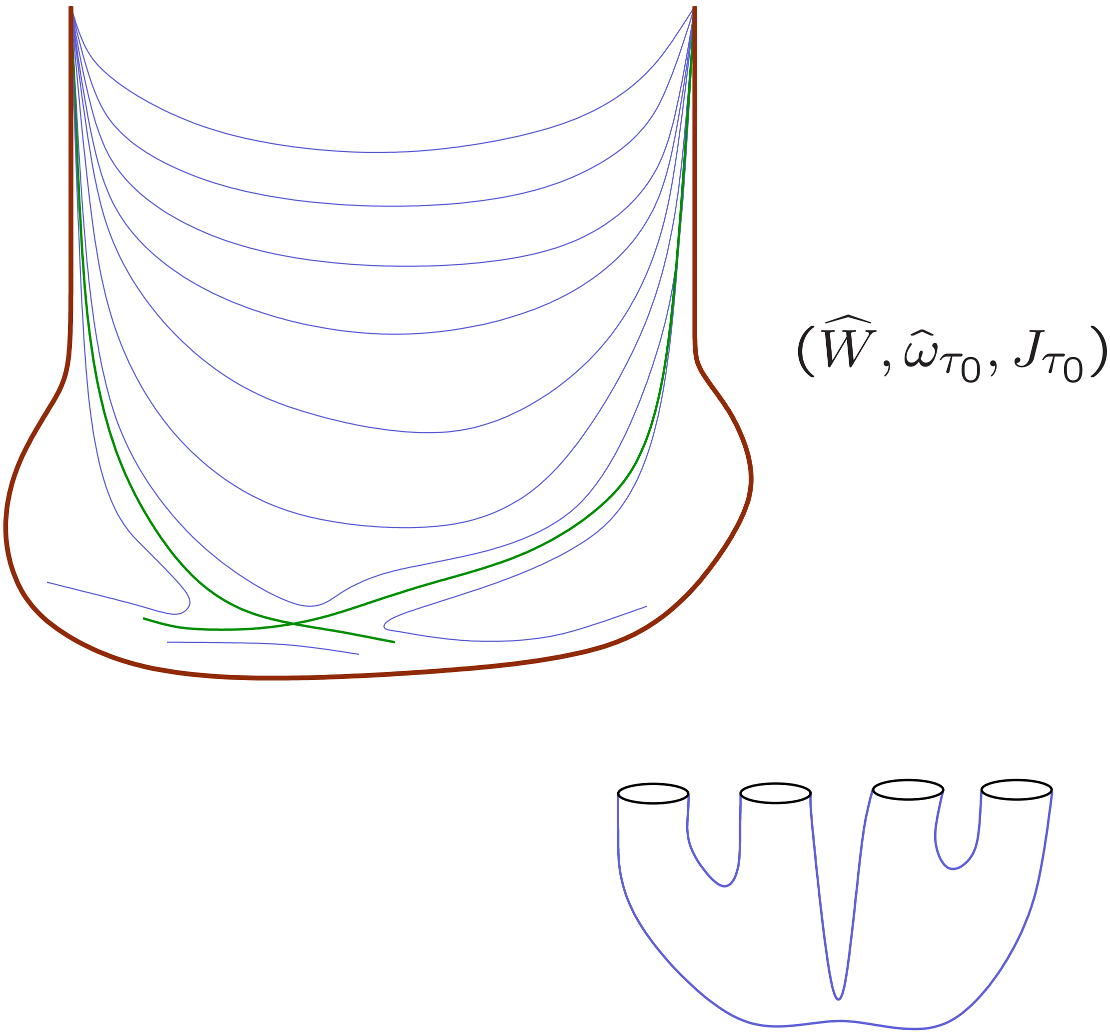
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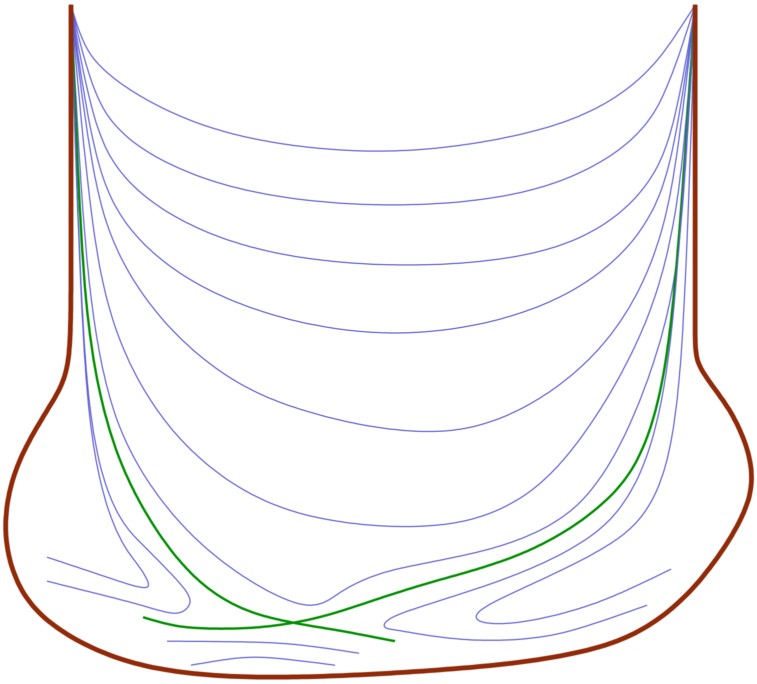

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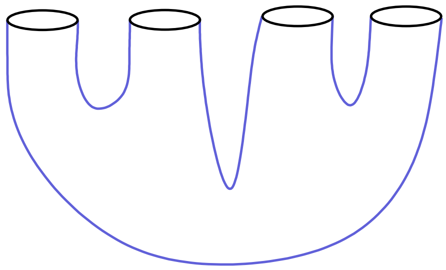

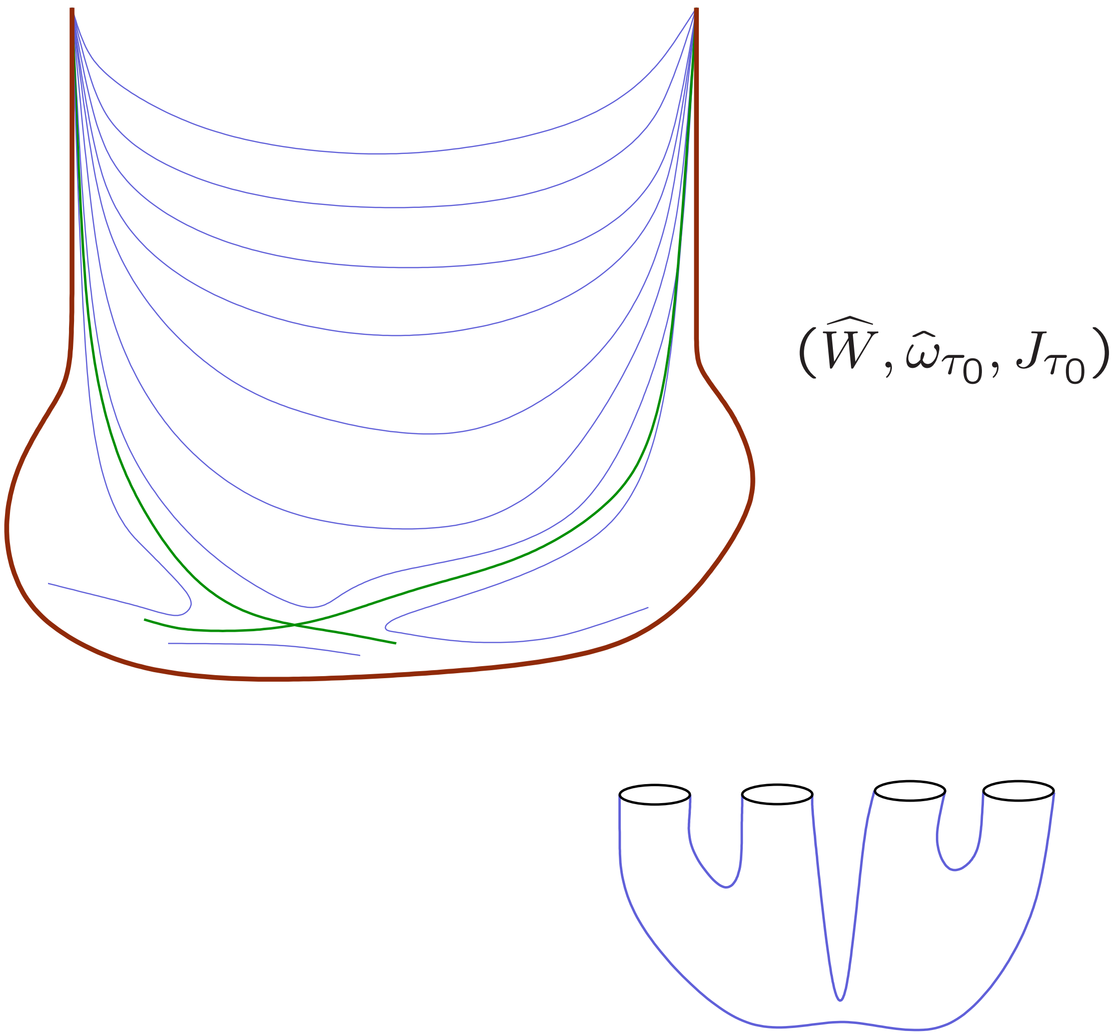
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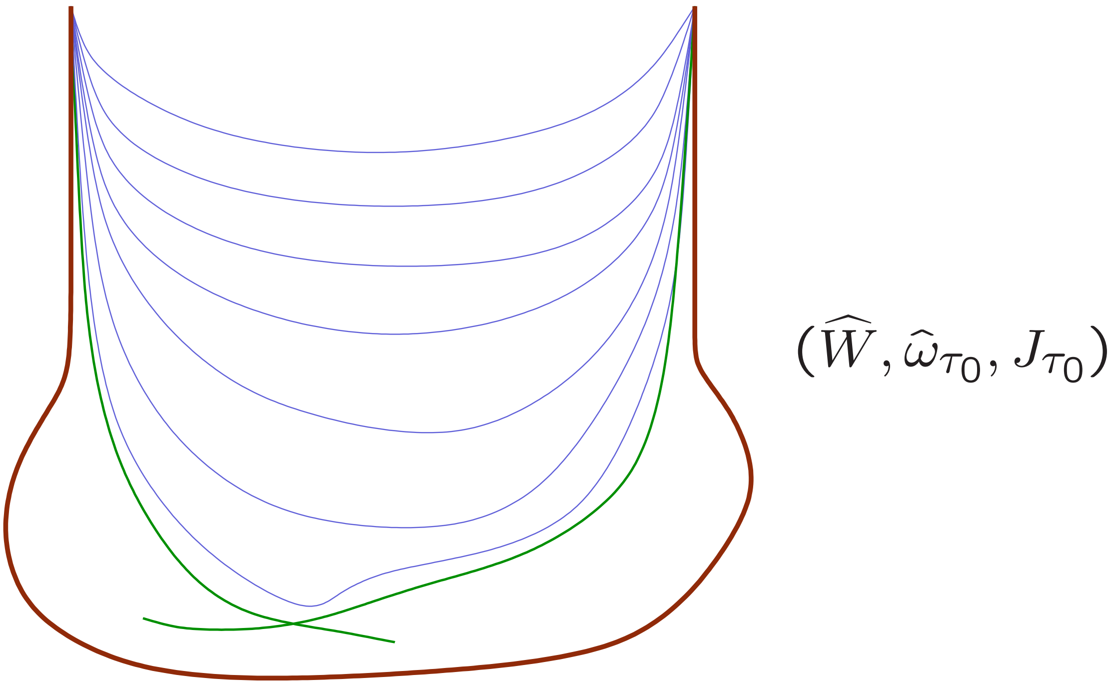
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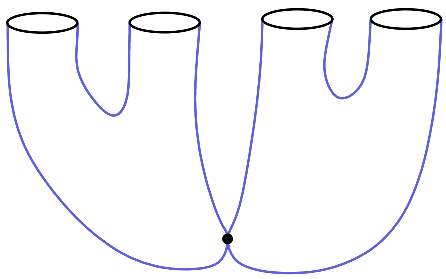

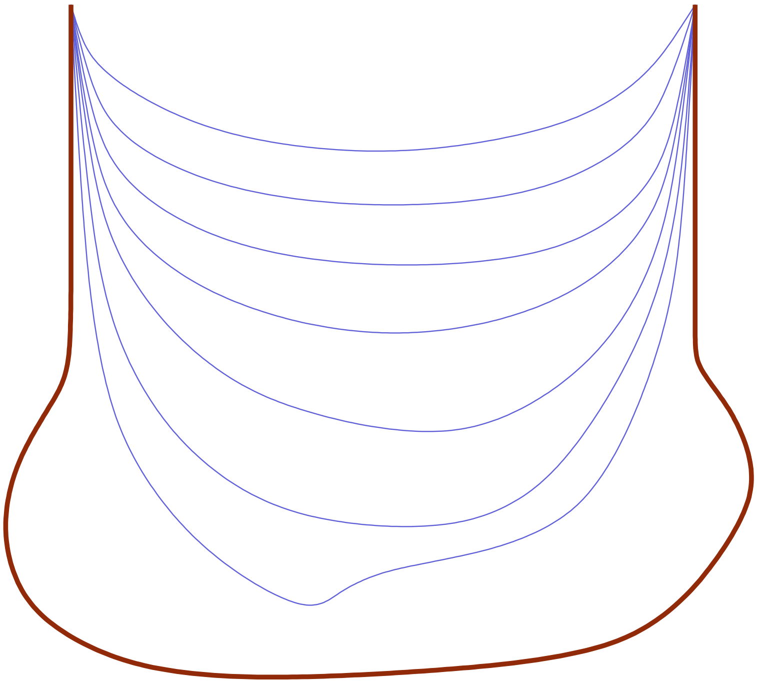

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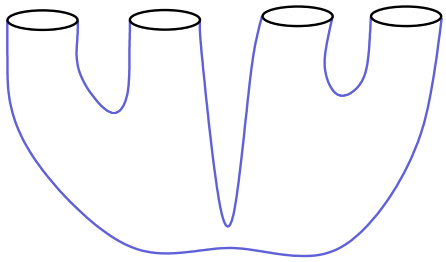

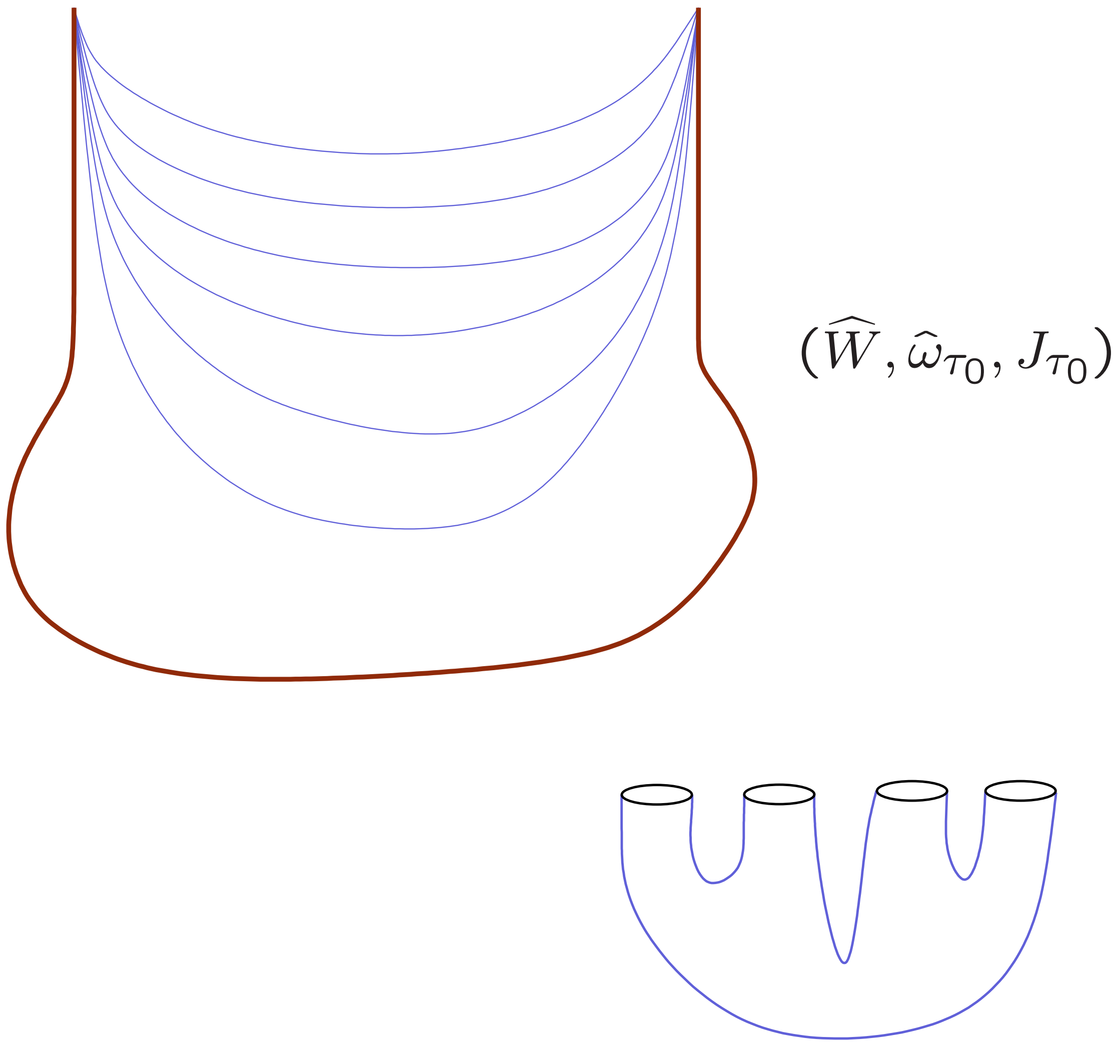
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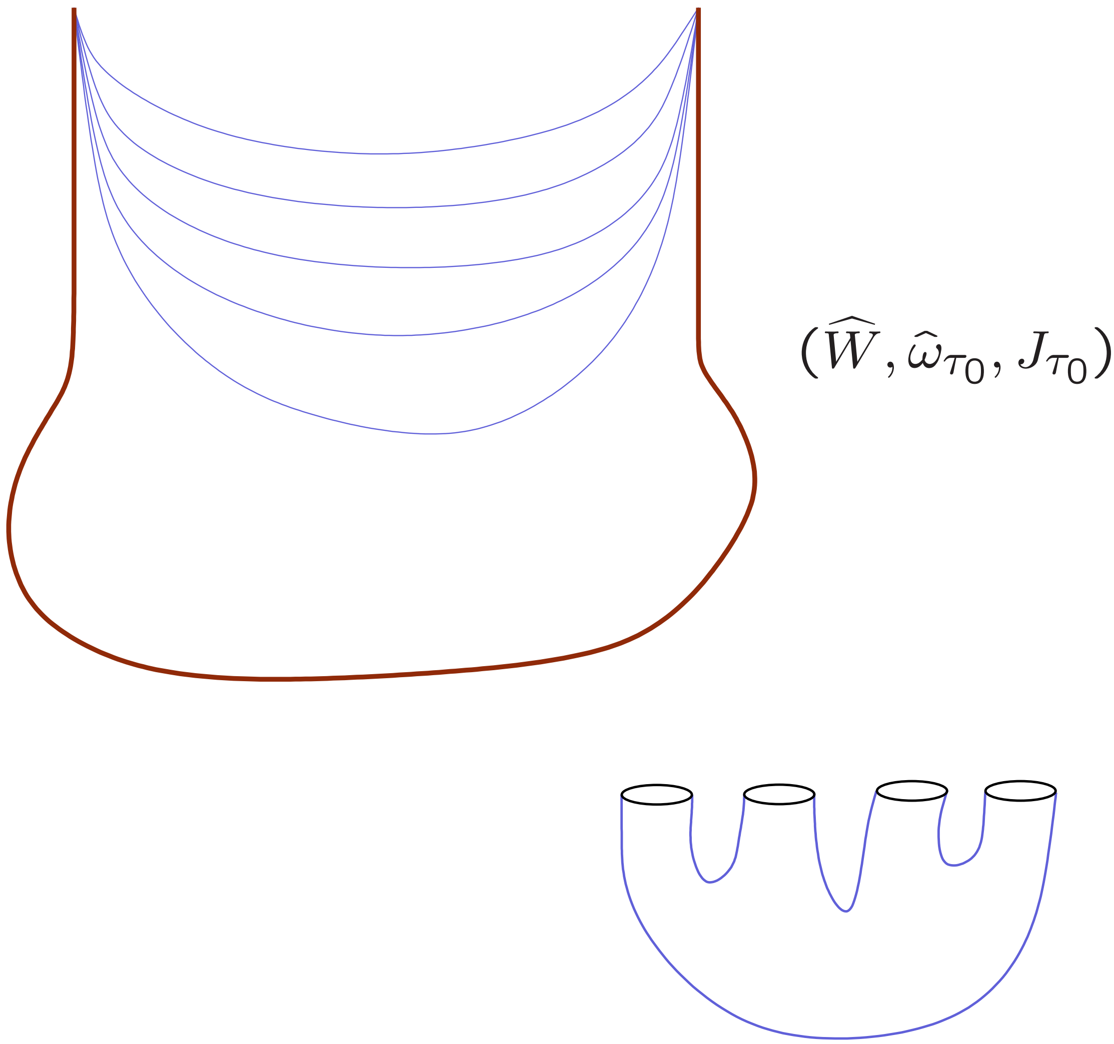

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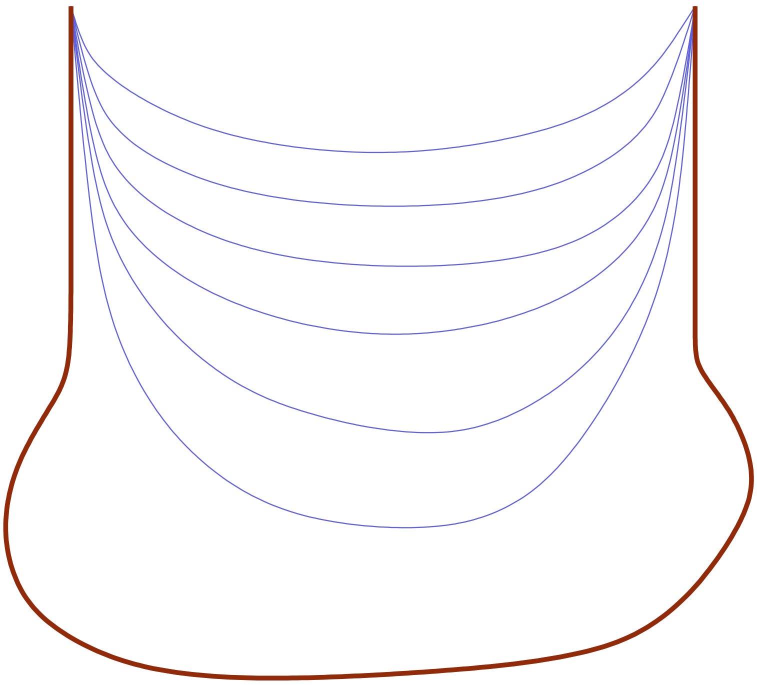
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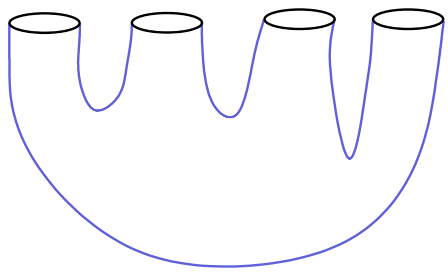

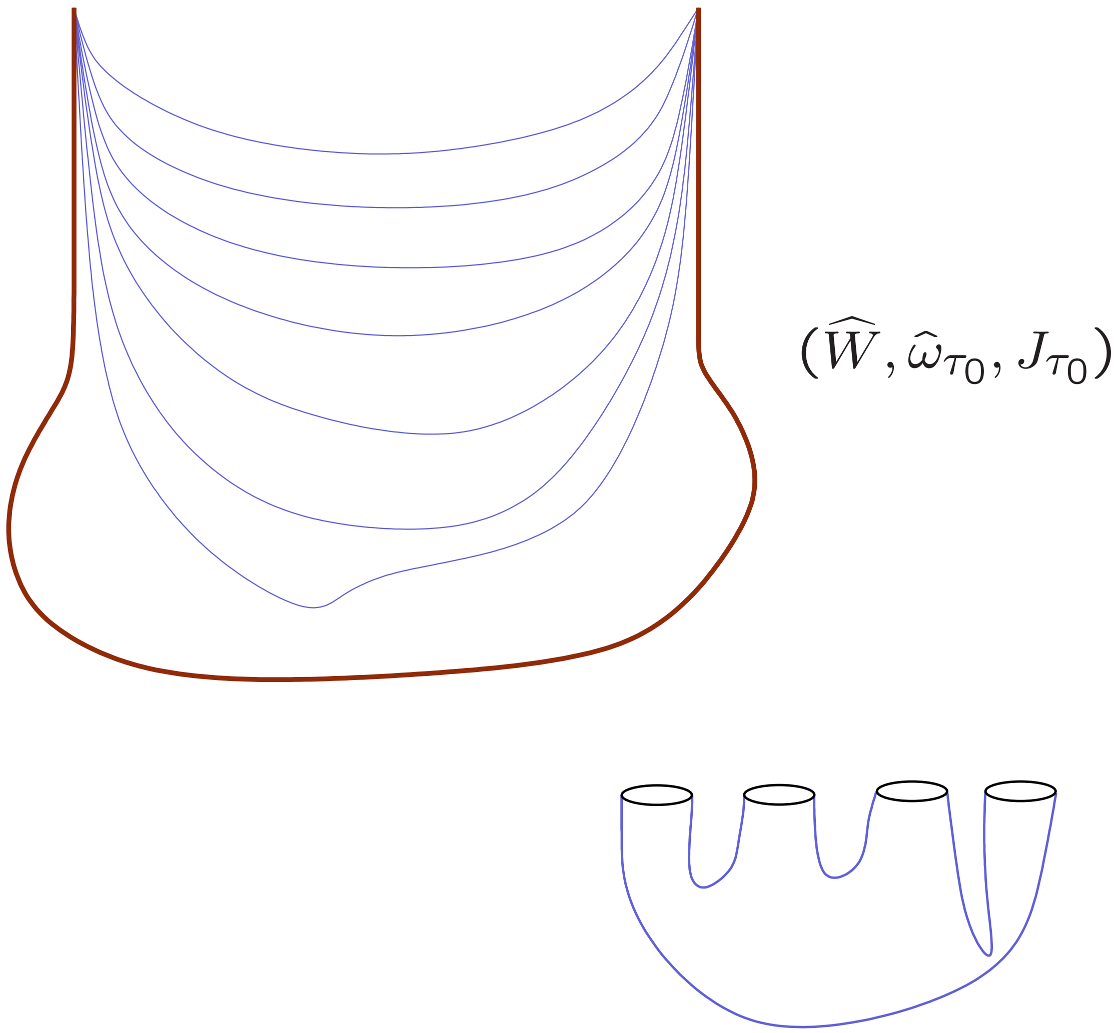
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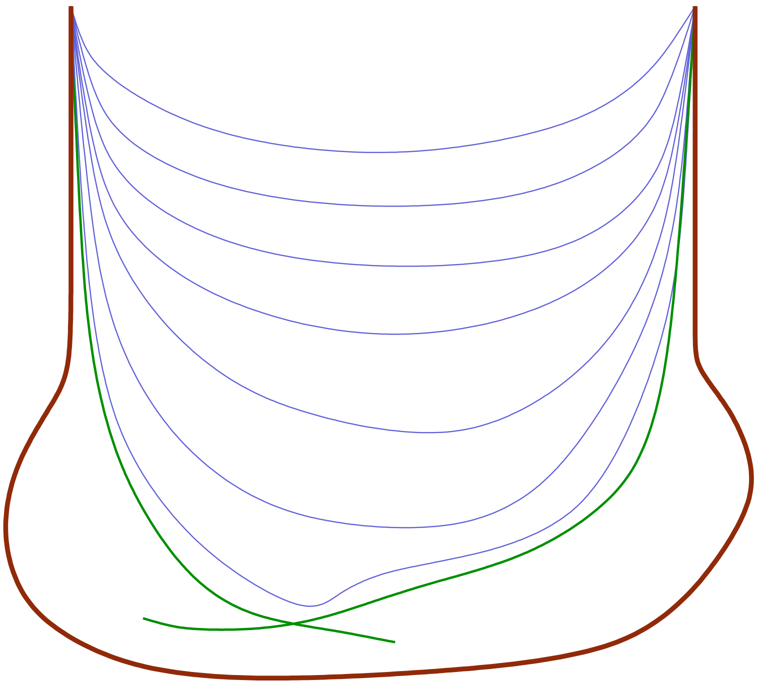

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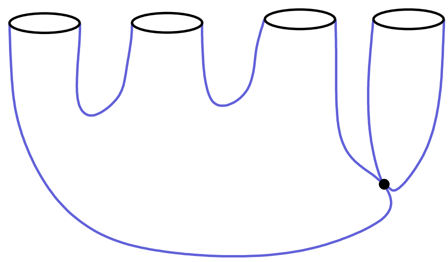

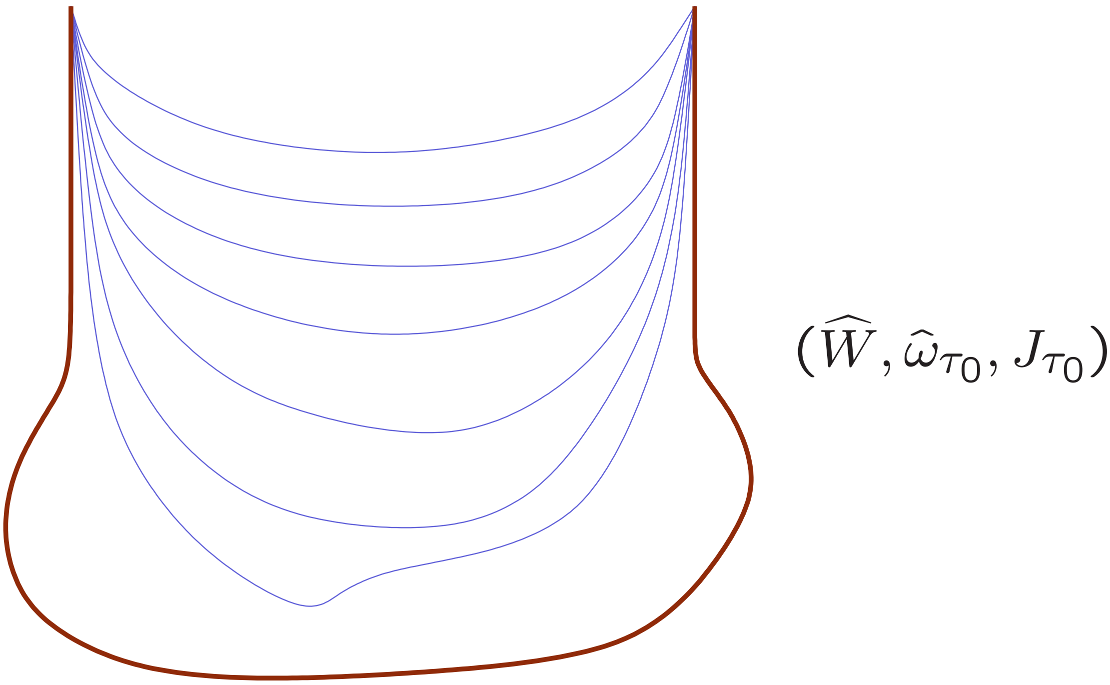
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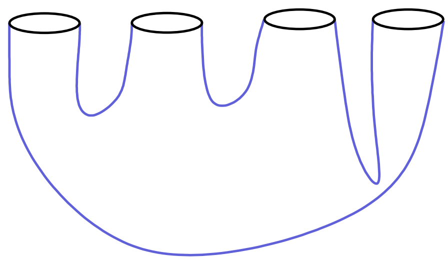

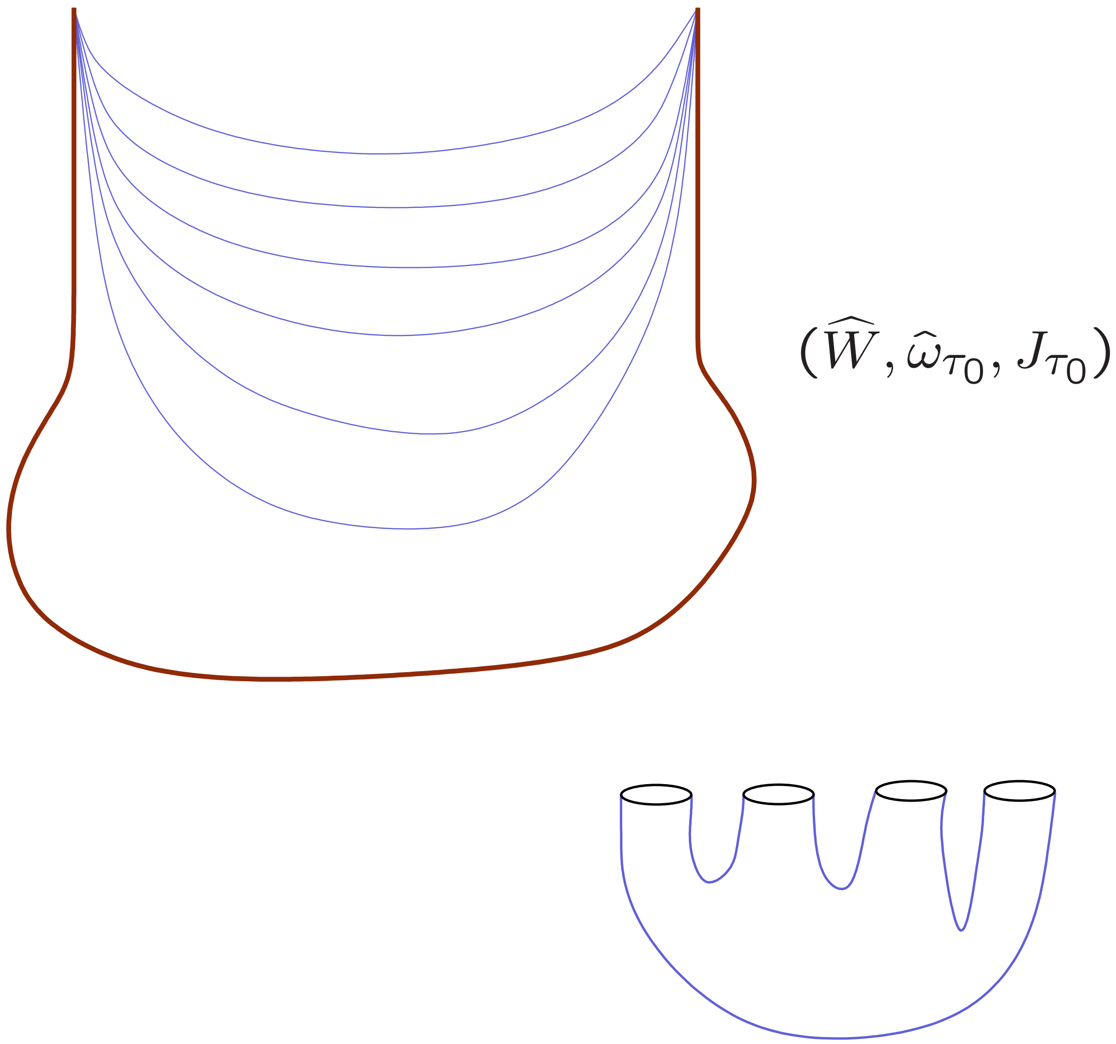
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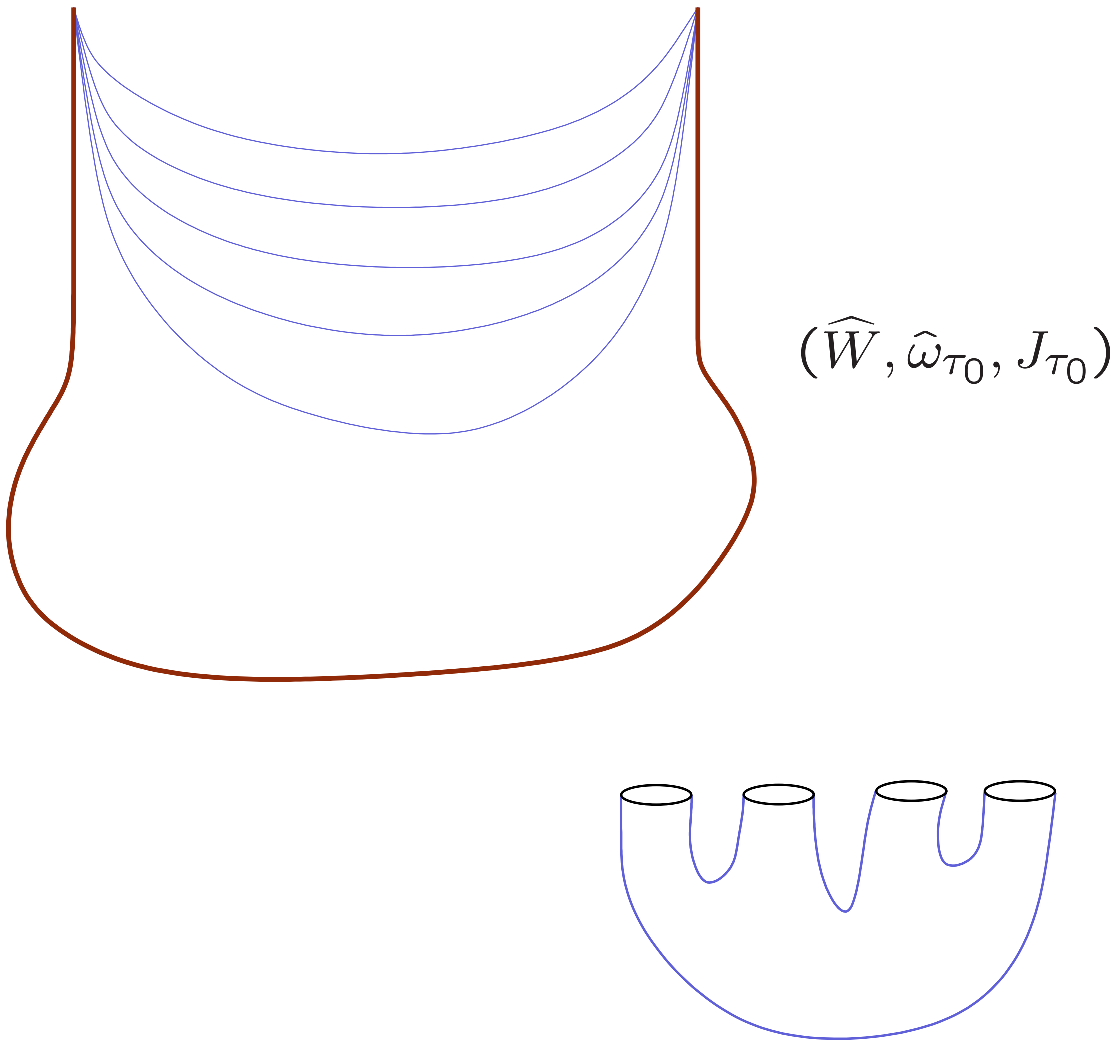
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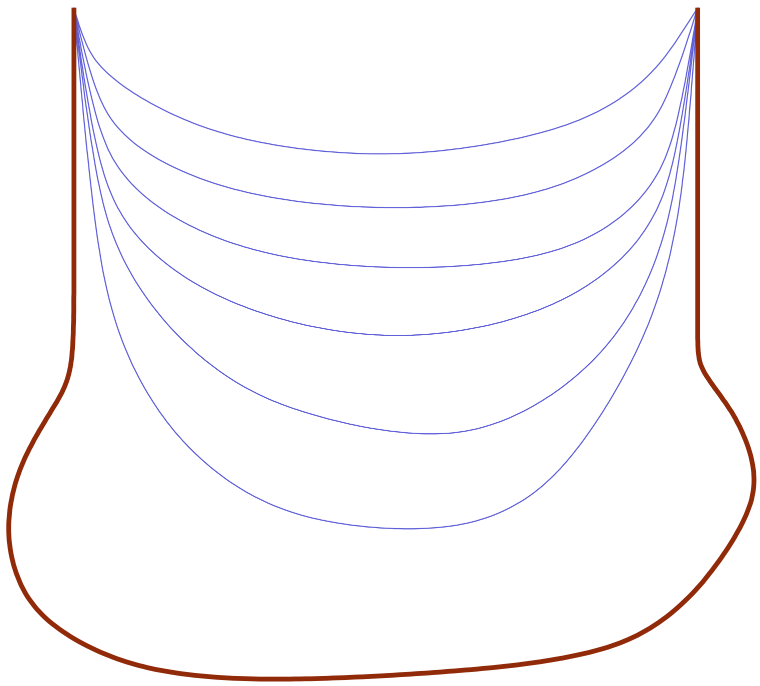

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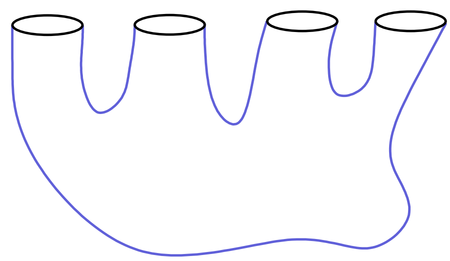

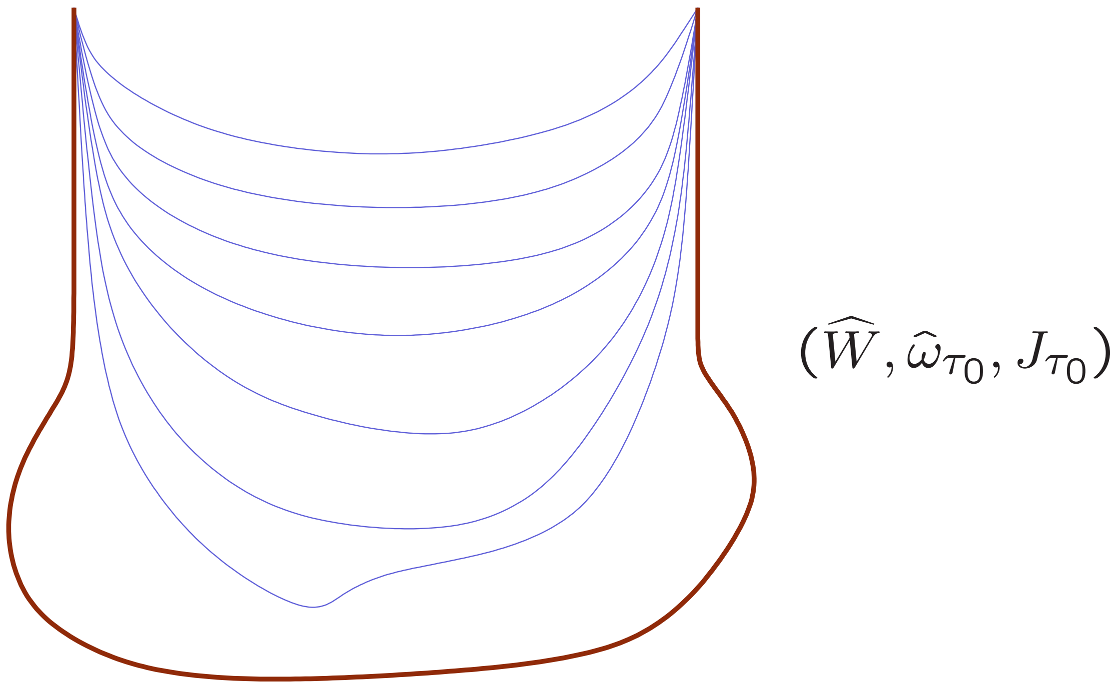
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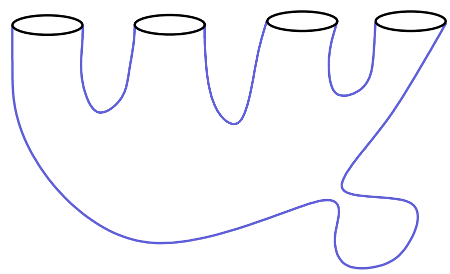

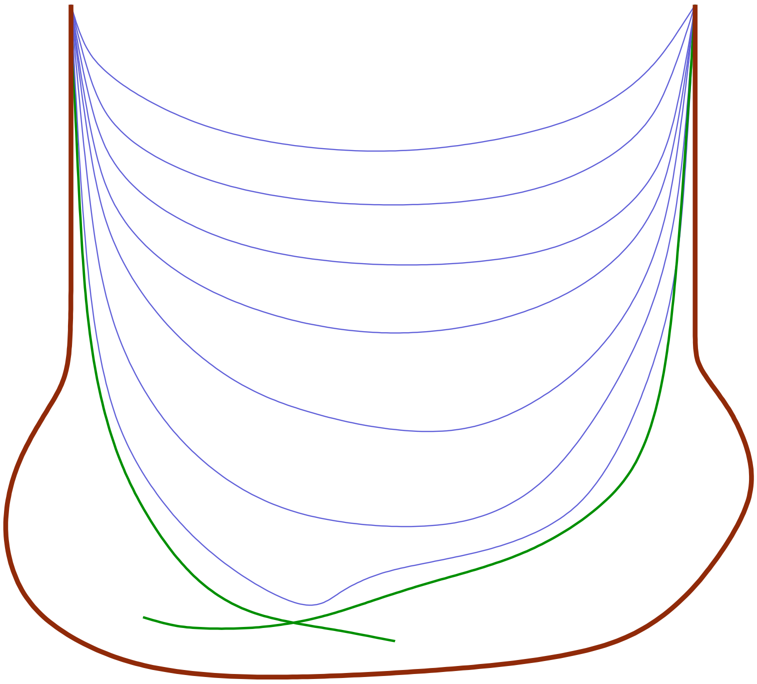

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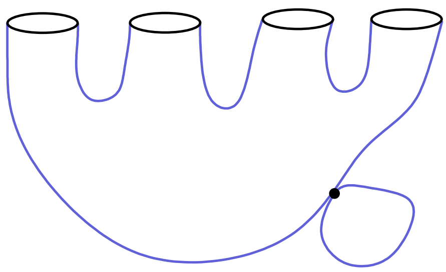
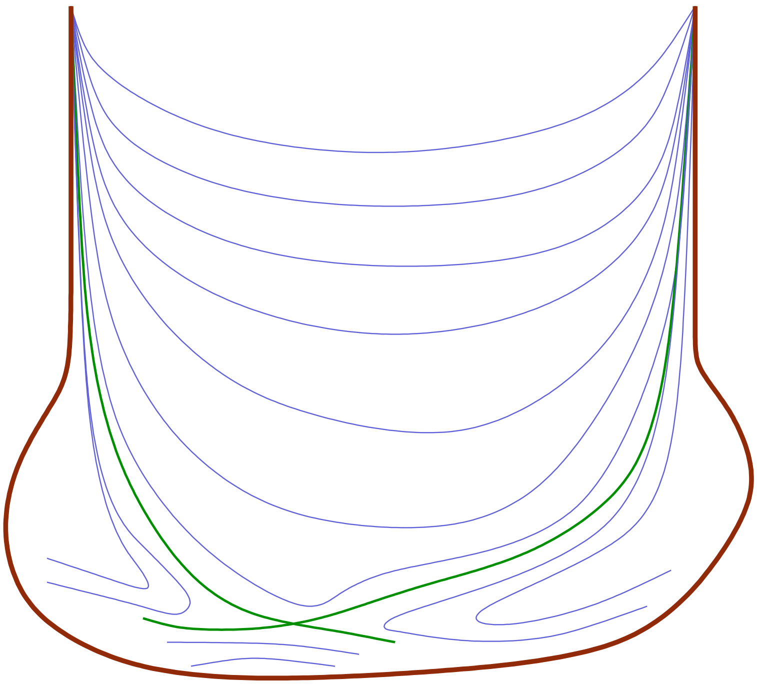


only if ω_{τ_0} not exact!



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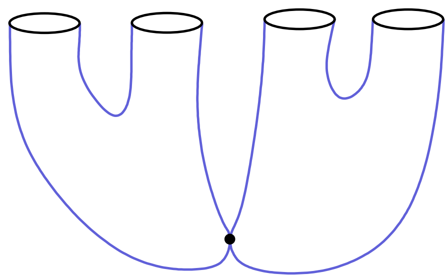
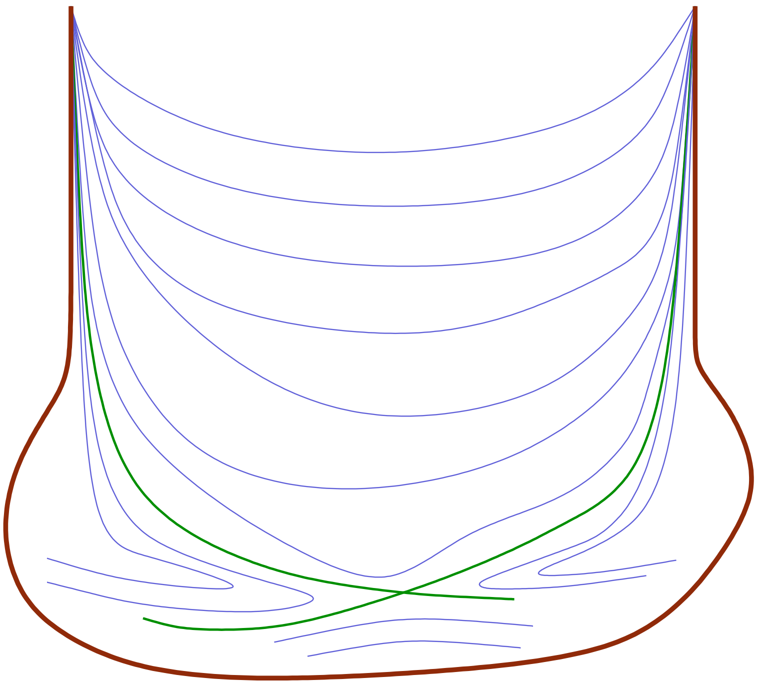
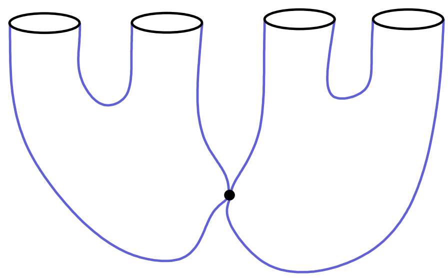
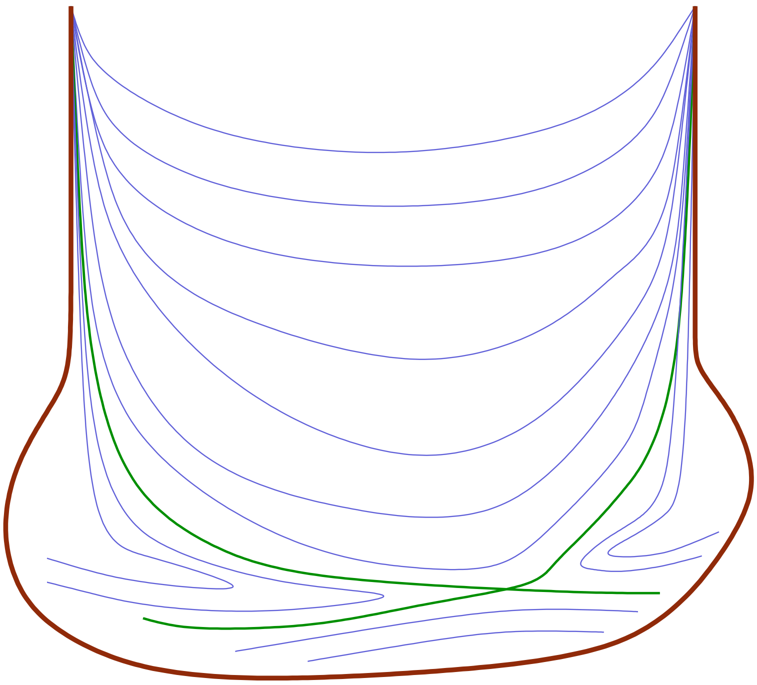


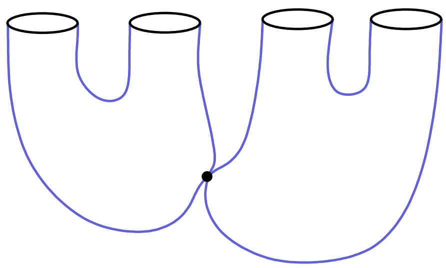


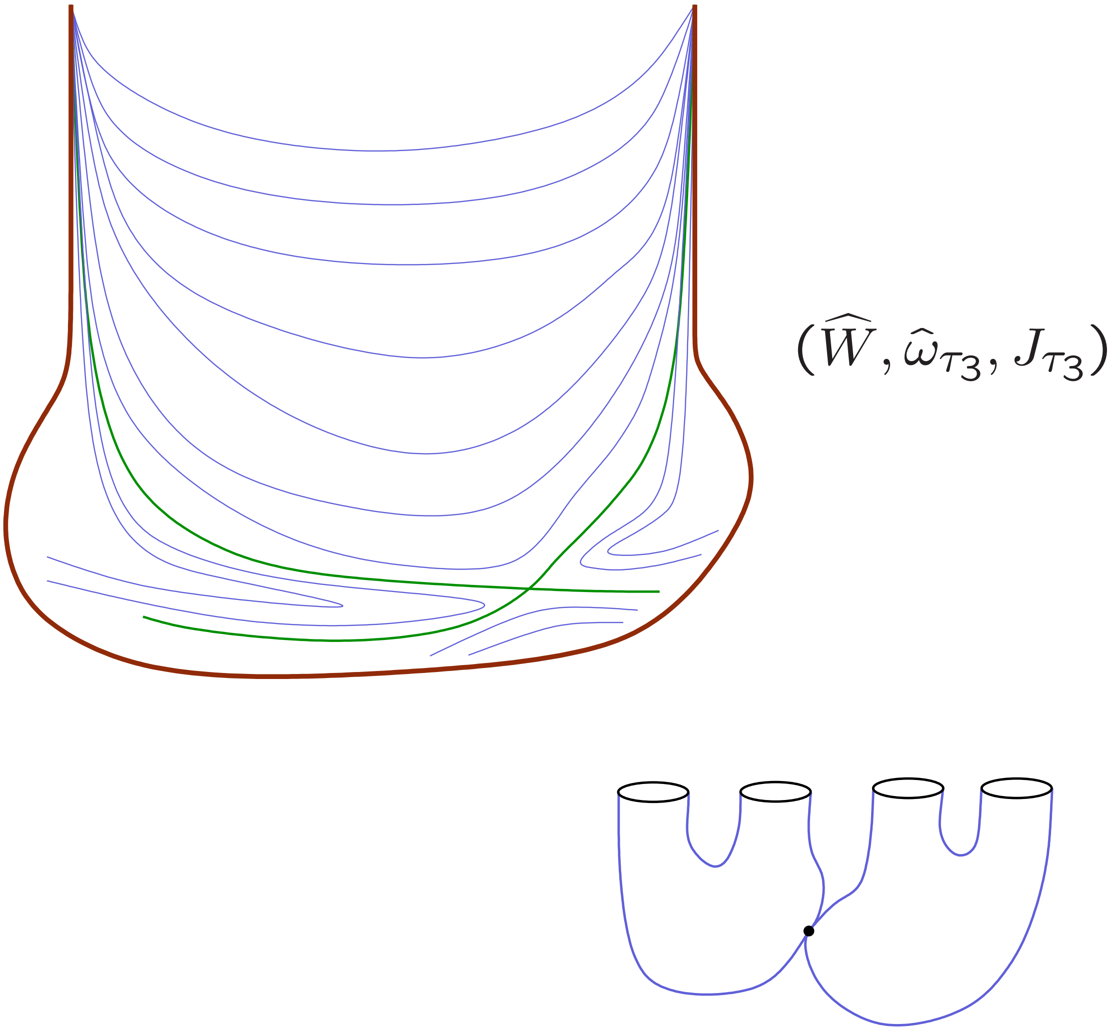
$(\widehat{W}, \widehat{\omega}_{\tau_1}, J_{\tau_1})$

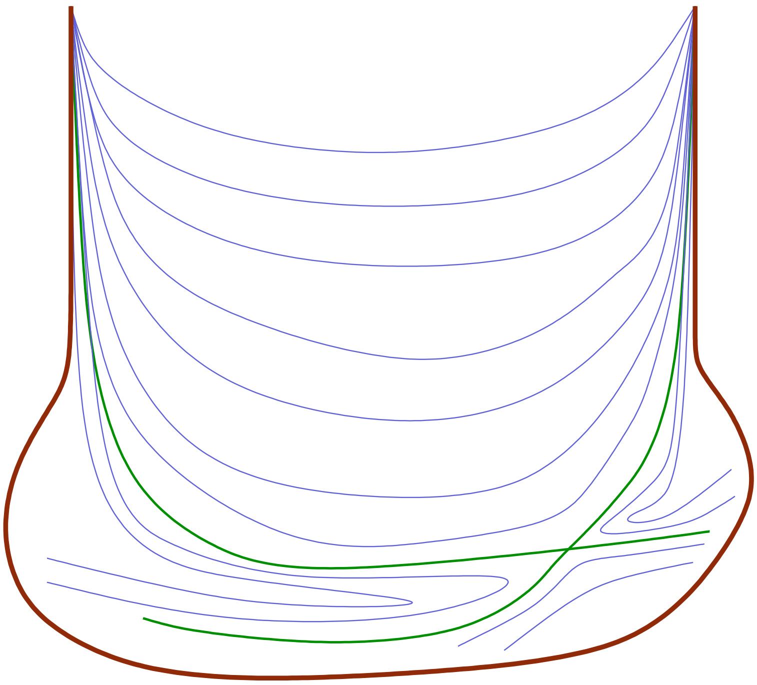




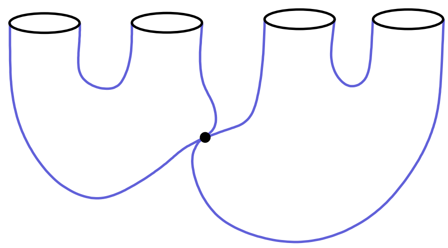
$(\widehat{W}, \widehat{\omega}_{\tau_2}, J_{\tau_2})$

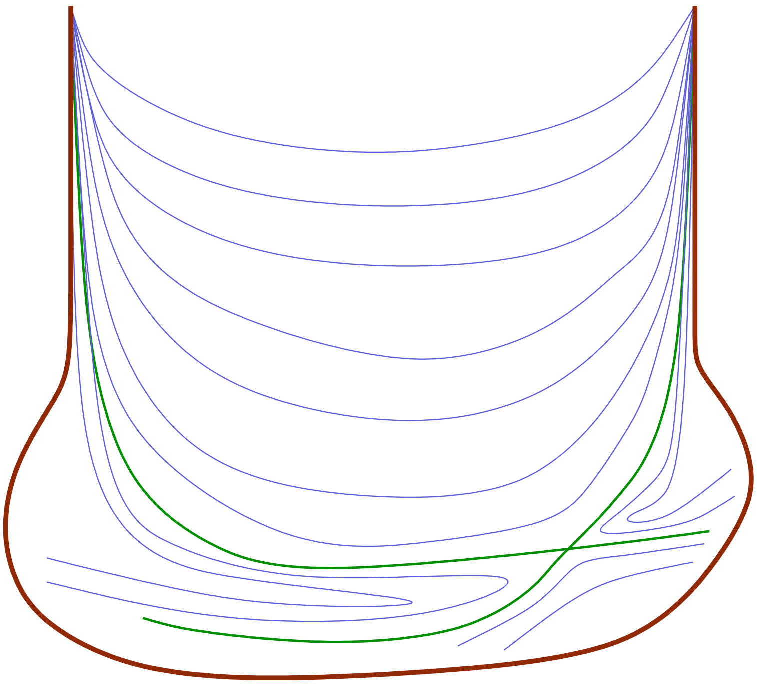



$(\widehat{W}, \widehat{\omega}_{\tau_3}, J_{\tau_3})$



$(\widehat{W}, \widehat{\omega}_{\tau_4}, J_{\tau_4})$





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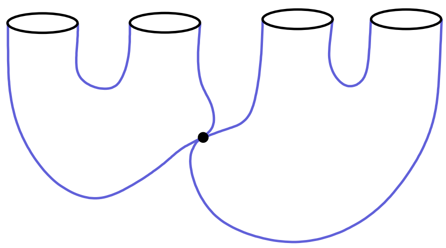
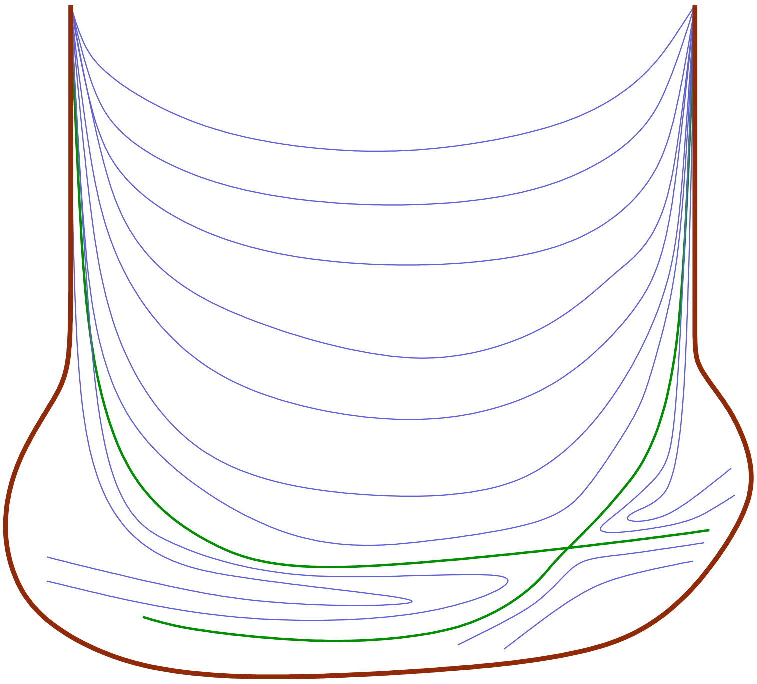


Remark: This does *not* work with higher-genus open books. Curves have index $2-2g$.



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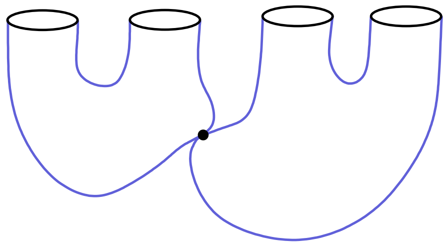

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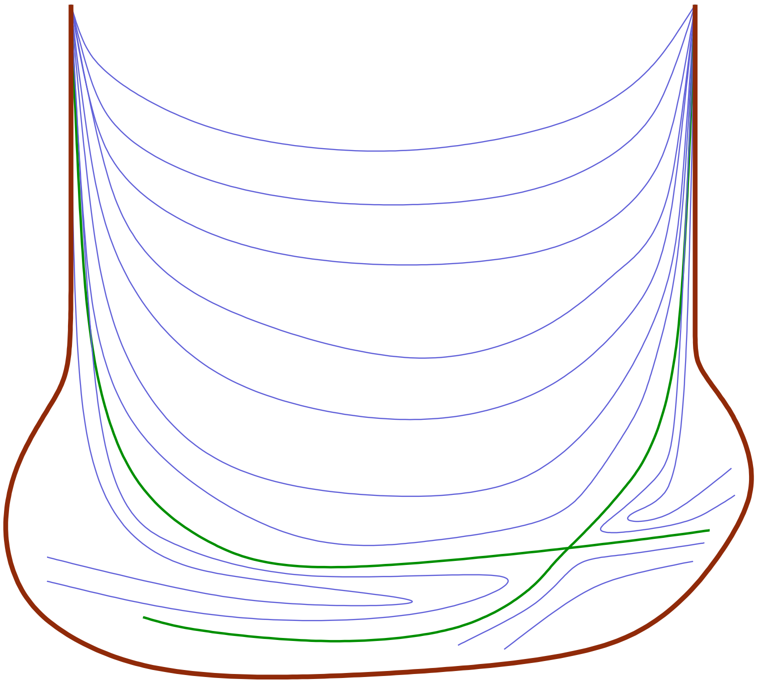
Proof of main theorem:



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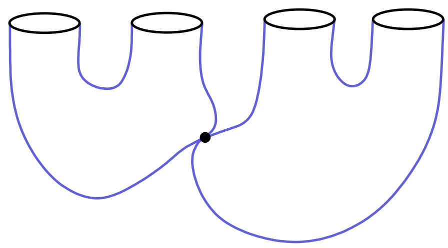

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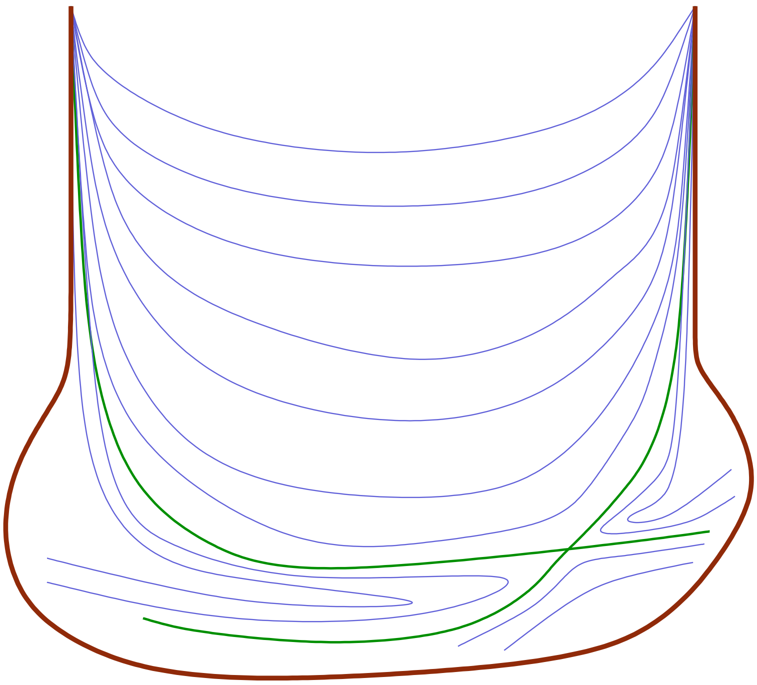
Proof of main theorem:
Symplectic deformation
$\Longrightarrow$ isotopy of Lefschetz fibrations



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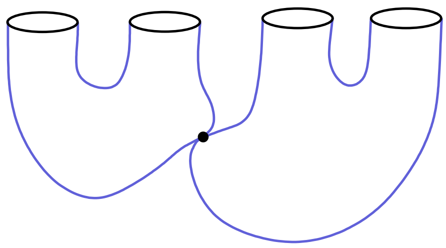

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Proof of main theorem:
Symplectic deformation
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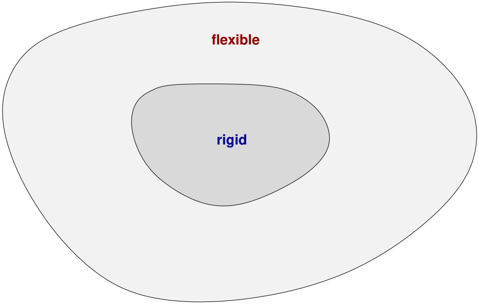
$\Longrightarrow$ homotopy of Stein structures. $\square$

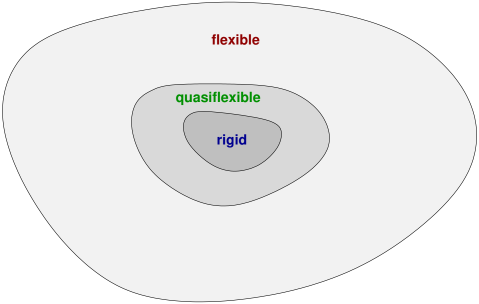
Conclusion

Even rigid structures can be. . .



Conclusion

Even rigid structures can be. . . **somewhat flexible**.



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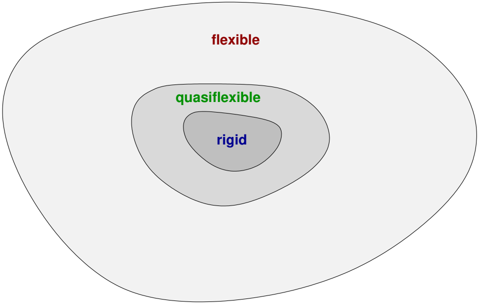
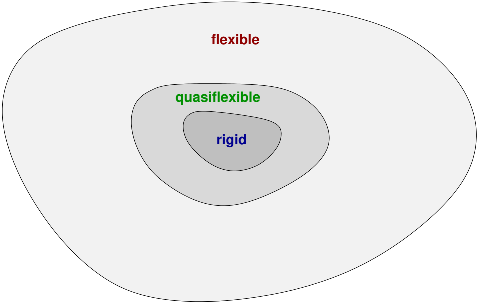


Some questions for the future

- Is there quasiflexibility in higher dimensions?

Conclusion

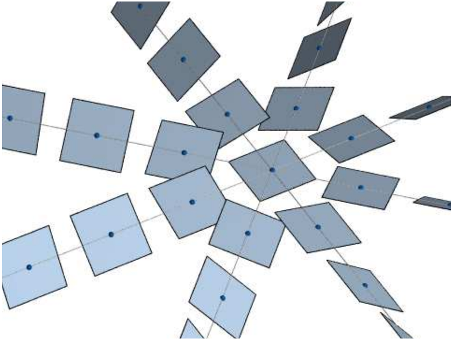
Even rigid structures can be... **somewhat flexible**.



Some questions for the future

- Is there quasiflexibility in higher dimensions?
- Is there a quasiflexible class of contact structures in dimension 3? *(planar?)*

Thank you for your attention!



Pictures of contact structures by Patrick Massot:

`https://www.math.u-psud.fr/~pmassot/exposition/gallerie_contact/`