COMPACTNESS FOR EMBEDDED PSEUDOHOLOMORPHIC CURVES IN 3–MANIFOLDS

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Abstract. We prove a compactness theorem for holomorphic curves in 4–dimensional symplectizations that have embedded projections to the underlying 3–manifold. It strengthens the cylindrical case of the SFT compactness theorem [BEH+03] by using intersection theory to show that degenerations of such sequences never give rise to multiple covers or nodes, so transversality is easily achieved. This has application to the theory of stable finite energy foliations introduced in [HWZ03], and also suggests a new approach to defining SFT-type invariants for contact 3–manifolds, or more generally, 3–manifolds with stable Hamiltonian structures.

Contents

1. Introduction and main results

Compactness arguments play a fundamental role in the application of pseudoholomorphic curves to problems in symplectic and contact geometry: in the closed case we have Gromov's compactness theorem, and more generally the compactness theorems of Symplectic Field Theory [BEH+03] for punctured holomorphic curves in noncompact symplectic cobordisms. As a rule, the singularities of the compactified moduli space have positive virtual codimension, which translates into algebraic invariants if transversality is achieved. In general however, even if the moduli space of smooth curves is regular, multiple covers can appear in the compactification and

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make transversality impossible without abstract perturbations, thus presenting a large technical complication.

The motivating idea of this paper is that by restricting to a certain geometrically natural class of holomorphic curves in low dimensional settings, one can use topological constraints to prevent the aforementioned analytical difficulties from arising—in fact the compactified moduli space turns out to have a miraculously nice structure. Examples of this phenomenon have been seen previously in the compactness arguments of [HWZ03] and [Wenb], both of which deal with stable finite energy foliations on contact 3–manifolds. Roughly speaking, a finite energy foliation on a contact manifold (M, ξ) is an $\mathbb{R}$–invariant collection of pseudoholomorphic curves in $\mathbb{R} \times M$ which project to a foliation of M outside some set of closed Reeb orbits. The foliation is called *stable* if it deforms smoothly under sufficiently small perturbations of the data on M; in particular, this requires that every leaf be parametrized by an embedded holomorphic curve of index 1 or 2. As is shown in [Wena], the class of holomorphic curves we consider here consists (in the positive index case) of precisely those curves which can be used to form finite energy foliations.

To illustrate the need for a compactness theorem, consider for a moment the following question: can a stable finite energy foliation be deformed smoothly under *generic homotopies* of the contact form or complex structure? Figure 1 shows that the answer in general is no. Here we see a homotopy of the contact form which moves two of the Reeb orbits that bound leaves of the foliation, and the families of leaves deform smoothly up until the isolated parameter value $\tau = 1/2$. At this value a non-generic index 0 leaf appears, producing a discontinuous change in the structure of the foliation. The remarkable fact is that, at least in this example, the foliation *survives* this discontinuous change: the leaves of the unstable foliation at parameter $\tau = 1/2$ can be glued to produce a stable foliation for $\tau = 1/2 + \epsilon$. To prove that this is what should happen in general, one needs two fundamental ingredients:

- *Compactness*: to show that the set of parameter values for which a foliation exists is closed
- *Fredholm/gluing theory*: to show that that set is also open

The second ingredient only works if the linearized Cauchy-Riemann operator achieves transversality: taking the homotopy to be sufficiently generic guarantees this, but only for *somewhere injective* holomorphic curves. In this regard, the standard compactness theory falls short, as it may in general allow all manner of multiply covered curves to appear. The result of this paper is to strengthen the standard compactness theory accordingly for the relevant class of holomorphic curves; this is a necessary step toward carrying out the homotopy argument described above.

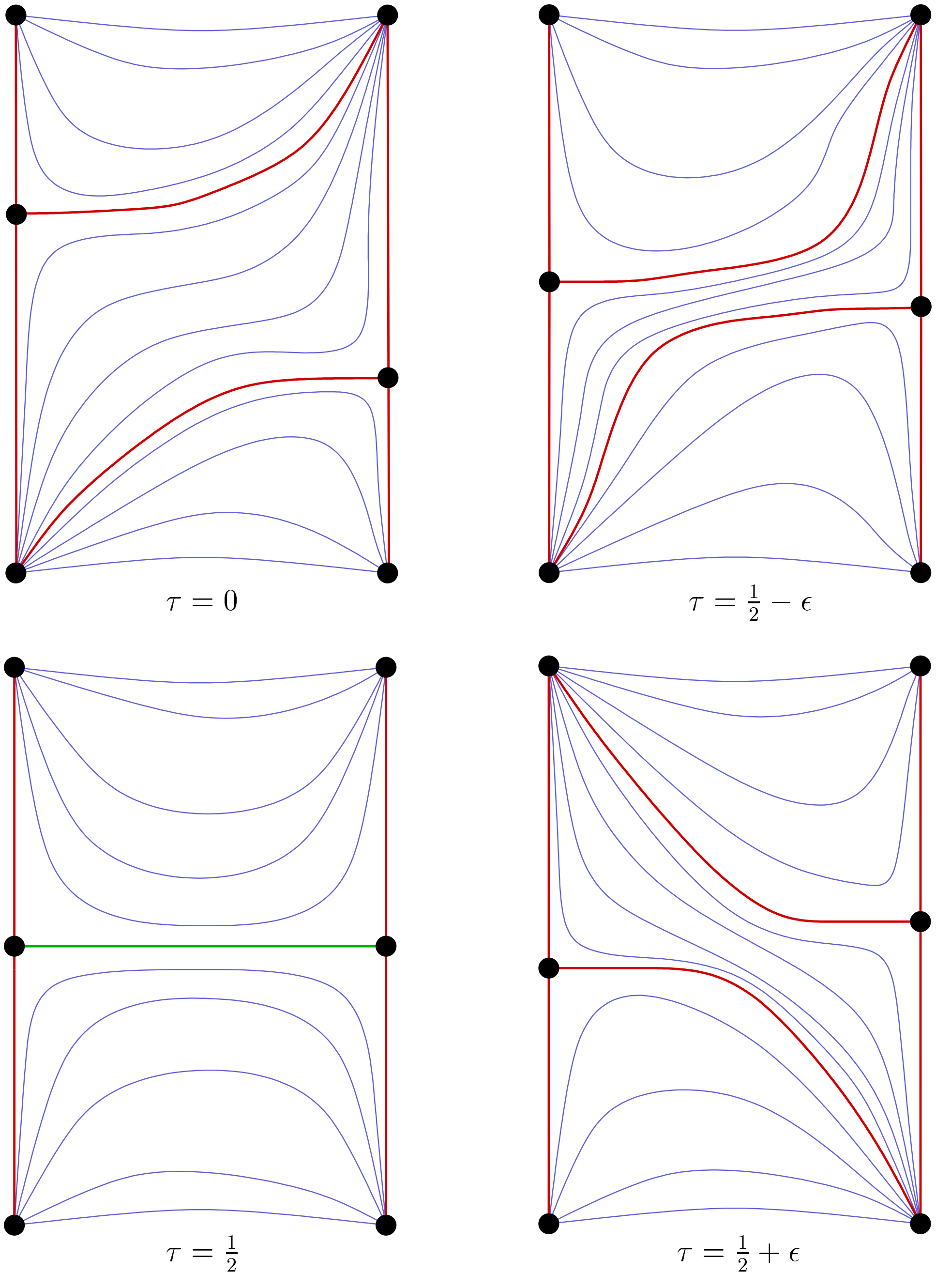

FIGURE 1. Four steps in the deformation of a stable finite energy foliation under a generic homotopy of contact forms $\{\lambda_\tau\}_{\tau\in[0,1]}$. Each picture is a cross section, consisting mainly of Reeb orbits that point through the page, index 1 holomorphic curves (which appear isolated) and index 2 holomorphic curves (which appear in 1–parameter families). The $\tau = 1/2$ picture also contains a non-generic index 0 holomorphic curve.

Along similar lines, M. Hutchings [Hut02] has proved a strong version of SFT-type compactness for a class of embedded index 1 and 2 curves in 4–dimensional symplectizations, a result which forms the analytical basis of Periodic Floer Homology and Embedded Contact Homology. The result proved here is different in several respects. The condition on our set of curves is seemingly stricter than that of Hutchings (though technically, neither implies the other), and the result is correspondingly stronger: where Hutchings' limits allow certain types of multiple covers (over trivial cylinders), ours do not. In a different sense, the setup for our main result is more general because it uses no genericity assumptions and is valid for arbitrary (also negative) Fredholm indices. The restriction on multiple covers in the limit arises from topological considerations, independent of analysis; in particular we make crucial use of the recently developed intersection theory for punctured holomorphic curves, due to R. Siefring [Sieb].

Hutchings' results suggest another possible application for our compactness theory: it may be possible to define specifically low-dimensional symplectic or contact invariants (as in Gromov-Witten or Symplectic Field Theory [EGH00]) by counting this restricted class of holomorphic curves. If such a theory exists, it has an immediate technical advantage over general SFT, in that it seemingly can be defined without any need for restrictive topological assumptions (e.g. semipositivity) or abstract perturbations.

The present work is part of a larger program involving compactness for a special class of embedded holomorphic curves in 4–dimensional symplectic cobordisms. We focus here on the special case where the target space is the $\mathbb{R}$–invariant symplectization of a 3–manifold M. The relevant class of holomorphic curves is then distinguished by the property of being not only embedded in $\mathbb{R} \times M$ but also having embedded projections to M. We'll give the required definitions and state simple versions of the main theorems in §1.1; these are implied by some slightly more technical results which we state and prove in §7, after developing the necessary machinery. We will also give some more details in §1.2 on the general program into which this work fits, and state some partial results for nontrivial symplectic cobordisms.

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1.1. **Setup and main results.** The following structure was introduced in [BEH$^+$03] as a general setting in which one has compactness results for punctured holomorphic curves. Let M be a closed, oriented 3–manifold. We define a *stable Hamiltonian structure* on M to be a tuple $\mathcal{H} = (\xi, X, \omega, J)$, where[1]

- ξ is a smooth cooriented 2–plane distribution on M
- ω is a smooth closed 2–form on M which restricts to a symplectic structure on the vector bundle $\xi \to M$
- X is a smooth vector field which is transverse to ξ, satisfies $\omega(X, \cdot) \equiv 0$, and whose flow preserves ξ
- J is a smooth complex structure on the bundle $\xi \to M$, compatible with ω in the sense that $\omega(\cdot, J\cdot)$ defines a bundle metric

It follows from these definitions that the flow of X also preserves the symplectic structure defined by ω on ξ, and the special 1–form λ associated to ξ and X by the conditions

$$\lambda(X) \equiv 1, \qquad \ker \lambda \equiv \xi,$$

satisfies $d\lambda(X, \cdot) \equiv 0$.

An important example of a stable Hamiltonian structure arises when λ is a *contact form* on M: then $d\lambda$ defines a symplectic structure on the contact structure $\xi := \ker \lambda$, so if X_λ is the corresponding Reeb vector field and J is any complex structure on ξ compatible with $d\lambda$, we obtain a stable Hamiltonian structure in the form $(\xi, X_\lambda, d\lambda, J)$. A few non-contact examples may be found in [BEH$^+$03], some of which have also appeared in applications, e.g. in [EKP06] and [Wenb].

We shall denote *periodic orbits* of X by $\gamma = (x, T)$, where $T > 0$ and $x : \mathbb{R} \to M$ satisfies $\dot{x} = X(x)$ and $x(T) = x(0)$. If $x, x' : \mathbb{R} \to M$ differ only by $x(t) = x'(t + c)$ for some $c \in \mathbb{R}$, we regard these as the same orbit $\gamma = (x, T) = (x', T)$. We say that γ has *covering number* $k \in \mathbb{N}$ if $T = k\tau$, where $\tau > 0$ is the *minimal period*, i.e. the smallest number $\tau > 0$ such that $x(\tau) = x(0)$. An orbit with covering number 1 is called *simply covered.* The *k–fold cover* of $\gamma = (x, T)$ will be denoted by

$$\gamma^k = (x, kT).$$

We shall occasionally abuse notation and regard γ as a subset of M; it should always be remembered that the orbit itself is specified by both this subset and the period.

The open 4–manifold $\mathbb{R} \times M$ is called the *symplectization* of M, and it has a natural $\mathbb{R}$–invariant almost complex structure $\tilde{J}$ associated to any stable Hamiltonian structure $\mathcal{H} = (\xi, X, \omega, J)$. This is defined by $\tilde{J}\partial_a = X$ and $\tilde{J}v = Jv$ for $v \in \xi$, where a denotes the coordinate on the $\mathbb{R}$–factor and ∂_a

[1]The tuple (ξ, X, ω), not including J, is equivalent to a *framed stable Hamiltonian structure* in the definition given by [Siea]. A similar definition appears in [EKP06] with the additional requirement that ω be exact. The inclusion of J in the data is not so natural geometrically, but convenient for our purposes.

is the unit vector in the $\mathbb{R}$–direction. We then consider pseudoholomorphic (or $\tilde{J}$*–holomorphic*) curves

$$\tilde{u} = (a, u) : (\dot{\Sigma}, j) \to (\mathbb{R} \times M, \tilde{J}),$$

where $\dot{\Sigma} = \Sigma \setminus \Gamma$, (Σ, j) is a closed Riemann surface, $\Gamma \subset \Sigma$ is a finite set of punctures, and by definition $\tilde{u}$ satisfies the nonlinear Cauchy-Riemann equation $T\tilde{u} \circ j = \tilde{J} \circ T\tilde{u}$. It is convenient to think of $(\dot{\Sigma}, j)$ as a Riemann surface with cylindrical ends, and we will sometimes refer to neighborhoods of the punctures as *ends* of $\dot{\Sigma}$.

The *energy* of a punctured pseudoholomorphic curve $\tilde{u} = (a, u) : (\dot{\Sigma}, j) \to (\mathbb{R} \times M, \tilde{J})$ is defined by

$$E(\tilde{u}) = E_\omega(\tilde{u}) + E_\lambda(\tilde{u}),$$

where

$$E_\omega(\tilde{u}) = \int_{\dot{\Sigma}} u^*\omega \tag{1.1}$$

is the so-called ω*–energy*, and

$$E_\lambda(\tilde{u}) = \sup_{\varphi \in \mathcal{T}} \int_{\Sigma} \tilde{u}^*(d\varphi \wedge \lambda),$$

with $\mathcal{T} := \{\varphi \in C^\infty(\mathbb{R}, [0, 1]) \mid \varphi' \geq 0\}$. An easy computation shows that both integrands are nonnegative whenever $\tilde{u}$ is $\tilde{J}$–holomorphic, and such a curve is constant if and only if $E(\tilde{u}) = 0$. When $\tilde{u}$ is proper, connected, $\tilde{J}$–holomorphic and satisfies $E(\tilde{u}) < \infty$, we call it a *finite energy surface.* As shown in [Hof, HWZ96], finite energy surfaces have *asymptotically cylindrical* behavior at the punctures: this means the map $\tilde{u} : \dot{\Sigma} \to \mathbb{R} \times M$ approaches $\{\pm\infty\} \times \gamma_z$ at each puncture $z \in \Gamma$, where γ_z is a (perhaps multiply covered) periodic orbit of X. (See Prop. 3.1 for a precise statement.) The sign in this expression partitions Γ into positive and negative punctures $\Gamma = \Gamma^+ \cup \Gamma^-$.

Definition 1.1. The *trivial cylinder* over a periodic orbit $\gamma = (x, T)$ of X is the finite energy surface with one positive and one negative puncture given by

$$\tilde{u} : \mathbb{R} \times S^1 \to \mathbb{R} \times M : (s, t) \mapsto (Ts, x(Tt)).$$

Let φ_X^t denote the time–t flow of X, and recall that a periodic orbit $\gamma = (x, T)$ of X is called *nondegenerate* if the linearized time–T return map $d\varphi_X^T(x(0))|_{\xi_{x(0)}}$ does not have 1 in its spectrum. Choosing a unitary trivialization Φ of ξ along x, one can associate to any nondegenerate orbit γ its *Conley-Zehnder index* $\mu_{\mathrm{CZ}}^\Phi(\gamma) \in \mathbb{Z}$. The odd/even parity of $\mu_{\mathrm{CZ}}^\Phi(\gamma)$ is independent of the choice Φ, and we call the orbit *odd* or *even* accordingly. Dynamically, even orbits are always hyperbolic, elliptic orbits are always odd, and there can also exist odd hyperbolic orbits, whose double covers are then even. The following piece of terminology is borrowed from Symplectic

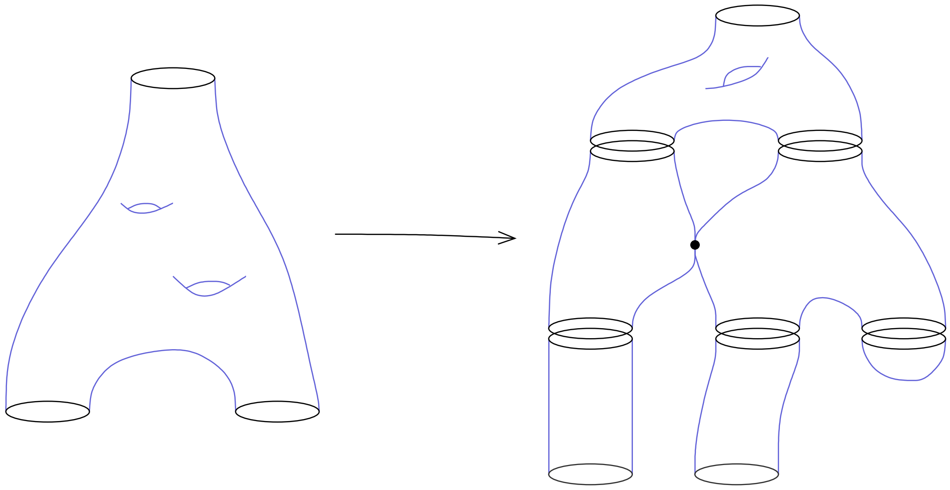

FIGURE 2. A sequence of finite energy surfaces of genus 2 converging to a stable holomorphic building with three levels and arithmetic genus 2. The middle level has a node.

Field Theory [EGH00], where the orbits in question are precisely those which must be excluded in order to define coherent orientations.

Definition 1.2. A *bad* orbit of X is an even periodic orbit which is a double cover of an odd hyperbolic orbit.

A stable Hamiltonian structure $\mathcal{H} = (\xi, X, \omega, J)$ will be called *nondegenerate* if all periodic orbits of X are nondegenerate, and we will say that a sequence $\mathcal{H}_k = (\xi_k, X_k, \omega_k, J_k)$ *converges* to $\mathcal{H} = (\xi, X, \omega, J)$ if each piece of the data converges in the C^∞–topology on M. We shall be concerned mainly with the following special class of holomorphic curves.

Definition 1.3. A finite energy surface $\tilde{u} = (a, u) : \dot{\Sigma} \to \mathbb{R} \times M$ will be called *nicely embedded* if the map $u : \dot{\Sigma} \to M$ is embedded.

By a compactness result in [BEH$^+$03], sequences of finite energy surfaces with uniformly bounded genus and energy have subsequences convergent to stable holomorphic *buildings* (see Figure 2). We will give precise definitions in §2; for now, let us simply recall that a holomorphic building $\tilde{u}$ consists of finitely many *levels*, each of which is a (possibly disconnected) nodal $\tilde{J}$–holomorphic curve with finite energy, and neighboring levels can be attached to each other along matching *breaking orbits.* Every holomorphic building $\tilde{u}$ defines a graph $G_{\tilde{u}}$ whose vertices correspond to the smooth connected components of $\tilde{u}$, with edges representing each node and breaking orbit. We say that the building $\tilde{u}$ is connected if the graph $G_{\tilde{u}}$ is connected.

Definition 1.4. For a holomorphic building $\tilde{u}$, a breaking orbit will be called *trivial* if deletion of the corresponding edge from $G_{\tilde{u}}$ divides the graph into two components, one of which only has vertices corresponding to trivial cylinders. Breaking orbits that do not have this property will be called *nontrivial.*

Definition 1.5. We say that a holomorphic building is *nicely embedded* if

(1) It has no nodes.
(2) Each connected component is either a trivial cylinder or is nicely embedded.
(3) If $\tilde{v}_1 = (b_1, v_1)$ and $\tilde{v}_2 = (b_2, v_2)$ are any two distinct connected components, then the maps v_1 and v_2 either are identical or have no intersections.
(4) Every nontrivial breaking orbit is even, and either simply covered or both doubly covered and bad.

Theorem 1. *Assume $\mathcal{H}_k = (\xi_k, X_k, \omega_k, J_k)$ is a sequence of stable Hamiltonian structures converging to a nondegenerate stable Hamiltonian structure $\mathcal{H} = (\xi, X, \omega, J)$, and $\tilde{u}_k = (a_k, u_k) : \dot{\Sigma} \to \mathbb{R} \times M$ are $\tilde{J}_k$-holomorphic finite energy surfaces which are nicely embedded and converge in the sense of* [BEH+03] *to a stable $\tilde{J}$-holomorphic building $\tilde{u}$. Then $\tilde{u}$ is nicely embedded.*

Note that this statement assumes nothing about the index of the curves $\tilde{u}_k$. We will therefore obtain a stronger statement by restricting attention to generic data and curves of positive index. Suppose $\tilde{u} : \dot{\Sigma} \to \mathbb{R} \times M$ is a finite energy surface with nondegenerate asymptotic orbits γ_z at the punctures $z \in \Gamma$, and Φ denotes a choice of unitary trivialization for ξ along each γ_z. Then the Conley-Zehnder index of $\tilde{u}$ with respect to Φ is defined to be the sum

$$\mu^{\Phi}(\tilde{u}) = \sum_{z \in \Gamma^+} \mu^{\Phi}_{\mathrm{CZ}}(\gamma_z) - \sum_{z \in \Gamma^-} \mu^{\Phi}_{\mathrm{CZ}}(\gamma_z). \tag{1.2}$$

Note that the parities of the orbits γ_z partition Γ into sets of *even* and *odd* punctures, which we denote by

$$\Gamma = \Gamma_0 \cup \Gamma_1.$$

The *Fredholm index* of $\tilde{u}$ is

$$\mathrm{ind}(\tilde{u}) = -\chi(\dot{\Sigma}) + 2c_1^{\Phi}(u^*\xi) + \mu^{\Phi}_{\mathrm{CZ}}(\tilde{u}), \tag{1.3}$$

where $c_1^{\Phi}(u^*\xi)$ denotes the relative first Chern number of the bundle $u^*\xi \to \dot{\Sigma}$ with respect to Φ, defined by counting zeros of a generic section that is constant with respect to Φ near the ends. As shown in [HWZ99], $\mathrm{ind}(\tilde{u})$ is indeed the index of the linearized Cauchy-Riemann operator, and gives the virtual dimension of the moduli space of finite energy surfaces in a neighborhood of $\tilde{u}$.

For the following definition, it is useful to observe from (1.3) that $\mathrm{ind}(\tilde{u}) + \Gamma_0$ is always even.

Definition 1.6. The *normal first Chern number* $c_N(\tilde{u}) \in \mathbb{Z}$ of a finite energy surface $\tilde{u}$ of genus g is defined by the relation

$$2c_N(\tilde{u}) = \mathrm{ind}(\tilde{u}) - 2 + 2g + \#\Gamma_0.$$

The meaning of $c_N(\tilde{u})$ is most easily seen by considering an immersed, closed curve $\tilde{u} = (a, u) : \Sigma \to \mathbb{R} \times M$: then $2c_N(\tilde{u}) = -\chi(\Sigma) + 2c_1(u^*\xi) - 2 + 2g = 2c_1(u^*T(\mathbb{R} \times M)) - 2\chi(\Sigma) = 2c_1(N_{\tilde{u}})$, where $N_{\tilde{u}} \to \Sigma$ is the normal bundle. More generally, for immersed curves with punctures, $c_N(\tilde{u})$ should be interpreted as the relative first Chern number of $N_{\tilde{u}} \to \dot{\Sigma}$ with respect to special trivializations at the asymptotic orbits; this notion will be made precise in §6. The "nicely embedded" condition is relevant to the normal first Chern number for the following reason: u is injective if and only if $\tilde{u}$ is embedded and there is never any intersection between $\tilde{u}$ and its $\mathbb{R}$–translations $\tilde{u}^c := (a + c, u)$ for $c \in \mathbb{R}$. In this case, $\tilde{u}$ belongs to a 1–parameter family of non-intersecting finite energy surfaces, which can be described via zero free sections of $N_{\tilde{u}}$. This implies morally that $c_N(\tilde{u}) = 0$, a statement which becomes literally true after applying the appropriate asymptotic constraints (see §5). In fact, one can show that for generic J (or generic parametrized families J_τ), the condition $c_N(\tilde{u}) = 0$ is necessary so that $\tilde{u}$ and all other finite energy surfaces nearby have embedded projections into M. A more detailed discussion of this may be found in [Wena]. Note also that a linearized version of positivity of intersections (see the discussion of $\mathrm{wind}_\pi(\tilde{u})$ in §4) implies $c_N(\tilde{u}) \geq 0$ for any nicely embedded curve—thus $c_N(\tilde{u}) = 0$ is a minimality condition.

According to Definition 1.6, the condition $c_N(\tilde{u}) = 0$ allows exactly two cases where $\tilde{u}$ can have positive index. We will say that a nicely embedded curve $\tilde{u}$ is *stable* if either

- $\tilde{u}$ has index 2, genus 0 and no even punctures, or
- $\tilde{u}$ has index 1, genus 0 and exactly one even puncture.

Theorem 2. *In addition to the assumptions of Theorem 1, suppose the choice of J in $\mathcal{H}$ is generic and the curves $\tilde{u}_k$ are stable. Then*

1. *If $\mathrm{ind}(\tilde{u}) = 1$, $\tilde{u}$ is a stable nicely embedded finite energy surface, hence the moduli space of such curves up to $\mathbb{R}$–translation is compact.*
2. *If $\mathrm{ind}(\tilde{u}) = 2$, then either $\tilde{u}$ is a nicely embedded finite energy surface, or it is a building with exactly two nicely embedded connected components, both stable with index 1, with projections that do not intersect each other in M, and connected to each other along a unique nontrivial breaking orbit.*

Figure 3 shows a possible limit of stable index 2 curves. Stranger things can happen if the genericity assumption is weakened: for example if J is not generic but belongs to a generic 1–parameter family $\{J_\tau\}_{\tau \in \mathbb{R}}$, then $\tilde{u}$ can contain index 0 components (arbitrarily many, in principle) with

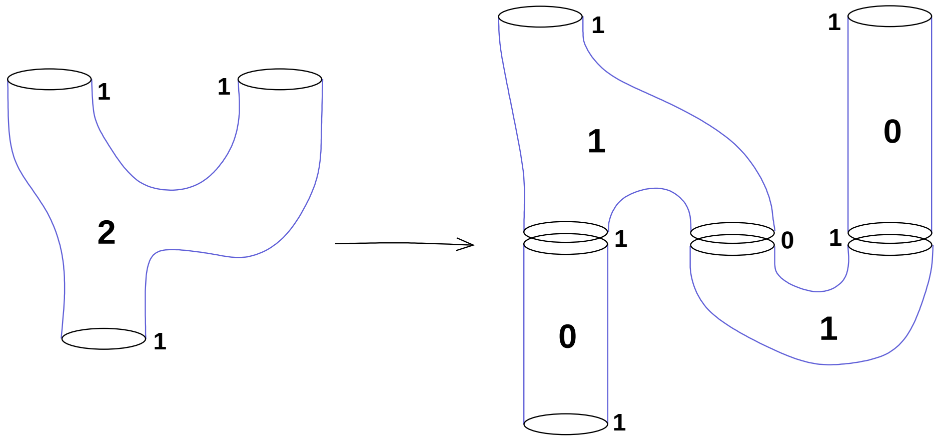

FIGURE 3. A sequence of stable nicely embedded finite energy surfaces degenerating in accordance with Theorem 2. The numbers indicate the Fredholm indices of the components and parities of the orbits. Note that each of the index 0 curves in the limit is a trivial cylinder, and the odd breaking orbits are thus trivial. The limit has exactly two nontrivial components (of index 1) and one nontrivial breaking orbit (even); the latter is also the unique even orbit for each of the index 1 components.

$\#\Gamma_0 = 2$, and there may be distinct nicely embedded components with identical images (Figure 4).

1.2. **Discussion.** The class of *stable* nicely embedded finite energy surfaces that we've defined above is of great interest in the theory of *stable finite energy foliations* introduced by Hofer, Wysocki and Zehnder in [HWZ03]. As shown in [Wena], these are precisely the curves whose moduli spaces form local foliations in both $\mathbb{R} \times M$ and M. Thus Theorem 1, when combined with some intersection theory and standard gluing analysis, can be seen as a tool for proving stability of holomorphic foliations under $\mathbb{R}$–invariant homotopies.

Likewise, Theorem 2 guarantees a particularly nice structure for the moduli space of leaves in a fixed foliation. This provides the first half of the proof of an informal conjecture suggested in [Wen05], that to every stable finite energy foliation $\mathcal{F}$ one can associate various SFT-type algebraic structures, in particular a Contact Homology algebra $\mathrm{HC}_*(\mathcal{F})$. In fact the result suggests more than this, since it does not assume the existence of any foliation: one might hope to encode this compactification of the space of nicely embedded index 2 curves algebraically as in SFT, thus defining new invariants that count nicely embedded index 1 curves. In this case the transversality problem is already solved for generic J, so one would not

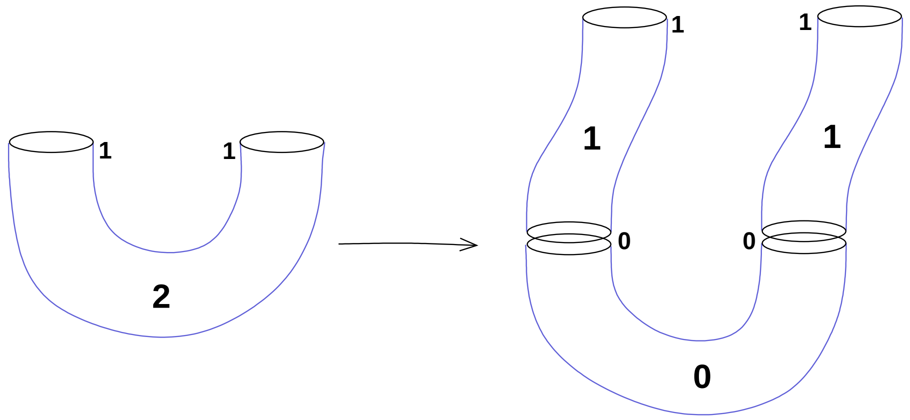

FIGURE 4. Convergence of stable nicely embedded index 2 curves in the absence of genericity: the limit can now have more than two nontrivial components, because nontrivial index 0 curves may appear. If the two odd orbits on the left are identical, it can also occur that the two components of the top level in the limit are identical curves, an outcome that is forbidden in the generic case.

need any abstract perturbations or restrictive assumptions on the target space.

To carry these ideas further, one needs a corresponding compactness theorem for punctured holomorphic curves $u : \dot{\Sigma} \to W$ in nontrivial symplectic cobordisms (W, ω) with compatible J. In this case the "nicely embedded" condition makes no sense, but one can formulate an appropriate generalization using the intersection theory of punctured holomorphic curves defined in [Sieb] (a conceptual summary without proofs may also be found in [Wenc]). In this theory, the standard adjunction formula for closed holomorphic curves has a generalization of the form

$$i(u, u) = 2\delta(u) + c_N(u) + \mathrm{cov}_\infty(u).$$

Here $i(u, u)$ and $\delta(u)$ are generalizations of the homological self intersection number and singularity number respectively: they are homotopy invariant integers that count intersections and singularities in addition to some nonnegative "asymptotic terms" (which vanish under generic perturbations). The normal first Chern number $c_N(u)$ is again the integer given by Definition 1.6, and $\mathrm{cov}_\infty(u)$ is a nonnegative integer that depends only on the asymptotic orbits of u: it is zero if and only if all the relevant *extremal eigenfunctions* are simply covered (cf. §3). Generically, one can now characterize moduli spaces of nicely embedded curves in symplectizations by the condition $i(\tilde{u}, \tilde{u}) = 0$, and this is a sensible condition to apply to certain curves in symplectic cobordisms as well. Of particular interest is the space of somewhere injective index 2 curves $u : \dot{\Sigma} \to W$ with $i(u, u) = 0$:

these are automatically embedded and satisfy $c_N(u) = \mathrm{cov}_\infty(u) = 0$. By a result in [Wena] in fact, such a curve is always regular and comes in a smooth 2–dimensional family of nonintersecting curves, which foliate a neighborhood of $u(\dot{\Sigma})$ in W.

We can now state two partial results that we conjecture to be special cases of a more general theorem. Assume for both that (W, J) is an asymptotically cylindrical almost complex manifold as in [BEH+03].

Theorem. *Suppose W is closed and $u_k : \Sigma \to W$ is a sequence of closed, somewhere injective J–holomorphic curves with $\mathrm{ind}(u_k) = 2$ and $i(u_k, u_k) = 0$, converging to a nodal curve u. Then u is either a smooth embedded curve or a nodal curve consisting of two embedded index 0 components that intersect each other once transversely. These fit together with all smooth curves close to u as a singular foliation of some neighborhood of the image of u, with the nodal point as an isolated singularity.*

Remark. We will not prove this here, but hope to include it in a future paper as a special case of a much harder theorem for symplectic cobordisms. The closed case is comparatively simple and requires no substantially new technology, only the adjunction formula and some covering relations for $i(u, u)$ and $c_N(u)$. It can also easily be generalized to apply to any 2–dimensional moduli space of curves that are embedded outside a set of marked points $z_1, \ldots, z_N$ satisfying fixed point constraints $u(z_j) = p_j \in W$; one must then assume that $i(u, u)$ has the smallest value allowed by the constraints. The local structure of such moduli spaces is studied in [Wena], showing that locally they form singular foliations in W. In this way one can also accomodate immersed curves if the images of the self intersections are fixed.

Theorem ([Wenc])**.** *Suppose J is generic and $u_k : \dot{\Sigma} \to W$ is a sequence of embedded, punctured finite energy J–holomorphic curves with $\mathrm{ind}(u_k) = 2$ and $i(u_k, u_k) = 0$, and they converge to a smooth multiple cover $u = v \circ \varphi$. Then v is an embedded index 0 curve with $i(v, v) = -1$ and u is immersed. Moreover, the moduli space of curves close to u is a smooth orbifold, all other curves close to u are embedded, and they fit together with v as a foliation on some neighborhood of the image of v.*

In both cases, as with Theorems 1 and 2, the upshot is that the degeneration in the limit is nice enough so that transversality can still be achieved—this is true even in the second case, despite the appearance of a multiple cover in the limit. (The latter can happen only in symplectic cobordisms that are both noncompact and nontrivial). The reason one obtains smoothness in this case has to do with the transversality results of Hofer, Lizan and Sikorav [HLS97], which are generalized in [Wenc]: specifically in dimension 4, one can sometimes use topological constraints to prove transversality for all J (not just generic choices). This does not depend on u being somewhere injective, though it is important that u is *immersed*, and in fact the proof of the latter fact is also based partly on such transversality arguments; see [Wenc] for details.

2. Holomorphic buildings in symplectizations

In this and the next few sections, we assemble some definitions and known results on punctured holomorphic curves and holomorphic buildings, fixing terminology and notation that will be used throughout.

Let $\mathcal{D}$ denote the open unit disk in $\mathbb{C}$, and write $\dot{\mathcal{D}} = \mathcal{D} \setminus \{0\}$. We define the *circle compactification* of $\dot{\mathcal{D}}$ as follows. Using the biholomorphic map

$$\varphi : (0, \infty) \times S^1 \to \mathcal{D} \setminus \{0\} : (s, t) \mapsto e^{-2\pi(s+it)}$$

to identify $\dot{\mathcal{D}}$ with the half-cylinder, define $\overline{\mathcal{D}} := \dot{\mathcal{D}} \cup (\{\infty\} \times S^1) \cong (0, \infty] \times S^1$. This is a topological surface with boundary, and has natural smooth structures over the interior $\operatorname{int} \overline{\mathcal{D}} = \dot{\mathcal{D}}$ as well as the boundary $\partial\overline{\mathcal{D}} = \delta_0 := \{\infty\} \times S^1$.

We use this to define a circle compactification $\overline{\Sigma}$ for $\dot{\Sigma} = \Sigma \setminus \Gamma$, where (Σ, j) is any Riemann surface with isolated punctures $\Gamma \subset \Sigma$. For each $z \in \Gamma$, choose coordinates to identify a neighborhood of z biholomorphically with $\mathcal{D}$, identify the punctured neighborhood as above with a half-cylinder and then add a *circle at infinity* $\delta_z \cong \{\infty\} \times S^1$ by replacing the half-cylinder with $(0, \infty] \times S^1$. The result is an oriented topological surface with boundary,

$$\overline{\Sigma} = \dot{\Sigma} \cup \left(\bigcup_{z \in \Gamma} \delta_z \right),$$

where the subsets $\operatorname{int} \overline{\Sigma} = \dot{\Sigma}$ and $\partial \overline{\Sigma} = \bigcup_{z \in \Gamma} \delta_z$ inherit natural smooth structures that are independent of the choices of holomorphic coordinates. The interior also has a conformal structure, and the complex structure on $T_z\Sigma$ for $z \in \Gamma$ defines a special class of diffeomorphisms $\varphi : S^1 \to \delta_z$, which are all related to each other by a constant shift, i.e. $\varphi_1(t) = \varphi_2(t + \text{const})$. For any two punctures $z_1, z_2 \in \Gamma$, an orientation reversing diffeomorphism $\psi : \delta_{z_1} \to \delta_{z_2}$ will be called *orthogonal* if it can be written as $\psi(t) = -t$ with respect to some choice of special diffeomorphisms $\delta_{z_i} \cong S^1$. Observe that $\overline{\Sigma}$ is compact if Σ is closed.

A closed *nodal Riemann surface* with marked points consists of the data

$$\mathbf{S} = (S, j, \Gamma, \Delta),$$

where (S, j) is a closed (but not necessarily connected) Riemann surface, and $\Gamma, \Delta \subset S$ are disjoint finite subsets with the following additional structure:

- Γ is ordered,
- elements of Δ are grouped into pairs $\overline{z}_1, \underline{z}_1, \ldots, \overline{z}_n, \underline{z}_n$.

We call Δ the *double points* of $\mathbf{S}$, and Γ the *marked points*. Let $\dot{S} = S \setminus (\Gamma \cup \Delta)$, with circle compactification $\overline{S}$. For a pair $\{\overline{z}, \underline{z}\} \subset \Delta$, a *decoration* at $\{\overline{z}, \underline{z}\}$ is an orientation reversing orthogonal diffeomorphism $\psi : \delta_{\overline{z}} \to \delta_{\underline{z}}$, and a decoration ψ of $\mathbf{S}$ is a choice of decorations at all pairs $\{\overline{z}, \underline{z}\} \subset \Delta$; we can regard this as a diffeomorphism on a certain subset of $\partial\overline{S}$. We call $\mathbf{S} := (S, j, \Gamma, \Delta, \psi)$ a *decorated nodal Riemann surface.*

Given $\mathbf{S}$ with decoration ψ, define

$$\overline{\mathbf{S}} = \overline{S}/\{z \sim \psi(z)\}.$$

This is an oriented topological surface with boundary, with a conformal structure that degenerates at $\partial\overline{\mathbf{S}} = \bigcup_{z\in\Gamma} \delta_z$ and also at a certain set of disjoint circles $\Theta_\Delta \subset \overline{\mathbf{S}}$, one for each double point pair $\{\overline{z}, \underline{z}\} \subset \Delta$. There is a natural inclusion of $\dot{S}$ into $\overline{\mathbf{S}}$ as the subset

$$\dot{S} = \operatorname{int} \overline{\mathbf{S}} \setminus \Theta_\Delta.$$

We say that the nodal surface $\mathbf{S}$ is *connected* if $\overline{\mathbf{S}}$ is connected, and define its *arithmetic genus* to be the genus of $\overline{\mathbf{S}}$. Neither of these definitions depends on the choice of decoration.

Let M be a closed 3–manifold with stable Hamiltonian structure $\mathcal{H} = (\xi, X, \omega, J)$ and associated almost complex structure $\tilde{J}$. If $\mathbf{S} = (S, j, \Gamma, \Delta)$ is a closed nodal Riemann surface with marked points, a *nodal $\tilde{J}$–holomorphic curve*

$$\tilde{u} : \mathbf{S} \to \mathbb{R} \times M$$

is a proper finite energy pseudoholomorphic map $\tilde{u} = (a, u) : (S \setminus \Gamma, j) \to (\mathbb{R} \times M, \tilde{J})$ such that for each pair $\{\overline{z}, \underline{z}\} \subset \Delta$, $\tilde{u}(\overline{z}) = \tilde{u}(\underline{z})$. In this context each pair $\{\overline{z}, \underline{z}\} \subset \Delta$ is called a *nodal pair*, or simply a *node* of $\tilde{u}$. The marked points Γ are called *punctures* of $\tilde{u}$, and the asymptotic behavior of $a : S \setminus \Gamma \to \mathbb{R}$ determines the *sign* of each, defining a partition $\Gamma = \Gamma^+ \cup \Gamma^-$. Observe that for any decoration ψ of $\mathbf{S}$, $\tilde{u} : S \setminus \Gamma \to M$ has a natural continuous extension

$$(\bar{a}, \bar{u}) : \overline{\mathbf{S}} \to [-\infty, \infty] \times M,$$

which is constant on each connected component of Θ_Δ and maps $\partial\overline{\mathbf{S}}$ to $\{\pm\infty\} \times M$; in particular the restriction of $\bar{u}$ to each $\delta_z \subset \partial\overline{\mathbf{S}}$ for $z \in \Gamma^\pm$ defines a positively/negatively oriented parametrization of a periodic orbit γ_z of X.

Consider next a collection of nodal $\tilde{J}$–holomorphic curves

$$\tilde{u}_m = (a_m, u_m) : \mathbf{S}_m = (S_m, j_m, \Gamma_m, \Delta_m) \to \mathbb{R} \times M$$

for $m = 1, \ldots, n$. Denote $\partial_\pm\overline{\mathbf{S}}_m = \bigcup_{z\in\Gamma_m^\pm} \delta_z$, and suppose there are orientation reversing orthogonal diffeomorphisms

$$\varphi_m : \partial_+\overline{\mathbf{S}}_m \to \partial_-\overline{\mathbf{S}}_{m+1}$$

for each $m = 1, \ldots, n-1$. Then the collection

$$\tilde{u} = (\tilde{u}_1, \ldots, \tilde{u}_n; \varphi_1, \ldots, \varphi_{n-1})$$

is called a *$\tilde{J}$–holomorphic building of height n* if for each $m = 1, \ldots, n-1$,

$$\bar{u}_m|_{\partial_+ \overline{\mathbf{S}}_m} = \bar{u}_{m+1} \circ \varphi_m.$$

The nodal curves $\tilde{u}_m$ are called *levels* of $\tilde{u}$. For each $m = 1, \ldots, n-1$ and $\underline{z} \in \Gamma_m^+$, there is a unique $\overline{z} \in \Gamma_{m+1}^-$ such that $\varphi_m(\delta_{\underline{z}}) = \delta_{\overline{z}}$. We then call the pair $\{\overline{z}, \underline{z}\}$ a *breaking pair*, and denote by $\gamma_{(\overline{z},\underline{z})}$ the *breaking orbit* parametrized by $\bar{u}_m|_{\delta_{\underline{z}}}$ and $\bar{u}_{m+1}|_{\delta_{\overline{z}}}$. Let Δ_{C} denote the set of all punctures in $\Gamma_1 \cup \ldots \cup \Gamma_n$ that belong to breaking pairs.

Define the *partially decorated* nodal Riemann surface $\mathbf{S} = (S, j, \Gamma, \Delta, \varphi)$, where (S, j) is the disjoint union of $(S_1, j_1), \ldots, (S_n, j_n)$, $\Gamma = \Gamma^+ \cup \Gamma^- := \Gamma_n^+ \cup \Gamma_1^-$, Δ is the union of the breaking pairs in Δ_{C} with the nodal pairs in $\Delta_{\mathrm{N}} := \Delta_1 \cup \ldots \cup \Delta_n$, and φ is the collection of decorations at the breaking pairs $\{\overline{z}, \underline{z}\}$ defined by $\varphi_m : \partial_+ \overline{\mathbf{S}}_m \to \partial_- \overline{\mathbf{S}}_{m+1}$. We will call $\mathbf{S}$ the *domain* of $\tilde{u}$, and indicate this via the shorthand notation

$$\tilde{u} : \mathbf{S} \to \mathbb{R} \times M.$$

Choosing arbitrary decorations ψ_m for each $\mathbf{S}_m$, these together with φ define a decoration ψ for $\mathbf{S}$, and $\overline{\mathbf{S}}$ is now the surface obtained from $\overline{\mathbf{S}}_1 \cup \ldots \cup \overline{\mathbf{S}}_n$ by gluing boundaries together via φ. There is then a continuous map

$$\bar{u} : \overline{\mathbf{S}} \to M$$

such that $\bar{u}|_{\overline{\mathbf{S}}_m} = \bar{u}_m$. The orbits γ_z parametrized by $\bar{u}|_{\delta_z}$ for $z \in \Gamma^\pm$ are called *asymptotic orbits* of $\tilde{u}$.

Remark 2.1. Technically, what we've defined should be called holomorphic buildings *with zero marked points*, since all the marked points of $\mathbf{S}$ are being viewed as punctures of $\tilde{u}$. One can also define holomorphic buildings with marked points, though we will not need them here; see [BEH+03] for details.

The relationship of a building $\tilde{u}$ with its domain $\mathbf{S}$ gives rise to a slightly more general notion which we will find useful.

Definition 2.2. Suppose $\mathbf{S} = (S, j, \Gamma, \Delta, \psi)$ is a nodal Riemann surface with Δ partitioned into two sets $\Delta_{\mathrm{C}} \cup \Delta_{\mathrm{N}}$, each organized in pairs, called *breaking* pairs and *nodal* pairs respectively, and ψ denotes a choice of decoration at each of the breaking pairs. A *generalized $\tilde{J}$–holomorphic building* $\tilde{u} : \mathbf{S} \to \mathbb{R} \times M$ is then a proper finite energy $\tilde{J}$–holomorphic map $\tilde{u} = (a, u) : S \setminus (\Gamma \cup \Delta_{\mathrm{C}}) \to \mathbb{R} \times M$ such that

1. For each nodal pair $\{\overline{z}, \underline{z}\} \subset \Delta_{\mathrm{N}}$, $\tilde{u}(\overline{z}) = \tilde{u}(\underline{z})$.
2. Completing ψ to a decoration of $\mathbf{S}$ by choosing arbitrary decorations at the nodal pairs, $u : S \setminus (\Gamma \cup \Delta)$ extends to a continuous map $\bar{u} : \overline{\mathbf{S}} \to M$.

Considering orientations, we see that each breaking pair $\{\overline{z}, \underline{z}\} \subset \Delta_C$ includes one positive and one negative puncture of $\tilde{u}$.

Just as with nodal Riemann surfaces, we say that the generalized building $\tilde{u} : \mathbf{S} \to \mathbb{R} \times M$ is *connected* if $\overline{\mathbf{S}}$ is connected, and its *arithmetic genus* is defined as the genus of $\overline{\mathbf{S}}$. A *connected component* of $\tilde{u}$ is the finite energy surface obtained by restricting the map $\tilde{u} : S \setminus (\Gamma \cup \Delta_C) \to \mathbb{R} \times M$ to any connected component of its domain. The sets $\Gamma^{\pm} \subset S$ are the positive and negative *punctures* of $\tilde{u}$. In general, each connected component may have some punctures that do not belong to Γ but are included among the breaking pairs Δ_C: we call these *breaking punctures.*

Every holomorphic building is also a generalized holomorphic building in an obvious way. The main difference is that the components of generalized buildings cannot in general be assigned *levels*, and every component may have positive and negative punctures which are not breaking punctures; holomorphic buildings have these only at the top and bottom levels.

Definition 2.3. A generalized building $\tilde{u} : \mathbf{S} \to \mathbb{R} \times M$ is *stable* if each connected component $\dot{S}_i \subset S \setminus (\Gamma \cup \Delta)$ on which $\tilde{u}$ is constant satisfies $\chi(\dot{S}_i) < 0$.

Notation. For any generalized holomorphic building $\tilde{u}$ and puncture $z \in \Gamma$, we will always denote by

$$\gamma_z := \bar{u}(\delta_z)$$

the asymptotic orbit at z, and for breaking pairs $\{\overline{z}, \underline{z}\} \subset \Delta_C$ denote the breaking orbit $\bar{u}(\delta_{\overline{z}}) = \bar{u}(\delta_{\underline{z}})$ by

$$\gamma_{(\overline{z},\underline{z})} = \gamma_{\overline{z}} = \gamma_{\underline{z}}.$$

Unless stated otherwise, the domain $\mathbf{S}$ will be assumed to consist of the data $(S, j, \Gamma, \Delta, \varphi)$ as defined above. When multiple domains are under discussion, we'll often use $\mathbf{S}'$ to denote a second domain $(S', j', \Gamma', \Delta', \varphi')$, or similarly $\mathbf{S}^{\mathrm{t}} = (S^{\mathrm{t}}, j^{\mathrm{t}}, \Gamma^{\mathrm{t}}, \Delta^{\mathrm{t}}, \varphi^{\mathrm{t}})$ and so forth.

Definition 2.4. Given a generalized holomorphic building $\tilde{u} : \mathbf{S} \to \mathbb{R} \times M$, an *augmentation* of $\tilde{u}$ at a puncture $z \in \Gamma^{\pm}$ is the generalized building $\tilde{u}' : \mathbf{S}' \to \mathbb{R} \times M$, defined as follows:

1. (S', j') is the disjoint union of (S, j) with a sphere $(S_T, j_T) := (S^2, i)$. Denote by $p_-, p_+ \in S'$ the points 0 and ∞ respectively in S_T.
2. $\Gamma' = (\Gamma \cup \{p_{\pm}\}) \setminus \{z\}$.
3. Δ'_C is Δ_C with the addition of one extra pair $\{z, p_{\mp}\}$.
4. $\Delta'_N = \Delta_N$.
5. $\tilde{u}'|_{S \setminus (\Gamma \cup \Delta_C)} = \tilde{u}$ and $\tilde{u}'|_{S_T \setminus \{p_+, p_-\}}$ is a trivial cylinder over γ_z. A decoration is then chosen at $\Delta'_C \setminus \Delta_C$ so that $\tilde{u}'$ is a generalized holomorphic building.

An augmentation at a breaking pair $\{\overline{z}, \underline{z}\} \subset \Delta_{\mathrm{C}}$ is defined in the same manner with the following changes:

(1) $\Gamma' = \Gamma$.
(2) Δ'_{C} is Δ_{C} with two additional pairs, $\{\underline{z}, p_-\}$ and $\{p_+, \overline{z}\}$.
(3) $\tilde{u}'|_{S_T \setminus \{p_+, p_-\}}$ is a trivial cylinder over $\gamma_{(\overline{z}, \underline{z})}$.

In general, an *augmentation* of $\tilde{u}$ is any generalized building obtained from $\tilde{u}$ by a finite sequence of these two operations.

Augmentation is essentially the operation of shifting levels of $\tilde{u}$ by inserting trivial cylinders. One should think of $\tilde{u}'$ as being *homotopic* to $\tilde{u}$, generalizing the fact that a finite energy surface $\tilde{v} = (b, v)$ is homotopic to any of its $\mathbb{R}$–translations $\tilde{v}^c = (b + c, v)$ for $c \in \mathbb{R}$; an augmentation is in some sense a sequence of infinite $\mathbb{R}$–translations.

Just as one can insert trivial cylinders in a generalized building, one can also "collapse" them.

Definition 2.5. Suppose $\tilde{u} : \mathbf{S} \to \mathbb{R} \times M$ is a generalized holomorphic building such that at least one connected component is not a trivial cylinder. The *core* of $\tilde{u}$ is then the unique generalized building $\tilde{u}^{\mathrm{K}} : \mathbf{S}^{\mathrm{K}} \to \mathbb{R} \times M$ such that $\tilde{u}$ is an augmentation of $\tilde{u}^{\mathrm{K}}$ and no connected component of $\tilde{u}^{\mathrm{K}}$ is a trivial cylinder.

Definition 2.6. Given a generalized building $\tilde{u} : \mathbf{S} \to \mathbb{R} \times M$, a *subbuilding* $\tilde{u}' : \mathbf{S}' \to \mathbb{R} \times M$ of $\tilde{u}$ is a generalized building such that

(1) S' is an open and closed subset of S, on which $j' = j$.
(2) Γ' is the union of $\Gamma \cap S'$ with all $z \in S'$ for which $\{z, z'\}$ is a breaking pair in Δ_{C} with $z' \notin S'$.
(3) Δ'_{C} is the set of all breaking pairs $\{\overline{z}, \underline{z}\}$ in Δ_{C} for which both $\overline{z}$ and $\underline{z}$ are in S', and ψ' is the restriction of ψ.
(4) Δ'_{N} is the set of all nodal pairs $\{\overline{z}, \underline{z}\}$ in Δ_{N} for which both $\overline{z}$ and $\underline{z}$ are in S'.
(5) $\tilde{u}' = \tilde{u}|_{S' \setminus (\Gamma' \cup \Delta'_{\mathrm{C}})}$

We refer to [BEH+03] for the detailed definition of what it means for a sequence of finite energy surfaces to converge to a holomorphic building. We will only need to use the following fact, immediate from the definition:

Proposition 2.7. *If* $\tilde{u}_k = (a_k, u_k) : (\dot{\Sigma}_k, j_k) \to (\mathbb{R} \times M, \tilde{J}_k)$ *is a sequence of finite energy surfaces converging to a* $\tilde{J}$*–holomorphic building* $\tilde{u} : \mathbf{S} \to \mathbb{R} \times M$*, then for sufficiently large* k *there exist homeomorphisms* $\varphi_k : \overline{\mathbf{S}} \to \overline{\Sigma}_k$*, restricting to smooth maps* $\overline{\mathbf{S}} \setminus (\partial \overline{\mathbf{S}} \cup \Theta_\Delta) \to \dot{\Sigma}_k$*, such that*

(1) $u_k \circ \varphi_k \to u$ *in* $C^\infty_{\mathrm{loc}}(\overline{\mathbf{S}} \setminus (\partial \overline{\mathbf{S}} \cup \Theta_\Delta), M)$*,*
(2) $\bar{u}_k \circ \varphi_k \to \bar{u}$ *in* $C^0(\overline{\mathbf{S}}, M)$*.*

The proof of Theorem 1 will rely heavily on our ability to control the normal first Chern number for components of a holomorphic building. Positivity of intersections guarantees that this number is nonnegative for any finite energy surface that is not preserved by the $\mathbb{R}$–action. The hardest part of the proof therefore involves the so-called *trivial curves*, for which this number can be negative.

Definition 2.8. A finite energy surface $\tilde{u} : \dot{\Sigma} \to \mathbb{R} \times M$ will be called a *trivial curve* if $E_\omega(\tilde{u}) = 0$, and *nontrivial* if $E_\omega(\tilde{u}) > 0$.

Examining the integrand in (1.1), one finds that a finite energy surface $\tilde{u} = (a, u) : \dot{\Sigma} \to \mathbb{R} \times M$ is trivial if and only if the image of $du(z)$ is everywhere tangent to X, which means $u(\dot{\Sigma})$ is contained in a single periodic orbit γ. If $\tilde{u}$ is not constant, then this implies one can always write $\tilde{u} = \tilde{v} \circ \varphi$ where $\varphi : \dot{\Sigma} \to \mathbb{R} \times S^1$ is a holomorphic branched cover and $\tilde{v}$ is the trivial cylinder over γ.

Definition 2.9. A holomorphic building or generalized holomorphic building will be called *trivial* if it is connected, has no nodes, and all its connected components are trivial curves. Additionally, such a building will be called *cylindrical* if it has arithmetic genus zero and exactly two punctures.

Observe that every nonconstant trivial curve has at least one positive and one negative puncture, and the same statement therefore holds for trivial (generalized) buildings. It follows that in general, a trivial building $\tilde{u} : \mathbf{S} \to \mathbb{R} \times M$ has $\chi(\overline{\mathbf{S}}) \le 0$, with equality if and only if $\tilde{u}$ is cylindrical.

Proposition 2.10. *A nonconstant trivial building is cylindrical if and only if it is an augmentation of a trivial cylinder.*

Proof. Observe first that the statement is true for any building $\tilde{u}$ with only one level (i.e. a finite energy surface), for then $\tilde{u}$ covers a trivial cylinder $\tilde{v}$ by a holomorphic map $\varphi : \mathbb{R} \times S^1 \to \mathbb{R} \times S^1$ of degree $k \in \mathbb{N}$, and every such map is of the form $\varphi(s,t) = (ks, kt)$ up to constant shifts in s and t.

For a more general trivial building $\tilde{u} : \mathbf{S} \to \mathbb{R} \times M$ with two punctures and $\chi(\overline{\mathbf{S}}) = 0$, we only need observe that under these assumptions, no connected component of $\tilde{u}$ can have nontrivial genus or more than two punctures. □

3. Asymptotic eigenfunctions

Let $\gamma = (x, T)$ be a periodic orbit of X, and writing $S^1 := \mathbb{R}/\mathbb{Z}$, define the parametrization

$$\mathbf{x} : S^1 \to M : t \mapsto x(Tt).$$

We can then view the normal bundle to γ as the induced bundle $\mathbf{x}^*\xi \to S^1$. Choosing any symmetric connection ∇ on M, we define the *asymptotic operator*

$$\mathbf{A}_\gamma : \Gamma(\mathbf{x}^*\xi) \to \Gamma(\mathbf{x}^*\xi) : v \mapsto -J(\nabla_t v - T\nabla_v X).$$

One can check that this expression doesn't depend on the choice ∇ and gives a well defined section of $\mathbf{x}^*\xi$. Morally, $\mathbf{A}_\gamma$ is the Hessian of a certain action functional on $C^\infty(S^1, M)$, whose critical points are the closed characteristics of X. As an unbounded operator on $L^2(\mathbf{x}^*\xi)$ with domain $H^1(\mathbf{x}^*\xi)$, $\mathbf{A}_\gamma$ is self adjoint, with spectrum $\sigma(\mathbf{A}_\gamma)$ consisting of isolated real eigenvalues of multiplicity at most two. We shall sometimes refer to the eigenfunctions of $\mathbf{A}_\gamma$ as *asymptotic eigenfunctions*. Recall that the orbit γ is degenerate if and only if $0 \in \sigma(\mathbf{A}_\gamma)$.

Operators of this form are fundamental in the asymptotic analysis of punctured holomorphic curves, as demonstrated by the following result proved in [HWZ96, Mor03, Siea]. Denote

$$\mathbb{R}_+ = [0, \infty), \quad \mathbb{R}_- = (-\infty, 0], \quad Z_\pm = \mathbb{R}_\pm \times S^1,$$

and assign to $Z_\pm$ the standard complex structure $i\frac{\partial}{\partial s} = \frac{\partial}{\partial t}$ in terms of the coordinates $(s,t) \in Z_\pm$. We will use the term *asymptotically constant reparametrization* to mean a smooth embedding $\varphi : Z_\pm \to Z_\pm$ for which there are constants $s_0 \in \mathbb{R}$ and $t_0 \in S^1$ such that $\varphi(s - s_0, t - t_0) - (s,t) \to 0$ with all derivatives as $s \to \pm\infty$.

Proposition 3.1. *Suppose $\tilde{u} = (a, u) : Z_\pm \to \mathbb{R}_\pm \times M$ is a proper finite energy half-cylinder asymptotic to a nondegenerate orbit $\gamma = (x, T)$. Then there is an asymptotically constant reparametrization $\varphi : Z_\pm \to Z_\pm$ such that for $|s|$ sufficiently large, $\tilde{u} \circ \varphi : Z_\pm \to \mathbb{R} \times M$ is either $(Ts, x(Tt))$ or satisfies the following asymptotic formula: there exists an eigenfunction $e \in \Gamma(\mathbf{x}^*\xi)$ of $\mathbf{A}_\gamma$ with negative/positive eigenvalue λ such that*

$$\tilde{u} \circ \varphi(s,t) = \exp_{(Ts, x(Tt))}\left[e^{\lambda s} \cdot (e(t) + r(s,t))\right],$$

where exp *is defined with respect to any $\mathbb{R}$–invariant connection on $\mathbb{R} \times M$ and $r(s,t) \in \xi_{x(Tt)}$ satisfies $r(s,t) \to 0$ with all derivatives as $s \to \pm\infty$.*

Since nontrivial eigenfunctions $e \in \Gamma(\mathbf{x}^*\xi)$ are never zero, this implies in particular that $u(s,t)$ is either contained in γ or never intersects γ for sufficiently large $|s|$. Similar formulas are proved in [Siea] for the relative asymptotic behavior of distinct holomorphic half-cylinders approaching the same orbit; this is fundamental to the intersection theoretic results that we will review in §4.

Prop. 3.1 obviously applies to finite energy surfaces $\tilde{u} : \dot{\Sigma} \to \mathbb{R} \times M$ by identifying a punctured disk-like neighborhood of each puncture $z \in \Gamma^\pm$ biholomorphically with $Z_\pm$.

Definition 3.2. For $\tilde{u} : Z_\pm \to \mathbb{R}_\pm \times M$ as in Prop. 3.1, if $\tilde{u}$ satisfies the asymptotic formula with a nontrivial eigenfunction e and eigenvalue λ, we call $|\lambda| > 0$ the *transversal convergence rate* of $\tilde{u}$ and say that e *controls* the asymptotic approach of $\tilde{u}$ to γ. Otherwise, if $\tilde{u}$ is simply a reparametrization of $(Ts, x(Tt))$ near infinity, we define the transversal convergence rate to be $+\infty$. Similar wording will be used also for more general punctured holomorphic curves $\tilde{u} : \dot{\Sigma} \to \mathbb{R} \times M$ approaching orbits γ_z at $z \in \Gamma$.

It will be important to understand the eigenfunctions of $\mathbf{A}_\gamma$ in greater detail. For $k \in \mathbb{N}$, define a parametrization of γ^k by

$$\mathbf{x}_k : S^1 \to M : t \mapsto x(kTt).$$

Choose a unitary trivialization Φ of $\mathbf{x}^*\xi$, and use Φ also to denote the natural trivialization induced on $\mathbf{x}_k^*\xi$. Every nowhere zero section $v \in \Gamma(\mathbf{x}^*\xi)$ now has a well defined *winding number*

$$\mathrm{wind}^\Phi(v) \in \mathbb{Z}.$$

By a result in [HWZ95], the winding number of a nontrivial eigenfunction e of $\mathbf{A}_\gamma$ depends only on its eigenvalue λ, thus we can sensibly write $\mathrm{wind}^\Phi(e) = \mathrm{wind}^\Phi(\lambda)$. In fact, the result in question proves:

Proposition 3.3. $\mathrm{wind}^\Phi : \sigma(\mathbf{A}) \to \mathbb{Z}$ *is a monotone increasing function, and for each $k \in \mathbb{Z}$, there are precisely two eigenvalues $\lambda \in \sigma(\mathbf{A}_\gamma)$ counted with multiplicity such that* $\mathrm{wind}^\Phi(\lambda) = k$.

Assuming γ to be nondegenerate, we now define the integers

$$\begin{aligned} \alpha_-^\Phi(\gamma) &= \max\{\mathrm{wind}^\Phi(\lambda) \mid \lambda \in \sigma(\mathbf{A}_\gamma),\ \lambda < 0\}, \\ \alpha_+^\Phi(\gamma) &= \min\{\mathrm{wind}^\Phi(\lambda) \mid \lambda \in \sigma(\mathbf{A}_\gamma),\ \lambda > 0\}, \\ p(\gamma) &= \alpha_+^\Phi(\gamma) - \alpha_-^\Phi(\gamma), \end{aligned} \tag{3.1}$$

noting that the *parity* $p(\gamma) \in \{0,1\}$ doesn't depend on Φ. Another result in [HWZ95] then gives the following formula for the Conley-Zehnder index:

$$\mu_{\mathrm{CZ}}^\Phi(\gamma) = 2\alpha_-^\Phi(\gamma) + p(\gamma) = 2\alpha_+^\Phi(\gamma) - p(\gamma). \tag{3.2}$$

Definition 3.4. We will say that a nontrivial eigenfunction e of $\mathbf{A}_\gamma$ is a *positive/negative extremal eigenfunction* of γ if $\mathrm{wind}^\Phi(e) = \alpha_\pm^\Phi(\gamma)$.

If $e \in \Gamma(\mathbf{x}^*\xi)$ satisfies $\mathbf{A}_\gamma e = \lambda e$, we define the k–fold cover $e^k \in \Gamma(\mathbf{x}_k^*\xi)$ by $e^k(t) = e(kt)$ and find $\mathbf{A}_{\gamma^k} e^k = k\lambda e^k$. In general, an eigenfunction f of $\mathbf{A}_\gamma$ is called a *k–fold cover* if there is an orbit ζ and eigenfunction e of $\mathbf{A}_\zeta$ such that $\zeta^k = \gamma$ and $e^k = f$. We say that f is *simply covered* if it is not a k–fold cover for any $k > 1$.

Lemma 3.5. *A nontrivial eigenfunction f of $\mathbf{A}_{\gamma^k}$ is a k–fold cover if and only if* $\mathrm{wind}^\Phi(f) \in k\mathbb{Z}$.

Proof. Clearly if $f = e^k$ then $\mathrm{wind}^\Phi(f) = k\,\mathrm{wind}^\Phi(e) \in k\mathbb{Z}$. To see the converse, note that by Prop. 3.3 there is a two-dimensional space of eigenfunctions e of $\mathbf{A}_\gamma$ having any given integer value of $\mathrm{wind}^\Phi(e)$. This gives rise to a two-dimensional space of k–fold covers e^k with winding $k\,\mathrm{wind}^\Phi(e)$. Since this attains all winding numbers in $k\mathbb{Z}$, every eigenfunction of $\mathbf{A}_{\gamma^k}$ that is *not* a k–fold cover has winding in $\mathbb{Z} \setminus k\mathbb{Z}$. □

Proposition 3.6. *Let γ be a simply covered periodic orbit, Φ a unitary trivialization of ξ along γ and $k \in \mathbb{N}$. Then a nontrivial eigenfunction e of $\mathbf{A}_{\gamma^k}$ is simply covered if and only if k and $\mathrm{wind}^\Phi(e)$ are relatively prime.*

Proof. From the lemma, we see that e is an n–fold cover if and only if n divides both k and $\mathrm{wind}^\Phi(e)$. So e is simply covered if and only if this is not true for any $n \in \{2, \ldots, k\}$. □

4. Intersection theory

If $\tilde{u} = (a, u) : \dot{\Sigma} \to \mathbb{R} \times M$ is a finite energy surface and $\pi : TM \to \xi$ is the fiberwise linear projection along X, then the composition $\pi \circ Tu : T\dot{\Sigma} \to \xi$ defines a section

$$\pi Tu : \dot{\Sigma} \to \mathrm{Hom}_{\mathbb{C}}(T\dot{\Sigma}, u^*\xi).$$

It is shown in [HWZ95] that πTu satisfies the similarity principle, thus it is either trivial or has only isolated positive zeros; the latter is the case unless $E_\omega(\tilde{u}) = 0$. Assuming $E_\omega(\tilde{u}) > 0$ and $\tilde{u}$ also has nondegenerate asymptotic orbits, the asymptotic formula of Prop. 3.1 implies that πTu has no zeros outside some compact subset, and its winding near infinity is controlled by eigenfunctions of asymptotic operators. We can thus define the integer

$$\mathrm{wind}_\pi(\tilde{u}) \ge 0$$

as the algebraic count of zeros of πTu. It follows from the nonlinear Cauchy-Riemann equation that $\mathrm{wind}_\pi(\tilde{u}) = 0$ if and only if $u : \dot{\Sigma} \to M$ is immersed and transverse to X.

Recalling the formula for $c_N(\tilde{u})$ from Definition 1.6, a result in [HWZ95] shows that all finite energy surfaces $\tilde{u}$ with $E_\omega(\tilde{u}) > 0$ satisfy $\mathrm{wind}_\pi(\tilde{u}) \le c_N(\tilde{u})$. Actually one can state this in a slightly stronger form as an equality. For $z \in \Gamma^\pm$, let e_z be an asymptotic eigenfunction that controls the approach of $\tilde{u}$ to γ_z, choose a unitary trivialization Φ of ξ along γ_z and define the *asymptotic defect* of $\tilde{u}$ at z to be the nonnegative integer

$$\mathrm{def}^z_\infty(\tilde{u}) = \left| \alpha^\Phi_\mp(\gamma_z) - \mathrm{wind}^\Phi(e_z) \right|.$$

This is zero if and only if the asymptotic approach to γ_z is controlled by an extremal eigenfunction. The total asymptotic defect of $\tilde{u}$ is then defined as

$$\mathrm{def}_\infty(\tilde{u}) = \sum_{z \in \Gamma} \mathrm{def}^z_\infty(\tilde{u}).$$

Now the argument in [HWZ95] implies:

Proposition 4.1. *For any finite energy surface $\tilde{u}$ with $E_\omega(\tilde{u}) > 0$ and nondegenerate asymptotic orbits,*

$$\mathrm{wind}_\pi(\tilde{u}) + \mathrm{def}_\infty(\tilde{u}) = c_N(\tilde{u}),$$

and both terms on the left hand side are nonnegative.

Next we collect some important results from the intersection theory of finite energy surfaces, due to R. Siefring. In the following, we assume all periodic orbits of X are nondegenerate and write punctured holomorphic disks as half-cylinders $Z_\pm \to \mathbb{R}_\pm \times M$. These statements, proved in [Sieb], are all based on the construction of homotopy invariant "asymptotic intersection numbers", which are well defined due to the relative asymptotic formulas proved in [Siea].

Proposition 4.2. *Suppose $\tilde{u} = (a, u) : Z_\pm \to \mathbb{R}_\pm \times M$ is a proper finite energy half-cylinder such that $u : Z_\pm \to M$ is embedded. Then any asymptotic eigenfunction controlling $\tilde{u}$ at infinity is simply covered.*

Proposition 4.3. *Suppose $\tilde{u} = (a, u) : Z_\pm \to \mathbb{R}_\pm \times M$ and $\tilde{v} = (b, v) : Z_\pm \to \mathbb{R}_\pm \times M$ are proper finite energy half-cylinders asymptotic to γ^m and γ^n respectively for some simply covered orbit γ and $m, n \in \mathbb{N}$. Assume also that u and v are both embedded and do not intersect each other. Then $m = n$, and the asymptotic eigenfunctions controlling $\tilde{u}$ and $\tilde{v}$ at infinity have the same winding number.*

Proposition 4.4. *Suppose $\tilde{u}_+ : (a_+, u_+) : Z_+ \to \mathbb{R}_+ \times M$ and $\tilde{u}_- : (a_-, u_-) : Z_- \to \mathbb{R}_- \times M$ are proper finite energy half-cylinders asymptotic to γ^{k_+} and γ^{k_-} respectively for some simply covered periodic orbit γ and $k_\pm \in \mathbb{N}$. Assume also u_+ and u_- are both embedded and do not intersect each other. Then both have asymptotic defect zero, and either*

(1) *γ is even and $k_+ = k_- = 1$, or*
(2) *γ is odd hyperbolic and $k_+ = k_- = 2$, hence $\gamma^{k_+} = \gamma^{k_-}$ is bad.*

5. Constraints at the asymptotic orbits

Given a periodic orbit γ, a positive/negative *asymptotic constraint* for γ is a real number $c \ge 0$ such that $\mp c \notin \sigma(\mathbf{A}_\gamma)$. We will say that a proper finite energy half-cylinder $\tilde{u} : Z_\pm \to \mathbb{R}_\pm \times M$ asymptotic to γ is *compatible* with this constraint if its transversal convergence rate (recall Definition 3.2) is strictly greater than c. Similarly, for a generalized holomorphic building $\tilde{u} : \mathbf{S} \to \mathbb{R} \times M$ with punctures $\Gamma = \Gamma^+ \cup \Gamma^-$, denote by $\mathbf{c} = \{c_z\}_{z \in \Gamma}$ an association of a positive/negative asymptotic constraint c_z to each asymptotic orbit γ_z for $z \in \Gamma^\pm$, and say that $\tilde{u}$ is compatible with $\mathbf{c}$ if for every $z \in \Gamma$, the corresponding end has transversal convergence rate strictly greater than c_z. Observe that the space of holomorphic buildings compatible with a given set of asymptotic constraints is a closed subset of the space of all holomorphic buildings.

Let γ be a nondegenerate orbit and fix a unitary trivialization Φ of ξ along γ. Then if c is a positive asymptotic constraint for γ, define the positive *constrained Conley-Zehnder index* by

$$\mu^\Phi_{\mathrm{CZ}}(\gamma; c) = \mu^\Phi_{\mathrm{CZ}}(\gamma) - \# \left(\sigma(\mathbf{A}_\gamma) \cap (-c, 0)\right), \tag{5.1}$$

where eigenvalues are counted with multiplicity. Similarly, for c a negative asymptotic constraint, define the negative constrained Conley-Zehnder index

$$\mu^{\Phi}_{\mathrm{CZ}}(\gamma; -c) = \mu^{\Phi}_{\mathrm{CZ}}(\gamma) + \# \left(\sigma(\mathbf{A}_\gamma) \cap (0, c)\right), \tag{5.2}$$

and if $\tilde{u} : \mathbf{S} \to \mathbb{R} \times M$ is a holomorphic building compatible with constraints $\mathbf{c}$, choose unitary trivializations Φ for ξ along all asymptotic orbits γ_z and define the total constrained Conley-Zehnder index

$$\mu^{\Phi}_{\mathrm{CZ}}(\tilde{u}; \mathbf{c}) = \sum_{z \in \Gamma^+} \mu^{\Phi}_{\mathrm{CZ}}(\gamma_z; c_z) - \sum_{z \in \Gamma^-} \mu^{\Phi}_{\mathrm{CZ}}(\gamma_z; -c_z).$$

The even/odd parity of the constrained indices $\mu^{\Phi}_{\mathrm{CZ}}(\gamma_z; \pm c_z)$ for $z \in \Gamma^{\pm}$ defines a *constrained parity* for each puncture, thus defining a new partition

$$\Gamma = \Gamma_0(\mathbf{c}) \cup \Gamma_1(\mathbf{c}).$$

Now define the *constrained Fredholm index*

$$\operatorname{ind}(\tilde{u}; \mathbf{c}) = -\chi(\overline{\mathbf{S}}) + 2c_1^{\Phi}(\bar{u}^*\xi) + \mu_{\mathrm{CZ}}(\tilde{u}; \mathbf{c}). \tag{5.3}$$

As shown in [Wena] (based on arguments in [HWZ99]), if $\tilde{u}$ is a finite energy surface, $\operatorname{ind}(\tilde{u}; \mathbf{c})$ is the virtual dimension of the moduli space of finite energy surfaces near $\tilde{u}$ that are compatible with $\mathbf{c}$.

The relation between Conley-Zehnder indices and winding numbers has a straightforward generalization to the constrained case. Given an orbit γ and $c \in \mathbb{R}$ with $-c \notin \sigma(\mathbf{A}_\gamma)$, define

$$\begin{aligned} \alpha^{\Phi}_-(\gamma; c) &= \max\{\operatorname{wind}^{\Phi}(\lambda) \mid \lambda \in \sigma(\mathbf{A}_\gamma),\ \lambda < -c\}, \\ \alpha^{\Phi}_+(\gamma; c) &= \min\{\operatorname{wind}^{\Phi}(\lambda) \mid \lambda \in \sigma(\mathbf{A}_\gamma),\ \lambda > -c\}, \\ p(\gamma; c) &= \alpha^{\Phi}_+(\gamma; c) - \alpha^{\Phi}_-(\gamma; c). \end{aligned} \tag{5.4}$$

Then combining (3.2) with (5.1) and (5.2), we have

$$\mu^{\Phi}_{\mathrm{CZ}}(\gamma; c) = 2\alpha^{\Phi}_-(\gamma; c) + p(\gamma; c) = 2\alpha^{\Phi}_+(\gamma; c) - p(\gamma; c). \tag{5.5}$$

A nontrivial eigenfunction e of $\mathbf{A}_\gamma$ will now be called a positive/negative *extremal* eigenfunction *with respect to the constraint* $|c|$ if $\operatorname{wind}^{\Phi}(e) = \alpha^{\Phi}_{\pm}(\gamma; c)$.

Now if $\tilde{u} : \dot{\Sigma} \to \mathbb{R} \times M$ is a finite energy surface compatible with $\mathbf{c}$ and $E_\omega(\tilde{u}) > 0$, define the *constrained asymptotic defect* at $z \in \Gamma^{\pm}$ by

$$\operatorname{def}^z_\infty(\tilde{u}; c_z) = \left|\alpha^{\Phi}_{\mp}(\gamma_z; \pm c_z) - \operatorname{wind}^{\Phi}(e_z)\right|,$$

where e_z is an eigenfunction controlling the asymptotic approach of $\tilde{u}$ to γ_z. The total *constrained asymptotic defect* is then

$$\operatorname{def}_\infty(\tilde{u}; \mathbf{c}) := \sum_{z \in \Gamma} \operatorname{def}^z_\infty(\tilde{u}; c_z).$$

This sum is nonnegative, and is zero if and only if $\tilde{u}$ is controlled by extremal eigenfunctions with respect to the constraints at every puncture.

If $\tilde{u}$ is a generalized building compatible with constraints $\mathbf{c}$, then every subbuilding $\tilde{u}_0$ is compatible with a natural set of *induced constraints* $\hat{\mathbf{c}}$ defined as follows. For each puncture z of $\tilde{u}_0$ that is also a puncture of $\tilde{u}$, set $\hat{c}_z = c_z$, and for all other punctures of $\tilde{u}_0$ (i.e. those which are only *breaking* punctures of $\tilde{u}$), set $\hat{c}_z = 0$. The following relation is easily verified using (5.3).

Proposition 5.1. *If $\tilde{u} : \mathbf{S} \to \mathbb{R} \times M$ is a generalized holomorphic building compatible with constraints $\mathbf{c}$ and it has connected components $\tilde{u}_i$ with induced constraints $\mathbf{c}_i$, then*

$$\operatorname{ind}(\tilde{u}; \mathbf{c}) = \sum_i \operatorname{ind}(\tilde{u}_i; \mathbf{c}_i) + \#\Delta_{\mathrm{N}}.$$

Suppose now that $\tilde{u}'$ is an augmentation of $\tilde{u}$: there is then a canonical bijection between the sets of punctures for each, so a set of asymptotic constraints $\mathbf{c}$ on either induces one on the other, which we'll also denote by $\mathbf{c}$. However if $\tilde{u}'$ is compatible with $\mathbf{c}$, it is *not* necessarily true that $\tilde{u}$ is as well. Indeed, it may happen that for a given puncture $z \in \Gamma$, the component of $\tilde{u}'$ containing z is a trivial cylinder, and is therefore compatible with arbitrarily strict asymptotic constraints, which is not necessarily true for $\tilde{u}$. On the other hand, if $\tilde{u}'$ arises as the limit of a sequence $\tilde{u}_k$ of finite energy surfaces compatible with $\mathbf{c}$, then in a neighborhood of $z \in \Gamma$, convergence to $\tilde{u}'$ and convergence to $\tilde{u}$ are equivalent notions. It follows that both $\tilde{u}$ and $\tilde{u}'$ are in this case compatible with $\mathbf{c}$: this is true in particular if $\tilde{u}$ is the core of $\tilde{u}'$.

6. The normal first Chern number

Suppose $\tilde{u} : \mathbf{S} \to \mathbb{R} \times M$ is a generalized holomorphic building compatible with asymptotic constraints $\mathbf{c} = \{c_z\}_{z \in \Gamma}$, and the asymptotic orbits γ_z are all nondegenerate. Fix a unitary trivialization Φ for ξ along each γ_z.

Definition 6.1. Define the *constrained normal first Chern number* of $\tilde{u}$ with respect to $\mathbf{c}$ as the integer

$$c_N(\tilde{u}; \mathbf{c}) = c_1^\Phi(\bar{u}^*\xi) - \chi(\overline{\mathbf{S}}) + \sum_{z \in \Gamma^+} \alpha_-^\Phi(\gamma_z; c_z) - \sum_{z \in \Gamma^-} \alpha_+^\Phi(\gamma_z; -c_z).$$

One can easily check that this doesn't depend on Φ, and a simple computation using (5.5) and (5.3) shows that

$$2c_N(\tilde{u}; \mathbf{c}) = \operatorname{ind}(\tilde{u}; \mathbf{c}) - 2 + 2g + \#\Gamma_0(\mathbf{c}), \tag{6.1}$$

where g is the arithmetic genus of $\tilde{u}$. The new formula is therefore consistent with Definition 1.6.

The following result is immediate from the definition.

Proposition 6.2. *If $\tilde{u}'$ is an augmentation of $\tilde{u}$ then $c_N(\tilde{u}'; \mathbf{c}) = c_N(\tilde{u}; \mathbf{c})$.*

We also have an immediate generalization of Prop. 4.1:

Proposition 6.3. *If $\tilde{u}$ is a finite energy surface compatible with* $\mathbf{c}$ *and* $E_\omega(\tilde{u}) > 0$, *then*

$$\mathrm{wind}_\pi(\tilde{u}) + \mathrm{def}_\infty(\tilde{u}; \mathbf{c}) = c_N(\tilde{u}; \mathbf{c}),$$

and both terms on the left hand side are nonnegative.

Proposition 6.4. *Suppose $\tilde{u} : \mathbf{S} \to \mathbb{R} \times M$ is a generalized holomorphic building compatible with asymptotic constraints* $\mathbf{c}$, *and $\tilde{u}_i : \dot{S}_i \to \mathbb{R} \times M$ are the connected components of $\tilde{u}$, with induced constraints $\mathbf{c}_i$ for $i = 1, \ldots, N$. Then*

$$c_N(\tilde{u}; \mathbf{c}) = \sum_{i=1}^{N} c_N(\tilde{u}_i; \mathbf{c}_i) + \sum_{\{\overline{z}, \underline{z}\} \subset \Delta_{\mathrm{C}}} p(\gamma_{(\overline{z}, \underline{z})}) + \#\Delta_{\mathrm{N}}.$$

Proof. We must check that c_N behaves appropriately under certain natural operations on generalized holomorphic buildings. The simplest such operation is the *disjoint union* of two buildings $\tilde{u} : \mathbf{S} \to \mathbb{R} \times M$ and $\tilde{u}' : \mathbf{S}' \to \mathbb{R} \times M$ with constraints $\mathbf{c}$ and $\mathbf{c}'$ respectively: this defines a building $\tilde{u} \sqcup \tilde{u}' : \mathbf{S} \sqcup \mathbf{S}' \to \mathbb{R} \times M$, compatible with the obvious union of constraints $\mathbf{c} \sqcup \mathbf{c}'$. Clearly then,

$$c_N(\tilde{u} \sqcup \tilde{u}'; \mathbf{c} \sqcup \mathbf{c}') = c_N(\tilde{u}; \mathbf{c}) + c_N(\tilde{u}'; \mathbf{c}').$$

Next, if $\tilde{u} : \mathbf{S} \to \mathbb{R} \times M$ is a building with two points $z, z' \in S \setminus (\Gamma \cup \Delta)$ such that $\tilde{u}(z) = \tilde{u}(z')$, we can add a node to $\tilde{u}$ and define $\odot_{(z,z')}\tilde{u} : \odot_{(z,z')}\mathbf{S} \to \mathbb{R} \times M$ by adding $\{z, z'\}$ to the set of nodal pairs. This decreases the Euler characteristic of $\overline{\mathbf{S}}$ by 2, thus

$$c_N(\odot_{(z,z')}\tilde{u}; \mathbf{c}) = c_N(\tilde{u}; \mathbf{c}) + 2.$$

Similarly, if there are punctures $\underline{z} \in \Gamma^+$ and $\overline{z} \in \Gamma^-$ for which $\gamma_{\underline{z}} = \gamma_{\overline{z}}$ and $c_{\underline{z}} = c_{\overline{z}} = 0$, then we can change $\tilde{u}$ by "gluing" these punctures, which means adding $\{\overline{z}, \underline{z}\}$ to the set of breaking pairs and choosing an appropriate decoration so that the result is a generalized holomorphic building $\boxplus_{(\overline{z},\underline{z})}\tilde{u} : \boxplus_{(\overline{z},\underline{z})}\mathbf{S} \to \mathbb{R} \times M$. By losing two unconstrained punctures, this operation subtracts $\alpha_-^\Phi(\gamma_{\underline{z}}) - \alpha_+^\Phi(\gamma_{\overline{z}}) = -p(\gamma_{(\overline{z},\underline{z})})$ from $c_N(\tilde{u}; \mathbf{c})$, hence

$$c_N(\boxplus_{(\overline{z},\underline{z})}\tilde{u}; \mathbf{c}) = c_N(\tilde{u}; \mathbf{c}) + p(\gamma_{(\overline{z},\underline{z})}).$$

Composing these operations as often as necessary gives the stated result. □

7. Proofs of the main results

We will now state and prove stronger, more technical versions of Theorems 1 and 2. Assume $\mathcal{H}_k = (\xi_k, X_k, \omega_k, J_k)$ is a sequence of stable Hamiltonian structures converging to $\mathcal{H} = (\xi, X, \omega, J)$, where the latter is nondegenerate, and $\tilde{u} : \mathbf{S} \to \mathbb{R} \times M$ is a $\tilde{J}$–holomorphic building compatible with asymptotic constraints $\mathbf{c} = \{c_z\}_{z\in\Gamma}$. Since the orbits γ_z are nondegenerate, for sufficiently large k there are unique periodic orbits $\gamma_{z,k}$ of X_k such that

$$\gamma_{z,k} \to \gamma_z,$$

in the sense that these orbits have parametrizations $S^1 \to M$ that converge in the C^∞–topology. We may also assume that for each $z \in \Gamma^\pm$, $\mp c_z \notin \sigma(\mathbf{A}_{\gamma_{z,k}})$.

Theorem 3. *Assume $\tilde{u}_k$ are nicely embedded $\tilde{J}_k$–holomorphic finite energy surfaces converging to the $\tilde{J}$–holomorphic building $\tilde{u}$, such that the $\tilde{u}_k$ are also compatible with $\mathbf{c}$ and $c_N(\tilde{u}_k;\mathbf{c}) = 0$. Then $\tilde{u}$ is nicely embedded, its core $\tilde{u}^{\mathrm{K}}$ is compatible with $\mathbf{c}$, and for every connected component $\tilde{v}_i$ of $\tilde{u}^{\mathrm{K}}$ with induced constraints $\mathbf{c}_i$, $c_N(\tilde{v}_i;\mathbf{c}_i) = 0$.*

Remark 7.1. If $\tilde{u}_k \to \tilde{u}$ under the assumptions of Theorem 1, then one can assume after taking a subsequence that all the $\tilde{u}_k$ are compatible with some choice of asymptotic constraints $\mathbf{c}$ such that $c_N(\tilde{u}_k;\mathbf{c}) = \mathrm{wind}_\pi(\tilde{u}_k) + \mathrm{def}_\infty(\tilde{u}_k;\mathbf{c}) = 0$. This is why Theorem 3 implies Theorem 1. Similarly, Theorem 2 is a special case of the next statement.

Theorem 4. *In addition to the assumptions of Theorem 3, suppose J is generic. Then $\mathrm{ind}(\tilde{u};\mathbf{c})$ is either 1 or 2. If it is 1, then $\tilde{u}$ is a finite energy surface, hence the moduli space of such curves with constraint $\mathbf{c}$ is compact. If $\mathrm{ind}(\tilde{u};\mathbf{c}) = 2$ and $\tilde{u}$ is not a finite energy surface, then it has exactly two nontrivial connected components $\tilde{v}_i = (b_i, v_i)$, both with $\mathrm{ind}(\tilde{v}_i;\mathbf{c}_i) = 1$, such that v_1 and v_2 have no intersections in M and they are connected to each other by a unique nontrivial breaking orbit.*

We begin now with some preparations for the proof of Theorem 3. By Prop. 2.7, we can assume without loss of generality that the curves $\tilde{u}_k$ have a fixed domain $\dot{\Sigma} = \Sigma \setminus \Gamma$ with varying complex structures j_k, and there is a fixed homeomorphism

$$\psi : \overline{\mathbf{S}} \to \overline{\Sigma}$$

such that $u_k \circ \psi \to u$ in $C^\infty_{\mathrm{loc}}(\overline{\mathbf{S}} \setminus (\partial\overline{\mathbf{S}} \cup \Theta_\Delta), M)$ and $\bar{u}_k \circ \psi \to \bar{u}$ in $C^0(\overline{\mathbf{S}}, M)$. The punctures $\Gamma^\pm$ of $\tilde{u}_k$ and $\tilde{u}$ are also identified via ψ, so we shall use the same notation for both: the asymptotic orbit of $\tilde{u}_k$ at $z \in \Gamma$ is then $\gamma_{z,k}$ for sufficiently large k.

Lemma 7.2. *The building $\tilde{u}$ has at least one nontrivial component.*

Proof. If $\tilde{u} : \mathbf{S} \to \mathbb{R} \times M$ is a trivial building, then $\bar{u} : \overline{\mathbf{S}} \to M$ represents the trivial homology class $[\bar{u}] = 0 \in H_2(M)$. Perturbing $\tilde{u}$ to $\tilde{u}_k$ with asymptotic orbits $\gamma_{z,k}$ for sufficiently large k, we also have $[\bar{u}_k] = 0 \in H_2(M)$, thus $E_{\omega_k}(\tilde{u}_k) = \int_{\dot{\Sigma}} u_k^*\omega_k = \langle [\omega_k], [\bar{u}_k] \rangle = 0$. This is a contradiction, since $\tilde{u}_k$ is assumed to be nicely embedded, and thus nontrivial. □

Lemma 7.3. *For each k, $u_k(\dot{\Sigma}) \subset M$ is disjoint from each of the orbits $\gamma_{z,k} \subset M$ for $z \in \Gamma$.*

Proof. Since u_k is embedded, it follows from the nonlinear Cauchy-Riemann equation that it is also transverse to X_k, thus any intersection with $\gamma_{z,k}$ is transverse and implies transverse intersections of u_k with its image in a neighborhood of z. $\square$

Lemma 7.4. *For every $z \in \Gamma^{\pm}$, the extremal negative/positive eigenfunctions of γ_z with respect to c_z are simply covered. Moreover if z and ζ are distinct punctures with the same sign and γ_z and γ_ζ cover the same simply covered orbit, then $\gamma_z = \gamma_\zeta$.*

Proof. Since $\operatorname{def}_\infty(\tilde{u}_k; \mathbf{c}) \le c_N(\tilde{u}_k; \mathbf{c}) = 0$, $\tilde{u}_k$ is controlled by extremal eigenfunctions with respect to $\mathbf{c}$ at each puncture, and these must then be simply covered by Prop. 4.2. Similarly Prop. 4.3 implies that distinct positive/negative ends of $\tilde{u}_k$ approaching covers of the same orbit must approach with the same covering number. Both statements hold also in the limit due to the nondegeneracy of the orbits γ_z. $\square$

Lemma 7.5. *Suppose $z_+ \in \Gamma^+$ and $z_- \in \Gamma^-$, such that γ_{z_+} and γ_{z_-} cover the same simply covered orbit. Then $\gamma_{z_+} = \gamma_{z_-}$ and it is either a simply covered even orbit or a doubly covered bad orbit with simply covered extremal eigenfunctions. Moreover, we can reset $c_{z_+} = c_{z_-} = 0$ without changing* $\operatorname{ind}(\tilde{u}; \mathbf{c})$ *or* $c_N(\tilde{u}; \mathbf{c})$.

Proof. For $\gamma_{z_\pm,k}$, the first part of the statement follows from Prop. 4.4 since u_k is embedded, and the second part results from the fact that $\gamma_{z_\pm,k}$ is therefore even and $\tilde{u}_k$ is controlled by extremal eigenfunctions (in the unconstrained sense) at both of these punctures, so $\mathbf{A}_{\gamma_{z_\pm}}$ can have no eigenvalues between $c_{z_\pm}$ and 0. The same result is true for $\gamma_{z_\pm}$ due to nondegeneracy. $\square$

Lemma 7.6. *For every connected component $\tilde{v}_i = (b_i, v_i) : \dot{S}_i \to \mathbb{R} \times M$ of $\tilde{u}$, either $\tilde{v}_i$ is a trivial curve or $v_i : \dot{S}_i \to M$ is injective. Moreover for any two such components $\tilde{v}_1$ and $\tilde{v}_2$ that are not trivial, $v_1(\dot{S}_1)$ and $v_2(\dot{S}_2)$ are either disjoint or identical, the latter if and only if $\tilde{v}_1$ can be obtained from $\tilde{v}_2$ by $\mathbb{R}$–translation (up to parametrization).*

Proof. Suppose $\tilde{v}_i = (b_i, v_i) : \dot{S}_i \to \mathbb{R} \times M$ is a nontrivial connected component of $\tilde{u}$, so $E_\omega(\tilde{v}_i) > 0$ and consequently the section

$$\pi T v_i : \dot{S}_i \to \operatorname{Hom}_{\mathbb{C}}(T\dot{S}_i, v_i^* \xi)$$

has only finitely many zeros, all positive. We claim first that $\tilde{v}_i$ is somewhere injective. If not, then there exists a somewhere injective finite energy surface $\tilde{w}_i = (\beta_i, w_i) : \dot{S}_i' \to \mathbb{R} \times M$ and a holomorphic branched cover $\varphi_i : \dot{S}_i \to \dot{S}_i'$ of degree $k \ge 2$ such that $\tilde{v}_i = \tilde{w}_i \circ \varphi_i$. We can therefore find an embedded loop $\alpha' : S^1 \to \dot{S}_i'$ which does not lift to $\dot{S}_i$, and by small perturbations of α', we may assume it misses all punctures and zeros of $\pi T w_i$. Now choose an embedded loop $\alpha : S^1 \to \dot{S}_i$ which projects to an n–fold cover of α' for some $n \geq 2$, and denote $C = \alpha(S^1) \subset \dot{S}_i$, $C' = \alpha'(S^1) \subset \dot{S}_i'$. Choose also an open neighborhood $\mathcal{U}'$ of C' and a corresponding neighborhood $\mathcal{U}$ of C such that $\varphi_i(\mathcal{U}) = \mathcal{U}'$. The restriction $\varphi_i|_{\mathcal{U}} : \mathcal{U} \to \mathcal{U}'$ is an n–fold covering map, and we may assume without loss of generality that $v_i|_{\mathcal{U}} : \mathcal{U} \to M$ and $w_i|_{\mathcal{U}'} : \mathcal{U}' \to M$ are both transverse to X. From this we can derive a contradiction. Indeed, any map $v' : \mathcal{U} \to M$ that's C^∞–close to $v_i|_{\mathcal{U}}$ can be written on some neighborhood of C as

$$v'(z) = \varphi_X^{f(z)}(v_i(z))$$

where φ_X^t denotes the flow of X and f is a smooth real valued function defined on some neighborhood of C. Choosing any nontrivial deck transformation $g : \mathcal{U} \to \mathcal{U}$ for the covering map $\varphi_i|_{\mathcal{U}}$, there is necessarily a point $z \in C$ at which $f(z) = f \circ g(z)$, and thus $v'(z) = v'(g(z))$. By Prop. 2.7, this is true in particular for a suitable restriction of u_k for k sufficiently large, contradicting the assumption that u_k is embedded. We conclude that $\tilde{v}_i$ is somewhere injective.

Now denote $\mathbb{R}$–translations of finite energy surfaces $\tilde{u} = (a, u)$ by $\tilde{u}^c := (a + c, u)$ for $c \in \mathbb{R}$. Suppose that $\tilde{v}_1$ and $\tilde{v}_2$ are two nontrivial components and $v_1(z_1) = v_2(z_2)$. This gives an intersection $\tilde{v}_1(z_1) = \tilde{v}_2^c(z_2)$ for some $c \in \mathbb{R}$. If the intersection is isolated then it is positive, and yields an isolated intersection of $\tilde{u}_k$ and $\tilde{u}_k^{c'}$ for some $c' \in \mathbb{R}$, again contradicting the fact that u_k is embedded. The alternative, since $\tilde{v}_1$ and $\tilde{v}_2$ are both somewhere injective, is that $\tilde{v}_1$ and $\tilde{v}_2^c$ are identical up to parametrization. The same argument applies to intersections of v_1 with itself: since $\tilde{v}_1$ is somewhere injective, the intersection of $\tilde{v}_1$ with $\tilde{v}_1^c$ is then necessarily isolated, otherwise $\tilde{v}_1$ and $\tilde{v}_1^c$ are identical up to parametrization; this is impossible in light of the asymptotic behavior described in Prop. 3.1. $\square$

We shall call a nonconstant trivial subbuilding of $\tilde{u}$ *maximal* if every component attached to it by a breaking orbit is nontrivial. Given such a subbuilding $\tilde{u}^{\mathrm{t}} : \mathbf{S}^{\mathrm{t}} \to \mathbb{R} \times M$, we introduce the following notation: write the punctures of $\tilde{u}^{\mathrm{t}}$ as

$$\widehat{\Gamma}^{\pm} = \widehat{\Gamma}^{\pm}_{\mathrm{C}} \cup \widehat{\Gamma}^{\pm}_{\mathrm{E}},$$

where $\widehat{\Gamma}_{\mathrm{E}} := \widehat{\Gamma} \cap \Gamma$ and $\widehat{\Gamma}_{\mathrm{C}}$ consists of all punctures of $\tilde{u}^{\mathrm{t}}$ that arise from *breaking* punctures of $\tilde{u}$. Assume $\#\widehat{\Gamma}^+_{\mathrm{C}} = p$, $\#\widehat{\Gamma}^-_{\mathrm{C}} = q$, $\#\widehat{\Gamma}^+_{\mathrm{E}} = r$ and $\#\widehat{\Gamma}^-_{\mathrm{E}} = s$; we have necessarily $\#\widehat{\Gamma}^+ = p + r > 0$, $\#\widehat{\Gamma}^- = q + s > 0$ and since $\tilde{u}$ is connected and has nontrivial components, $\#\widehat{\Gamma}_{\mathrm{C}} = p + q > 0$. Every asymptotic orbit of $\tilde{u}^{\mathrm{t}}$ covers the same simply covered orbit γ, so denote the orbit at $z \in \widehat{\Gamma}$ by

$$\gamma_z = \gamma^{m_z}$$

for some multiplicity $m_z \in \mathbb{N}$. Each $z \in \widehat{\Gamma}^{\pm}_{\mathrm{C}}$ belongs to a breaking pair $\{z, \hat{z}\} \subset \Delta_{\mathrm{C}}$ of $\tilde{u}$, and the component $\tilde{v}_z = (b_z, v_z)$ of $\tilde{u}$ containing $\hat{z}$ is necessarily nontrivial, and negatively/positively asymptotic to γ^{m_z} at $\hat{z}$.

We know now from Lemma 7.6 that each of the maps v_z is injective (thus embedded near the punctures), and any two of them are either disjoint or identical. Then by the intersection theoretic results of §4, all the m_z for $z \in \widehat{\Gamma}_{\mathrm{C}}$ equal a fixed number $m_{\mathrm{C}} \in \mathbb{N}$, and the asymptotic approach of each $\tilde{v}_z$ to $\gamma_{m_{\mathrm{C}}}$ is controlled by eigenfunctions e_z with the same winding $\mathrm{wind}^{\Phi}(e_z) := w_{\mathrm{C}} \in \mathbb{Z}$. Likewise for $z \in \widehat{\Gamma}_{\mathrm{E}}^{\pm}$, Lemmas 7.4 and 7.5 imply that all m_z equal a fixed multiplicity $m_{\mathrm{E}} \in \mathbb{N}$, and there is a fixed extremal winding number $w_{\mathrm{E}} := \alpha_{\mp}^{\Phi}(\gamma^{m_{\mathrm{E}}}; c_z)$. Note that if both $\widehat{\Gamma}_{\mathrm{E}}^{+}$ and $\widehat{\Gamma}_{\mathrm{E}}^{-}$ are nonempty, then $\alpha_{+}^{\Phi}(\gamma^{m_{\mathrm{E}}}; c_z) = \alpha_{+}^{\Phi}(\gamma^{m_{\mathrm{E}}}) = \alpha_{-}^{\Phi}(\gamma^{m_{\mathrm{E}}}) = \alpha_{-}^{\Phi}(\gamma^{m_{\mathrm{E}}}; c_z)$; this follows from Lemma 7.5.

Lemma 7.7. *For the maximal trivial subbuilding $\tilde{u}^{\mathrm{t}}$ described above, $m_{\mathrm{C}} = m_{\mathrm{E}}$ and $w_{\mathrm{C}} = w_{\mathrm{E}}$.*

Proof. There's nothing to prove if $\widehat{\Gamma}_{\mathrm{E}} = \emptyset$, so assume $r + s > 0$. Define the compact subset $\overline{\Sigma}^{\mathrm{t}} = \psi(\overline{\mathbf{S}}^{\mathrm{t}}) \subset \overline{\Sigma}$ and recall that for sufficiently large k, $\bar{u}_k \circ \psi|_{\overline{\mathbf{S}}^{\mathrm{t}}}$ is C^0–close to $\bar{u}^{\mathrm{t}}$. Let γ_k be the unique simply covered orbit of X_k for sufficiently large k such that $\gamma_k \to \gamma$. Then some cover of γ_k is an asymptotic orbit of $\tilde{u}_k$, thus by Lemma 7.3, we can assume $\bar{u}_k(\overline{\Sigma}^{\mathrm{t}})$ lies in a fixed tubular neighborhood N_γ of γ but without intersecting γ_k. We can also arrange that $\bar{u}_k$ have the following behavior at each component of $\partial\overline{\Sigma}^{\mathrm{t}}$:

- for $z \in \widehat{\Gamma}_{\mathrm{C}}^{\pm}$, $\mathrm{wind}^{\Phi}(\bar{u}_k(\psi(\delta_z))) = \pm w_{\mathrm{C}}$,
- for $z \in \widehat{\Gamma}_{\mathrm{E}}^{\pm}$, $\mathrm{wind}^{\Phi}(\bar{u}_k(\psi(\delta_z))) = \pm w_{\mathrm{E}}$.

The crucial observation is now that $\bar{u}_k(\overline{\Sigma}^{\mathrm{t}})$ realizes a homology in $H_2(N_\gamma \setminus \gamma_k) \cong H_2(T^2)$. From this we obtain the relations

$$
\begin{aligned}
p m_{\mathrm{C}} + r m_{\mathrm{E}} &= q m_{\mathrm{C}} + s m_{\mathrm{E}}, \\
p w_{\mathrm{C}} + r w_{\mathrm{E}} &= q w_{\mathrm{C}} + s w_{\mathrm{E}},
\end{aligned}
$$

and consequently

$$
\begin{pmatrix} m_{\mathrm{C}} & m_{\mathrm{E}} \\ w_{\mathrm{C}} & w_{\mathrm{E}} \end{pmatrix} \begin{pmatrix} p - q \\ r - s \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.
$$

If $p = q$ and $r = s$, then both of these are nonzero and Prop. 4.4 implies that either γ is even and $m_{\mathrm{C}} = m_{\mathrm{E}} = 1$ or γ is odd hyperbolic and $m_{\mathrm{C}} = m_{\mathrm{E}} = 2$, with $w_{\mathrm{C}} = w_{\mathrm{E}} = \alpha_{+}^{\Phi}(\gamma^{m_{\mathrm{C}}}) = \alpha_{-}^{\Phi}(\gamma^{m_{\mathrm{C}}})$ in either case. Otherwise the determinant of the matrix above must vanish, so $m_{\mathrm{C}} w_{\mathrm{E}} = m_{\mathrm{E}} w_{\mathrm{C}}$. However, by Prop. 4.2, w_{C} and w_{E} are winding numbers of simply covered eigenfunctions for $\gamma^{m_{\mathrm{C}}}$ and $\gamma^{m_{\mathrm{E}}}$ respectively, thus Prop. 3.6 implies that m_{C} and w_{C} are relatively prime, as are m_{E} and w_{E}. This implies $m_{\mathrm{C}} = m_{\mathrm{E}}$ and $w_{\mathrm{C}} = w_{\mathrm{E}}$ □

Corollary 7.8. *If $\tilde{u}^{\mathrm{t}} : \mathbf{S}^{\mathrm{t}} \to \mathbb{R} \times M$ is the maximal trivial subbuilding above with induced asymptotic constraints $\hat{\mathbf{c}} = \{\hat{c}_z\}_{z \in \widehat{\Gamma}}$, then*

$$c_N(\tilde{u}^{\mathrm{t}}; \hat{\mathbf{c}}) + \sum_{z \in \widehat{\Gamma}_{\mathrm{C}}} \left[p(\gamma_z) + \operatorname{def}_\infty^{\hat{z}}(\tilde{v}_z) \right] = -\chi(\overline{\mathbf{S}}^{\mathrm{t}}).$$

In particular this sum is nonnegative, and is zero if and only if $\tilde{u}^{\mathrm{t}}$ is cylindrical.

Proof. By the lemma we have $\#\widehat{\Gamma}^+ = \#\widehat{\Gamma}^-$ and can write $m := m_{\mathrm{E}} = m_{\mathrm{C}}$ and $w := w_{\mathrm{E}} = w_{\mathrm{C}} = \alpha_\mp^\Phi(\gamma^m; c_z)$ for each $z \in \widehat{\Gamma}_{\mathrm{E}}^\pm$. Then, noting that $c_1^\Phi((\bar{u}^{\mathrm{t}})^*\xi) = 0$,

$$\begin{aligned}
c_N(\tilde{u}^{\mathrm{t}}; \hat{\mathbf{c}}) &+ \sum_{z \in \widehat{\Gamma}_{\mathrm{C}}} \left[p(\gamma_z) + \operatorname{def}_\infty^{\hat{z}}(\tilde{v}_z) \right] \\
&= -\chi(\overline{\mathbf{S}}^{\mathrm{t}}) + \sum_{z \in \widehat{\Gamma}^+} \alpha_-^\Phi(\gamma^m; \hat{c}_z) - \sum_{z \in \widehat{\Gamma}^-} \alpha_+^\Phi(\gamma^m; \hat{c}_z) \\
&\qquad + \sum_{z \in \widehat{\Gamma}_{\mathrm{C}}} \left[\alpha_+^\Phi(\gamma^m) - \alpha_-^\Phi(\gamma^m) \right] \\
&\qquad + \sum_{z \in \widehat{\Gamma}_{\mathrm{C}}^+} \left[w - \alpha_+^\Phi(\gamma^m) \right] + \sum_{z \in \widehat{\Gamma}_{\mathrm{C}}^-} \left[\alpha_-^\Phi(\gamma^m) - w \right] \\
&= -\chi(\overline{\mathbf{S}}^{\mathrm{t}}) + \sum_{z \in \widehat{\Gamma}^+} w - \sum_{z \in \widehat{\Gamma}^-} w = -\chi(\overline{\mathbf{S}}^{\mathrm{t}}).
\end{aligned}$$

□

We shall handle constant components of $\tilde{u}$ similarly. Call a connected subbuilding $\tilde{u}^{\mathrm{c}} : \mathbf{S}^{\mathrm{c}} \to \mathbb{R} \times M$ of $\tilde{u}$ a *constant subbuilding* if every connected component of $\tilde{u}^{\mathrm{c}}$ is constant. Further, call it a *maximal* constant subbuilding if every constant component of $\tilde{u}$ that is attached by a node to some component of $\tilde{u}^{\mathrm{c}}$ is also in $\tilde{u}^{\mathrm{c}}$. Note that constant subbuildings cannot have punctures, thus $\overline{\mathbf{S}}^{\mathrm{c}}$ is closed. Denote by $\widehat{\Delta}_{\mathrm{N}} \subset S^{\mathrm{c}}$ the set of points $z \in S^{\mathrm{c}}$ that belong to nodal pairs $\{z, z'\} \subset \Delta_{\mathrm{N}}$ of $\mathbf{S}$ such that $\tilde{u}$ is not constant near z'; this set is necessarily nonempty since $\tilde{u}$ is connected. Then the stability condition on $\tilde{u}$ implies

$$\chi(\overline{\mathbf{S}}^{\mathrm{c}} \setminus \widehat{\Delta}_{\mathrm{N}}) < 0.$$

Thus $c_N(\tilde{u}^{\mathrm{c}}) + 2\#\widehat{\Delta}_{\mathrm{N}} = -\chi(\overline{\mathbf{S}}^{\mathrm{c}}) + \#\widehat{\Delta}_{\mathrm{N}} + \#\widehat{\Delta}_{\mathrm{N}} = -\chi(\overline{\mathbf{S}}^{\mathrm{c}} \setminus \widehat{\Delta}_{\mathrm{N}}) + \#\widehat{\Delta}_{\mathrm{N}} > \#\widehat{\Delta}_{\mathrm{N}} > 0$. We've proved:

Lemma 7.9. *For any maximal constant subbuilding $\tilde{u}^{\mathrm{c}}$ of $\tilde{u}$ with nodes $\widehat{\Delta}_{\mathrm{N}}$ connecting it to nonconstant components of $\tilde{u}$,*

$$c_N(\tilde{u}^{\mathrm{c}}) + 2\#\widehat{\Delta}_{\mathrm{N}} > 0.$$

All the ingredients are now in place.

Proof of Theorem 3. By the above results, $\tilde{u}$ consists of the following pieces:

(1) Maximal constant subbuildings $\tilde{u}^{\mathrm{c}}$ such that $c_N(\tilde{u}^{\mathrm{c}}) + 2\#\widehat{\Delta}_{\mathrm{N}} > 0$.
(2) Maximal trivial subbuildings $\tilde{u}^{\mathrm{t}}$ with induced asymptotic constraints $\mathbf{c}^{\mathrm{t}}$, for which the sum of $c_N(\tilde{u}^{\mathrm{t}}; \mathbf{c}^{\mathrm{t}}) + \sum_{z\in\widehat{\Gamma}_{\mathrm{C}}} p(\gamma_z)$ with the asymptotic defects of all neighboring nontrivial ends is nonnegative, and zero if and only if $\tilde{u}^{\mathrm{t}}$ is cylindrical.
(3) Nontrivial connected components $\tilde{v} = (b, v)$ with v injective.

Note that each nontrivial component $\tilde{v}$ is compatible with induced asymptotic constraints $\hat{\mathbf{c}}$ and satisfies $c_N(\tilde{v}; \hat{\mathbf{c}}) - \mathrm{def}_\infty(\tilde{v}; \hat{\mathbf{c}}) = \mathrm{wind}_\pi(\tilde{v}) \ge 0$.

Since $c_N(\tilde{u}; \mathbf{c}) = 0$, we conclude from Prop. 6.4 that $\tilde{u}$ contains no constant subbuildings or nodes, every trivial subbuilding is cylindrical and every nontrivial component $\tilde{v}$ has $\mathrm{wind}_\pi(\tilde{v}) = 0$. Such components $\tilde{v}$ are therefore nicely embedded. A slightly stronger statement results from the observation that the core $\tilde{u}^{\mathrm{K}}$ is also compatible with $\mathbf{c}$ and only contains nicely embedded components. Thus each of these components $\tilde{v}$ satisfies $c_N(\tilde{v}; \hat{\mathbf{c}}) = 0$, where $\hat{\mathbf{c}}$ are now the constraints induced on $\tilde{v}$ as a subbuilding of $\tilde{u}^{\mathrm{K}}$. □

Proof of Theorem 4. For a given set of asymptotic constraints $\mathbf{c}$, the $\mathbb{R}$–invariance of $\tilde{J}$ together with a standard transversality argument (cf. [Wena]) imply that for generic ω–compatible choices of J, all nontrivial somewhere injective finite energy surfaces $\tilde{w}$ compatible with $\mathbf{c}$ satisfy $\mathrm{ind}(\tilde{w}; \mathbf{c}) \ge 1$. Moreover, $c_N(u; \mathbf{c}) = 0$ implies $\mathrm{ind}(u; \mathbf{c}) \le 2$ due to (6.1), thus this index can only be 1 or 2. Likewise each connected component of $\tilde{u}^{\mathrm{K}}$ has constrained index at least 1, and by Prop. 5.1, these add up to $\mathrm{ind}(\tilde{u}; \mathbf{c})$. We conclude there is exactly one component if $\mathrm{ind}(\tilde{u}; \mathbf{c}) = 1$, and at most two if $\mathrm{ind}(\tilde{u}; \mathbf{c}) = 2$. In the latter case, both nontrivial components $\tilde{v}_i$ have $\mathrm{ind}(\tilde{v}_i; \mathbf{c}_i) = 1$, so by (6.1), each has a unique puncture whose constrained parity is even: this is therefore the unique breaking puncture. Since the ends of $\tilde{v}_1$ and $\tilde{v}_2$ approaching this breaking orbit have opposite signs, $\tilde{v}_1$ and $\tilde{v}_2$ cannot be the same up to $\mathbb{R}$–translation, thus their projections to M are disjoint. □

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