Chapter 2

Bundles

Contents

2.1 Vector bundles: definitions and examples

Roughly speaking, a smooth vector bundle is a family of vector spaces that are "attached" to a manifold in some smoothly varying manner. We will present this idea more rigorously in Definition 2.8 below. First though, it's worth looking at some examples to understand why such an object might be of interest.

Example 2.1 (The tangent bundle). If M is a smooth n-dimensional manifold, its tangent bundle

$$TM = \bigcup_{x \in M} T_x M$$

associates to every $x \in M$ the n-dimensional vector space $T_x M$. All of these vector spaces are isomorphic (obviously, since they have the same dimension), though it's important to note that there is generally no *natural* choice of isomorphism between $T_x M$ and $T_y M$ for $x \neq y$. There is however a sense in which the association of $x \in M$ with the vector space $T_x M$ can be thought of as a "smooth" function of x. We'll be more precise about this later, but intuitively one can imagine M as the sphere S^2, embedded smoothly in $\mathbb{R}^3$. Then each tangent space is a 2-dimensional subspace of $\mathbb{R}^3$, and the planes $T_x S^2 \subset \mathbb{R}^3$ vary smoothly as x varies over S^2. Of course, the definition of "smoothness" for a bundle should ideally not depend on any embedding of M in a larger space, just as the smooth manifold structure of M is independent of any such embedding. The general definition will have to reflect this.

In standard bundle terminology, the tangent bundle is an example of a *smooth vector bundle of rank* n *over* M. We call M the *base* of this bundle, and the $2n$-dimensional manifold TM itself is called its *total space*. There is a natural *projection* map $\pi : TM \to M$ which, for each $x \in M$, sends every vector $X \in T_x M$ to x. The preimages $\pi^{-1}(x) = T_x M$ are called *fibers* of the bundle, for reasons that might not be obvious at this stage, but will become more so as we see more examples.

Example 2.2 (Trivial bundles). This is the simplest and least interesting example of a vector bundle, but is nonetheless important. For any manifold M of dimension n, let $E = M \times \mathbb{R}^m$ for some $m \in \mathbb{N}$. This is a manifold of dimension $n + m$, with a natural projection map

$$\pi : M \times \mathbb{R}^m \to M : (x, v) \mapsto x.$$

We associate with every point $x \in M$ the set $E_x := \pi^{-1}(x) = \{x\} \times \mathbb{R}^m$, which is called the *fiber over* x and carries the structure of a real m-dimensional vector space. The manifolds E and M, together with the projection map $\pi : E \to M$, are collectively called the *trivial real vector bundle of rank* m *over* M. Any map $s : M \to E$ of the form

$$s(x) = (x, f(x))$$

is called a *section* of the bundle $\pi : E \to M$; here $f : M \to \mathbb{R}^m$ is an arbitrary map. The terminology comes from the geometric interpretation of the image $s(M) \subset E$ as a "cross section" of the space $M \times \mathbb{R}^m$. Sections can also be characterized as maps $s : M \to E$ such that $\pi \circ s = \mathrm{Id}_M$, a definition which will generalize nicely to other bundles.

Remark 2.3. A word about notation before we continue: since the essential information of a vector bundle is contained in the projection map π, one usually denotes a bundle by "$\pi : E \to M$". This notation is convenient because it also mentions the manifolds E and M, the *total space* and *base* respectively. When there's no ambiguity, we often omit π itself and denote a bundle simply by $E \to M$, or even just E.

Consider for one more moment the first example, the tangent bundle $\pi : TM \to M$. A *section* of the bundle $\pi : TM \to M$ is now defined to be any map $s : M \to TM$ that associates with each $x \in M$ a vector in the fiber $T_x M$; equivalently, $s : M \to TM$ is a section if $\pi \circ s = \mathrm{Id}_M$. Thus "section of the tangent bundle" is simply another term for a *vector field* on M. We denote the space of vector fields by $\mathrm{Vec}(M)$.

If M is a surface embedded in $\mathbb{R}^3$, then it is natural to think of each tangent space $T_x M$ as a 2-dimensional subspace of $\mathbb{R}^3$. This makes $\pi : TM \to M$ a *smooth subbundle* of the trivial bundle $M \times \mathbb{R}^3 \to M$, in that each fiber is a linear subspace of the corresponding fiber of the trivial bundle.

Example 2.4 (**The cotangent bundle**). Associated with the tangent bundle $TM \to M$, there is a "dual bundle" $T^*M \to M$, called the *cotangent bundle*: its fibers are the vector spaces

$$T_x^* M = \mathrm{Hom}_{\mathbb{R}}(TM, \mathbb{R}),$$

consisting of real linear maps $T_x M \to \mathbb{R}$, also known as *dual vectors*. The sections of T^*M are thus precisely the differential 1-forms on M. This is our first example of an important vector bundle that cannot easily be visualized, though one can still imagine that the vector spaces T^*M vary smoothly in some sense with respect to $x \in M$.

Observe that T^*M has the same rank as TM, and indeed there are always isomorphisms $T_x M \cong T_x^* M$ for each x, but the choice of such an isomorphism is not canonical. This is not a trivial comment: once we define what "isomorphism" means for vector bundles, it will turn out that TM and T^*M are often *not* isomorphic as bundles, even though the individual fibers $T_x M$ and $T_x^* M$ always are.

Example 2.5 (**Tensor bundles**). The tangent and cotangent bundles are both examples of a more general construction, the *tensor bundles* $T_\ell^k M \to M$, whose sections are the *tensor fields of type* (k, ℓ) on M. For integers $k \geq 0$ and $\ell \geq 0$, the fiber $(T_\ell^k M)_x$ over $x \in M$ is defined to be the vector space

$$\left(\bigotimes_{j=1}^{\ell} T_x^* M \right) \otimes \left(\bigotimes_{j=1}^{k} T_x M \right),$$

which consists of multilinear maps

$$\underbrace{T_xM \times \ldots \times T_xM}_{\ell} \times \underbrace{T_x^*M \times \ldots \times T_x^*M}_{k} \to \mathbb{R}.$$

Thus $(T_\ell^k M)_x$ has dimension $n^{k+\ell}$, and

$$T_\ell^k M \to M$$

is a real vector bundle of rank $n^{k+\ell}$. In particular, $T_0^1 M = TM$ has rank n, and so does $T_1^0 M = T^*M$. For $k = \ell = 0$, we obtain the *trivial line bundle* $T_0^0 M = M \times \mathbb{R}$, whose sections are simply real valued functions on M. For $k \ge 0$, $T_k^0 M$ contains an important subbundle

$$T_k^0 M \supset \Lambda^k T^* M \to M$$

of rank $\frac{n!}{k!(n-k)!}$, whose sections are the differential k-forms on M.[1]

Notation. In all that follows, we will make occasional use of the *Einstein summation convention*, which already made an appearance in the introduction, §1.2. This is a notational shortcut by which a summation is implied over any index that appears as a pair of upper and lower indices. So for instance if the index j runs from 1 to n,

$$\lambda_j \, dx^j := \sum_{j=1}^n \lambda_j \, dx^j, \quad \text{and} \quad X^j \frac{\partial}{\partial x^j} := \sum_{j=1}^n X^j \frac{\partial}{\partial x^j};$$

in the latter expression, the index in ∂x^j counts as a "lower" index by convention because it's in the denominator. The summation convention and the significance of "upper" vs. "lower" indices is discussed more fully in Appendix A.

Example 2.6 (**Distributions**)**.** Suppose λ is a smooth 1-form on the manifold M which is everywhere nonzero. Then we claim that at every point $p \in M$, the kernel of $\lambda|_{T_pM}$ is an $(n-1)$-dimensional subspace $\xi_p \subset T_pM$. To see this, choose coordinates $(x^1, \ldots, x^n)$ near a point p_0 so that λ can be written as

$$\lambda = \lambda_j \, dx^j,$$

using the set of real-valued "component functions" $\lambda_1(p), \ldots, \lambda_n(p)$ defined near p_0. By assumption the vector $(\lambda_1(p), \ldots, \lambda_n(p)) \in \mathbb{R}^n$ is never zero, and the kernel of λ at p is precisely the set of tangent vectors

$$X = X^j \frac{\partial}{\partial x^j} \in T_pM$$

[1] The notions of multilinear maps and tensor products used here are discussed in detail in Appendix A.

such that $\lambda(X) = \lambda_j(p)X^j = 0$, i.e. it is represented in coordinates by the *orthogonal complement* of $(\lambda_1(p), \dots, \lambda_n(p)) \in \mathbb{R}^n$, an $(n-1)$-dimensional subspace.

The union of the subspaces $\ker \lambda|_{T_pM}$ for all p defines a smooth rank-$(n-1)$ subbundle

$$\xi \to M$$

of the tangent bundle, also known as an $(n-1)$-dimensional *distribution* on M. More generally, one can define smooth m-dimensional distributions for any $m \le n$, all of which are rank-m subbundles of TM. In this case, "smoothness" means that in some neighborhood of any point $p \in M$, one can find a set of smooth vector fields $X_1, \dots, X_m$ that are linearly independent and span ξ_q at every point q close to p. A section of ξ is by definition a vector field $X \in \mathrm{Vec}(M)$ such that $X(p) \in \xi_p$ for all $p \in M$.

Example 2.7 (**A complex line bundle over** S^3)**.** All of the examples so far have been *real* vector bundles, but one can just as well define bundles with fibers that are *complex* vector spaces. For instance, define S^3 as the unit sphere in $\mathbb{R}^4$, and identify the latter with $\mathbb{C}^2$ via the correspondence

$$(x_1, y_1, x_2, y_2) \longleftrightarrow (x_1 + iy_1, x_2 + iy_2).$$

Then if $\langle\ ,\ \rangle_{\mathbb{C}^2}$ denotes the standard complex inner product on $\mathbb{C}^2$, one finds that the real inner product $\langle\ ,\ \rangle_{\mathbb{R}^4}$ on $\mathbb{R}^4$ is given by

$$\langle X, Y\rangle_{\mathbb{R}^4} = \mathrm{Re}\langle X, Y\rangle_{\mathbb{C}^2},$$

and in particular $\langle X, X\rangle_{\mathbb{R}^4} = \langle X, X\rangle_{\mathbb{C}^2}$. Thus by this definition,

$$S^3 = \{z \in \mathbb{C}^2 \mid \langle z, z\rangle_{\mathbb{C}^2} = 1\}.$$

The tangent space T_zS^3 is then the *real* orthogonal complement of the vector $z \in \mathbb{C}^2 = \mathbb{R}^4$ with respect to the inner product $\langle\ ,\ \rangle_{\mathbb{R}^4}$. But there is also a *complex* orthogonal complement

$$\xi_z := \{v \in \mathbb{C}^2 \mid \langle z, v\rangle_{\mathbb{C}^2} = 0\},$$

which is both a real 2-dimensional subspace of T_zS^3 and a complex 1-dimensional subspace of $\mathbb{C}^2$. The union of these for all $z \in S^3$ defines a vector bundle $\xi \to S^3$, which can be thought of as either a real rank-2 subbundle of $TS^3 \to S^3$ or as a complex rank-1 subbundle of the trivial complex bundle $S^3 \times \mathbb{C}^2 \to S^3$.[2]

[2]The distribution $\xi \subset TS^3$ is well known in contact geometry as the *standard contact structure* on S^3.

With these examples in mind, we are ready for the general definition. The rigorous discussion will begin in a purely topological context, where the notion of continuity is well defined but smoothness is not—smooth structures on vector bundles will be introduced in the next section.

Notation. In everything that follows, we choose a field

$$\mathbb{F} = \text{either } \mathbb{R} \text{ or } \mathbb{C},$$

and assume unless otherwise noted that all vector spaces and linear maps are $\mathbb{F}$-linear. In this way the real and complex cases can be handled simultaneously.

Definition 2.8. A *vector bundle* of rank m consists of a pair of topological spaces E and M, with a continuous surjective map $\pi : E \to M$ such that for each $x \in M$, the subset $E_x := \pi^{-1}(x) \subset E$ is a vector space isomorphic to $\mathbb{F}^m$, and for every point $x \in M$ there exists an open neighborhood $x \in \mathcal{U} \subset M$ and a *local trivialization*

$$\Phi : \pi^{-1}(\mathcal{U}) \to \mathcal{U} \times \mathbb{F}^m.$$

Here Φ is a homeomorphism which restricts to a linear isomorphism $E_y \to \{y\} \times \mathbb{F}^m$ for each $y \in \mathcal{U}$.

We call E the *total space* of the bundle $\pi : E \to M$, and M is called the *base*. For each $x \in M$, the set $E_x = \pi^{-1}(x) \subset E$ is called the *fiber* over x.

Definition 2.9. A vector bundle of rank 1 is called a *line bundle*.

A bundle of rank m is sometimes also called an *m-dimensional* vector bundle, or an *m-plane bundle*. The use of the term "line bundle" for $m = 1$ is quite intuitive when $\mathbb{F} = \mathbb{R}$; one must keep in mind however that in the complex case, the fibers should be visualized as *planes* rather than lines.

Given a vector bundle $\pi : E \to M$ and any subset $\mathcal{U} \subset M$, denote $E|_{\mathcal{U}} = \pi^{-1}(\mathcal{U})$. As in the definition above, a *trivialization* over $\mathcal{U}$ is a homeomorphism

$$\Phi : E|_{\mathcal{U}} \to \mathcal{U} \times \mathbb{F}^m$$

that restricts linearly to the fibers as isomorphisms

$$E_x \to \{x\} \times \mathbb{F}^m$$

for each $x \in \mathcal{U}$. One cannot expect such a trivialization to exist for every subset $\mathcal{U} \subset M$; when it does exist, we say that $\pi : E \to M$ is *trivializable over $\mathcal{U}$*. Thus every vector bundle is required to be *locally trivializable*, in the sense that trivializations exist over some neighborhood of each point in the base. The bundle is called (globally) *trivializable*, or simply *trivial*, if there exists a trivialization over the entirety of M. We've already seen the archetypal example of a globally trivial bundle: the product $E = M \times \mathbb{F}^m$. Nontrivial bundles are of course more interesting, and are also in some sense more common: e.g. the tangent bundle of any closed surface other than a torus is nontrivial, though we're not yet in a position to prove this. Example 2.12 below shows a nontrivial line bundle that is fairly easy to understand.

Definition 2.10. A *section* of the bundle $\pi : E \to M$ is a map $s : M \to E$ such that $\pi \circ s = \mathrm{Id}_M$.

The space of continuous sections on a vector bundle $\pi : E \to M$ is denoted by

$$\Gamma(E) = \{s \in C(M, E) \mid \pi \circ s = \mathrm{Id}_M\}.$$

This is an infinite dimensional vector space, with addition and scalar multiplication defined pointwise. In particular, every vector bundle has a preferred section $0 \in \Gamma(E)$, called the *zero section*, which maps each point $x \in M$ to the zero vector in E_x. This gives a natural embedding $M \hookrightarrow E$.

Remark 2.11. As we mentioned already in Example 2.2, the term "section" refers to the geometric interpretation of the image $s(M)$ as a subset of the total space E. One can alternately define a continuous section of the bundle $\pi : E \to M$ to be any subset $\Sigma \subset E$ such that $\pi|_\Sigma : \Sigma \to M$ is a homeomorphism.

Example 2.12 (**A nontrivial line bundle**)**.** Identify S^1 with the unit circle in $\mathbb{C}$, and define a bundle $\ell \to S^1$ to be the union of the sets $\{e^{i\theta}\} \times \ell_{e^{i\theta}} \subset S^1 \times \mathbb{R}^2$ for all $\theta \in \mathbb{R}$, where the fibers $\ell_{e^{i\theta}}$ are the 1-dimensional subspaces

$$\ell_{e^{i\theta}} = \mathbb{R} \begin{pmatrix} \cos(\theta/2) \\ \sin(\theta/2) \end{pmatrix} \subset \mathbb{R}^2.$$

If we consider the subset

$$\{(e^{i\theta}, v) \in \ell \mid \theta \in \mathbb{R}, \ |v| \le 1\}$$

consisting only of vectors of length at most 1, we obtain a Möbius strip. Observe that this bundle does not admit any continuous section that is nowhere zero. It follows then from Exercise 2.13 below that ℓ is not globally trivializable. *Local* trivializations are, however, easy to construct: e.g. for any point $p \in S^1$, define

$$\Phi : \ell|_{S^1 \setminus \{p\}} \to (S^1 \setminus \{p\}) \times \mathbb{R} : \left(e^{i\theta}, c \begin{pmatrix} \cos(\theta/2) \\ \sin(\theta/2) \end{pmatrix}\right) \mapsto (e^{i\theta}, c).$$

But this trivialization can never be extended continuously to the point p. (Spend a little time convincing yourself that this is true.)

Exercise 2.13. Show that a line bundle $\pi : E \to M$ is trivial if and only if there exists a continuous section $s : M \to E$ that is nowhere zero.

The example above shows why one often thinks of nontrivial vector bundles as being "twisted" in some sense—this word is often used in geometry and topology as well as in physics to describe theories that depend crucially on the *nontriviality* of some bundle.

Given two vector bundles $E \to M$ and $F \to M$, of rank m and ℓ respectively, a *linear bundle map* $A : E \to F$ is any continuous map that restricts to a linear map $E_x \to F_x$ for each $x \in M$. A *bundle isomorphism* is a homeomorphism which is also a linear bundle map—in this case, the inverse is a linear bundle map as well. Hence $E \to M$ is trivializable if and only if it is isomorphic to a trivial bundle $M \times \mathbb{F}^m \to M$.

There is such a thing as a *nonlinear bundle map* as well, also called a *fiber preserving* map. The map $f : E \to F$ is called *fiber preserving* if for all $x \in M$, $f(E_x) \subset F_x$.

Definition 2.14. Let $\pi : E \to M$ be a vector bundle. A *subbundle* of E is a subset $F \subset E$ such that the restriction $\pi|_F : F \to M$ is a vector bundle, and the inclusion $F \hookrightarrow E$ is a continuous linear bundle map.

Example 2.15. The line bundle of Example 2.12 is a subbundle of the trivial 2-plane bundle $S^1 \times \mathbb{R}^2 \to S^1$.

Real vector bundles can be characterized as orientable or non-orientable. Recall that an *orientation* for a real m-dimensional vector space V is a choice of an equivalence class of ordered bases

$$(v_{(1)}, \dots, v_{(m)}) \subset V,$$

where two such bases are equivalent if one can be deformed into the other through a continuous family of bases. A basis in the chosen equivalence class is called a *positively oriented* basis, while others are called *negatively oriented*. There are always two choices of orientation, owing to the fact that the general linear group $\mathrm{GL}(m, \mathbb{R})$ has two connected components, distinguished by the sign of the determinant.[3] If V and W are oriented vector spaces, then an isomorphism $A : V \to W$ is called *orientation preserving* if it maps every positively oriented basis of V to a positively oriented basis of W; otherwise, it is *orientation reversing*. For the space $\mathbb{R}^m$ with its natural orientation given by the standard basis of unit vectors $(e_{(1)}, \dots, e_{(m)})$, an invertible matrix $A : \mathbb{R}^m \to \mathbb{R}^m$ is orientation preserving if and only if $\det A > 0$.

This notion can be extended to a real vector bundle $\pi : E \to M$ by choosing orientations on each fiber E_x so that the orientations "vary continuously with x". More precisely:

[3] By contrast, $\mathrm{GL}(m, \mathbb{C})$ is connected, which is why orientation makes no sense for complex vector spaces.

Definition 2.16. An *orientation* of the real vector bundle $\pi : E \to M$ is a choice of orientation for every fiber E_x, such that for any local trivialization $\Phi : E|_{\mathcal{U}} \to \mathcal{U} \times \mathbb{R}^m$, the induced isomorphisms $E_x \to \{x\} \times \mathbb{R}^m$ for $x \in \mathcal{U}$ are all either orientation preserving or orientation reversing.

Not every real bundle admits an orientation; those that do are called *orientable.*

Exercise 2.17. Show that a real line bundle is orientable if and only if it is trivializable. In particular, the bundle of Example 2.12 is not orientable.

A smooth manifold M is called orientable if its tangent bundle is orientable. This is equivalent to a more general notion of orientation which is also defined for *topological* manifolds, even without a well defined tangent bundle. The more general definition is framed in terms of homology theory; see for example [Hat02].

Example 2.18. Using again the nontrivial line bundle $\ell \to S^1$ of Example 2.12, let

$$\Sigma = \{(e^{i\theta}, v) \in \ell \mid |v| \le 1\}.$$

This is a well known manifold with boundary: the *Möbius strip*. To see that it is non-orientable, imagine placing your thumb and index finger on the surface to form an oriented basis of the tangent space at some point: assume the thumb is the first basis vector and the index finger is the second. Now imagine moving your hand once around, deforming the basis continuously as it goes. When it comes back to the same point, you will have a basis that cannot be deformed back to the original one—you would have to rotate your hand to recover its original position, pointing in directions not tangent to Σ along the way.

Other popular examples of non-orientable surfaces include the *projective plane* and the *Klein bottle*, cf. [Spi99].

Definition 2.19. Let $\pi : E \to M$ be a vector bundle of rank m, and choose a subset $\mathcal{U} \subset M$ of the base. A *frame*

$$(v_{(1)}, \dots, v_{(m)})$$

over $\mathcal{U}$ is an ordered set of continuous sections $v_{(j)} \in \Gamma(E|_{\mathcal{U}})$, such that for every $x \in \mathcal{U}$ the set $(v_{(1)}(x), \dots, v_{(m)}(x))$ forms a basis of E_x.

A *global frame* is a frame over the entire base M; as the next statement shows, this can only exist if the bundle is trivializable.

Proposition 2.20. *A frame $(v_{(1)}, \dots, v_{(m)})$ over $\mathcal{U} \subset M$ for the bundle $\pi : E \to M$ determines a trivialization*

$$\Phi : E|_{\mathcal{U}} \to \mathcal{U} \times \mathbb{F}^m$$

such that for each $x \in \mathcal{U}$ and $j \in \{1, \dots, m\}$, $\Phi(v_{(j)}(x)) = (x, e_{(j)})$, where $(e_{(1)}, \dots, e_{(m)})$ is the standard basis of unit vectors in $\mathbb{F}^m$.

Proof. One simply defines $\Phi(v_{(j)}(x)) = (x, e_{(j)})$ and extends Φ to the rest of the fiber E_x by linearity: thus using the summation convention to express arbitrary vectors $X = X^j v_{(j)} \in E_x$ in terms of components $X^j \in \mathbb{F}$,

$$\Phi(X) = (x, X^j e_{(j)}) = \left(x, \begin{pmatrix} X^1 \\ \vdots \\ X^m \end{pmatrix} \right).$$

The resulting map on $E|_{\mathcal{U}}$ is clearly continuous. □

Obvious as this result seems, it's useful for constructing and for visualizing trivializations, since frames are in some ways more intuitive objects.

2.2 Smoothness

So far this is all purely topological. To bring things into the realm of differential geometry, some notion of "differentiability" is needed.

To define this, we note first that every vector bundle admits a *system of local trivializations*, i.e. a set of trivializations $\Phi_\alpha : E|_{\mathcal{U}_\alpha} \to \mathcal{U}_\alpha \times \mathbb{F}^m$ such that the open sets $\{\mathcal{U}_\alpha\}$ cover M. Such a system defines a set of continuous *transition maps*

$$g_{\beta\alpha} : \mathcal{U}_\alpha \cap \mathcal{U}_\beta \to \mathrm{GL}(m, \mathbb{F}),$$

so that each of the maps $\Phi_\beta \circ \Phi_\alpha^{-1} : (\mathcal{U}_\alpha \cap \mathcal{U}_\beta) \times \mathbb{F}^m \to (\mathcal{U}_\alpha \cap \mathcal{U}_\beta) \times \mathbb{F}^m$ takes the form

$$(x, v) \mapsto (x, g_{\beta\alpha}(x)v).$$

Exercise 2.21. Show that a real vector bundle $\pi : E \to M$ is orientable if and only if it admits a system of local trivializations for which the transition maps $g_{\beta\alpha} : \mathcal{U}_\alpha \cap \mathcal{U}_\beta \to \mathrm{GL}(m, \mathbb{R})$ all satisfy $\det g_{\beta\alpha}(x) > 0$.

Definition 2.22. Suppose M is a smooth manifold and $\pi : E \to M$ is a vector bundle. A *smooth structure* on $\pi : E \to M$ is a maximal system of local trivializations which are *smoothly compatible*, meaning that all transition maps $g_{\beta\alpha} : \mathcal{U}_\alpha \cap \mathcal{U}_\beta \to \mathrm{GL}(m, \mathbb{F})$ are smooth.

The bundle $\pi : E \to M$ together with a smooth structure is called a *smooth vector bundle.*

The smoothness of the transition maps $g_{\beta\alpha} : \mathcal{U}_\alpha \cap \mathcal{U}_\beta \to \mathrm{GL}(m, \mathbb{F})$ is judged by choosing local coordinates on M and treating $\mathrm{GL}(m, \mathbb{F})$ as an open subset of the vector space of $\mathbb{F}$-linear transformations on $\mathbb{F}^m$.

Remark 2.23. One can also define vector bundles *of class C^k*, by requiring the transition maps to be k-times differentiable. For this to make sense when $k \geq 1$, the base M must also have the structure of a C^k-manifold, or better. By this definition, all vector bundles are bundles of class C^0

(M need not even be a manifold when $k = 0$, only a topological space). Almost all statements below that use the word "smooth" can be adapted to assume only finitely many derivatives.

Proposition 2.24. *If $\pi : E \to M$ is a smooth vector bundle of rank m and $\dim M = n$, then the total space E admits a natural smooth manifold structure such that π is a smooth map. The dimension of E is $n + m$ if $\mathbb{F} = \mathbb{R}$, or $n + 2m$ if $\mathbb{F} = \mathbb{C}$.*

Exercise 2.25. Prove Proposition 2.24. *Hint:* for $x \in \mathcal{U}_\alpha$ and $v \in E_x$, use the trivialization Φ_α and a coordinate chart for some open neighborhood $x \in \mathcal{U} \subset \mathcal{U}_\alpha$ to define a coordinate chart for the open neighborhood $v \in E|_{\mathcal{U}} \subset E$.

The trivial bundle $M \times \mathbb{R}^m \to M$ obviously has a natural smooth structure if M is a smooth manifold. We can construct a smooth structure for the tangent bundle $TM \to M$ as follows. Pick an open covering $M = \bigcup_\alpha \mathcal{U}_\alpha$ and a collection of smooth charts $\varphi_\alpha : \mathcal{U}_\alpha \to \Omega_\alpha \subset \mathbb{R}^n$. Then define local trivializations $\Phi_\alpha : TM|_{\mathcal{U}_\alpha} \to \mathcal{U}_\alpha \times \mathbb{R}^n$ by

$$\Phi_\alpha(X) = (x, d\varphi_\alpha(x)X),$$

for $x \in \mathcal{U}_\alpha$ and $X \in T_xM$. Since $\varphi_\beta \circ \varphi_\alpha^{-1}$ is always a smooth diffeomorphism between open subsets of $\mathbb{R}^n$, the transition maps

$$g_{\beta\alpha}(x) = d(\varphi_\beta \circ \varphi_\alpha^{-1})(\varphi_\alpha(x)) \in \mathrm{GL}(n, \mathbb{R})$$

are smooth.

Exercise 2.26. Verify that the tensor bundles and distributions from Examples 2.5, 2.6 and 2.7 are all smooth vector bundles.

In the context of smooth vector bundles, it makes sense to require that all sections and bundle maps be smooth. We thus redefine $\Gamma(E)$ as the vector space of all *smooth sections*,

$$\Gamma(E) = \{s \in C^\infty(M, E) \mid \pi \circ s = \mathrm{Id}_M\}.$$

Note that this definition requires the smooth manifold structure of the total space E, using Proposition 2.24. The following exercise demonstrates an alternative definition which is also quite revealing, and makes no reference to the total space.

Exercise 2.27. Show that the following notion of a smooth section is equivalent to the definition given above: a smooth section is a map $s : M \to E$ such that for every smooth local trivialization $\Phi : E|_{\mathcal{U}} \to \mathcal{U} \times \mathbb{F}^m$, the map $\Phi \circ s|_{\mathcal{U}} : \mathcal{U} \to \mathcal{U} \times \mathbb{F}^m$ takes the form

$$\Phi \circ s(x) = (x, f(x))$$

for some smooth map $f : \mathcal{U} \to \mathbb{F}^m$.

For bundles of class C^k with $k \leq \infty$, it makes sense to speak also of C^r-sections for any $r \leq k$; these can be defined analogously to smooth sections, by either of the two approaches outlined above. The space of sections of class C^r is denoted by $C^r(E)$.

Similar remarks apply to linear bundle maps between differentiable vector bundles $E \to M$ and $F \to M$ of rank m and ℓ respectively. In particular, a smooth bundle map $A : E \to F$ can be expressed in local trivializations as a map of the form

$$(x, v) \mapsto (x, B(x)v)$$

for some smooth function $B(x)$ with values in $\mathrm{Hom}(\mathbb{F}^m, \mathbb{F}^\ell)$.

Proposition 2.20 can now be ammended to say that any *smooth* frame $(v_{(1)}, \ldots, v_{(m)})$ over an open subset or submanifold $\mathcal{U} \subset M$ determines a smooth trivialization over $\mathcal{U}$.

We also have an addendum to Definition 2.14: a subbundle $F \subset E$ is called a *smooth subbundle* if $\pi|_F : F \to M$ admits a smooth structure such that the inclusion $F \hookrightarrow E$ is a smooth linear bundle map.

Exercise 2.28. If $\xi \subset TM$ is a smooth distribution on the manifold M, show that $\xi \to M$ is a smooth subbundle of $TM \to M$.

Exercise 2.29. If M is a smooth submanifold of $\mathbb{R}^N$, show that $TM \to M$ is a smooth subbundle of the trivial bundle $M \times \mathbb{R}^N \to M$.

Many of the statements that follow can be applied in obvious ways to either general vector bundles or smooth vector bundles. Since we will need to assume smoothness when connections are introduced in Chapter 3, the reader may as well assume for simplicity that from now on all bundles, sections and bundle maps are *smooth*—with the understanding that this assumption is not always necessary.

2.3 Operations on vector bundles

There are many ways in which one or two vector bundles can be combined or enhanced to create a new bundle. We now outline the most important examples; in all cases it is a straightforward exercise to verify that the new objects admit (continuous or smooth) vector bundle structures.

Bundles of linear maps

For two vector spaces V and W, denote by

$$\mathrm{Hom}(V, W)$$

the vector space of linear maps $V \to W$. This notation assumes that both spaces are defined over the same field $\mathbb{F}$, in which case $\mathrm{Hom}(V, W)$ is also an $\mathbb{F}$-linear vector space. If one or both spaces are complex, one can always still define a space of *real linear* maps

$$\mathrm{Hom}_{\mathbb{R}}(V, W)$$

by regarding any complex space as a real space of twice the dimension. Note that $\mathrm{Hom}_{\mathbb{R}}(V, W)$ is actually a *complex* vector space if W is complex, since the scalar multiplication

$$(\lambda A)(v) = \lambda A(v)$$

then makes sense for $\lambda \in \mathbb{C}$, $A \in \mathrm{Hom}_{\mathbb{R}}(V, W)$ and $v \in V$. If V and W are both complex spaces, we can distinguish $\mathrm{Hom}(V, W)$ from $\mathrm{Hom}_{\mathbb{R}}(V, W)$ by using $\mathrm{Hom}_{\mathbb{C}}(V, W)$ to denote the (complex) space of complex linear maps. This notation will be used whenever there is potential confusion. If $V = W$, we use the notation

$$\mathrm{End}(V) = \mathrm{Hom}(V, V),$$

with obvious definitions for $\mathrm{End}_{\mathbb{R}}(V)$ and $\mathrm{End}_{\mathbb{C}}(V)$ when V is a complex vector space.

Now for two vector bundles $E \to M$ and $F \to M$ over the same base, we define the *bundle of linear maps*

$$\mathrm{Hom}(E, F) \to M,$$

whose fibers are $\mathrm{Hom}(E, F)_x = \mathrm{Hom}(E_x, F_x)$. This has a natural smooth bundle structure if both $E \to M$ and $F \to M$ are smooth. The sections $A \in \Gamma(\mathrm{Hom}(E, F))$ are precisely the smooth linear bundle maps $A : E \to F$. We similarly denote by

$$\mathrm{End}(E) \to M$$

the bundle of linear maps from E to itself. The bundles $\mathrm{Hom}_{\mathbb{F}}(E, F) \to M$ and $\mathrm{End}_{\mathbb{F}}(E)$ for $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ have natural definitions, relating to the discussion above.

When V and W are both complex vector spaces, we can regard $\mathrm{Hom}_{\mathbb{C}}(V, W)$ as a complex subspace of $\mathrm{Hom}_{\mathbb{R}}(V, W)$; indeed,

$$\mathrm{Hom}_{\mathbb{C}}(V, W) = \{A \in \mathrm{Hom}_{\mathbb{R}}(V, W) \mid A(iv) = iA(v) \text{ for all } v \in V\}.$$

Another interesting subspace is the space of complex *antilinear* maps

$$\overline{\mathrm{Hom}}_{\mathbb{C}}(V, W) = \{A \in \mathrm{Hom}_{\mathbb{R}}(V, W) \mid A(iv) = -iA(v) \text{ for all } v \in V\}.$$

Thus for complex vector bundles $E \to M$ and $F \to M$, we can define bundles of complex antilinear maps $E \to F$ and $E \to E$ respectively:

$$\overline{\mathrm{Hom}}_{\mathbb{C}}(E, F) \to M \qquad \text{and} \qquad \overline{\mathrm{End}}_{\mathbb{C}}(E) \to M.$$

These are subbundles of $\mathrm{Hom}_{\mathbb{R}}(E, F) \to M$ and $\mathrm{End}_{\mathbb{R}}(E) \to M$ respectively.

The dual bundle

The *dual space* V^* of a vector space V is $\mathrm{Hom}(V, \mathbb{F})$, and its elements are called *dual vectors*. The extension of this concept to bundles is a special case of the definition above: we choose F to be the trivial line bundle $M \times \mathbb{F} \to M$, and define the *dual bundle* of $E \to M$ by

$$E^* = \mathrm{Hom}(E, M \times \mathbb{F}) \to M.$$

For each $x \in M$, the fiber $E_x^* = \mathrm{Hom}(E_x, \mathbb{F})$ is the dual space of E_x.

For example, the dual of a tangent bundle $TM \to M$ is the *cotangent bundle* $T^*M \to M$, whose sections are the 1-forms on M.

Direct sums

The *direct sum* of a pair of vector spaces V and W is the vector space

$$V \oplus W = \{(v, w) \mid v \in V, w \in W\}$$

with addition defined by $(v, w) + (v', w') = (v + v', w + w')$ and scalar multiplication $c(v, w) = (cv, cw)$. Of course, as a set (and as a group under addition), $V \oplus W$ is the same thing as $V \times W$. The reason for the redundant notation is that $V \oplus W$ gives rise to a natural operation on two vector bundles $E \to M$ and $F \to M$, producing a new bundle over M that really is not the Cartesian product $E \times F$. The direct sum bundle

$$E \oplus F \to M$$

is defined to have fibers $(E \oplus F)_x = E_x \oplus F_x$ for $x \in M$.

Exercise 2.30. For two complex vector bundles $E \to M$ and $F \to M$, show that $\mathrm{Hom}_{\mathbb{R}}(E, F) = \mathrm{Hom}_{\mathbb{C}}(E, F) \oplus \overline{\mathrm{Hom}}_{\mathbb{C}}(E, F)$.

It's surprisingly complicated to visualize the total space of $E \oplus F$: notably, it's certainly not $E \times F$. The latter consists of all pairs (v, w) such that $v \in E_x$ and $w \in F_y$ for any $x, y \in M$, whereas $(v, w) \in E \oplus F$ only if $x = y$, i.e. v and w are in fibers over the *same point* of the base.

One *can* define a bundle whose total space is $E \times F$; in fact, for this we no longer need assume that E and F have the same base. Given $E \to M$ and $F \to N$, one can define a Cartesian product bundle $E \times F \to M \times N$ with fibers $(E \times F)_{(x,y)} = E_x \oplus F_y$. For our purposes however, this operation is less interesting than the direct sum.

Tensor products

For vector spaces V and W of dimension m and n respectively, the *tensor product* $V \otimes W$ is a vector space of dimension mn. There are various ways to define this, details of which we leave to Appendix A. For now let us simply recall the most important example of a tensor product that arises in differential geometry:

$$V_\ell^k = \left(\bigotimes_{i=1}^{\ell} V^* \right) \otimes \left(\bigotimes_{j=1}^{k} V \right)$$

is the $n^{k+\ell}$-dimensional vector space of all multilinear maps

$$\underbrace{V \times \ldots \times V}_{\ell} \times \underbrace{V^* \times \ldots \times V^*}_{k} \to \mathbb{F}.$$

Tensor products extend to bundles in the obvious way: given $E \to M$ and $F \to M$, their tensor product is the bundle

$$E \otimes F \to M$$

whose fibers are $(E \otimes F)_x = E_x \otimes F_x$. There is potential confusion here if both bundles are complex, so we sometimes distinguish the bundles

$$E \otimes_{\mathbb{R}} F \to M \qquad \text{and} \qquad E \otimes_{\mathbb{C}} F \to M.$$

The former has fibers that are tensor products of E_x and F_x, treating both as *real* vector spaces.

Exercise 2.31. If V and W are complex vector spaces of dimension p and q respectively, show that $\dim_{\mathbb{C}}(V \otimes_{\mathbb{C}} W) = pq$, while $\dim_{\mathbb{R}}(V \otimes_{\mathbb{R}} W) = 4pq$. Thus these two spaces cannot be identified with one another.

Having defined dual bundles and tensor products, the idea of the tensor bundles $T_\ell^k M$ on a manifold can be extended to arbitrary vector bundles by defining

$$E_\ell^k = \left(\bigotimes_{i=1}^{\ell} E^* \right) \otimes \left(\bigotimes_{j=1}^{k} E \right).$$

The fiber $(E_\ell^k)_x$ for $x \in M$ is then the vector space of all multilinear maps

$$\underbrace{E_x \times \ldots \times E_x}_{\ell} \times \underbrace{E_x^* \times \ldots \times E_x^*}_{k} \to \mathbb{F}.$$

A little multilinear algebra (see Appendix A) shows that there is a canonical bundle isomorphism

$$E_\ell^k = \mathrm{Hom}\left(\otimes_{i=1}^{\ell} E, \otimes_{j=1}^{k} E \right);$$

in particular, $E_0^1 = E$, $E_1^0 = E^*$ and $E_1^1 = \mathrm{End}(E)$.

The tensor product bundle $E_k^0 = \otimes_{j=1}^k E^*$ has an important subbundle

$$\Lambda^k E^* \to M,$$

whose fibers $\Lambda^k E_x^*$ are spaces of alternating k-forms on E_x. One can similarly define an alternating tensor product $\Lambda^k E \subset E_0^k$, consisting of alternating k-forms on E^*. A somewhat more abstract (but also mathematically cleaner) definition of $\Lambda^k E$ is given in Appendix A.

Exercise 2.32. Show that a real vector bundle $E \to M$ of rank m is orientable if and only if the line bundle $\Lambda^m E^* \to M$ is trivializable. And that this is equivalent to the line bundle $\Lambda^m E \to M$ being trivializable.

Complexification

Just as any complex vector bundle can be treated as a real bundle with twice the rank, any real bundle determines a related complex bundle of the *same* rank. For an individual real vector space V, the *complexification* is defined as

$$V_{\mathbb{C}} = V \otimes_{\mathbb{R}} \mathbb{C},$$

where $\mathbb{C}$ is regarded as a 2-dimensional real vector space. The tensor product is a real vector space with twice the dimension of V; indeed, any $\xi \in V_{\mathbb{C}}$ can be written as a sum

$$\xi = (v_1 \otimes 1) + (v_2 \otimes i)$$

for some unique pair of vectors $v_1, v_2 \in V$. More importantly, $V_{\mathbb{C}}$ can be regarded as a *complex* vector space by defining scalar multiplication

$$c(v \otimes z) := v \otimes (cz)$$

for $c \in \mathbb{C}$. Then $\dim_{\mathbb{C}} V_{\mathbb{C}} = \dim_{\mathbb{R}} V$, and any real basis of V becomes also a complex basis of $V_{\mathbb{C}}$.

The same trick can be used do complexify a real vector bundle $E \to M$, setting

$$E_{\mathbb{C}} = E \otimes_{\mathbb{R}} (M \times \mathbb{C}),$$

where $M \times \mathbb{C} \to M$ is the trivial complex line bundle, treated as a real vector bundle of rank 2.

Pullback bundles

Suppose $\pi : E \to M$ is a smooth vector bundle and $f : N \to M$ is a smooth map, with N a smooth manifold. The bundle on M can then be "pulled back" via f to a smooth vector bundle

$$f^*\pi : f^*E \to N,$$

with fibers

$$(f^*E)_x = E_{f(x)}.$$

This is often called an *induced bundle.*

Exercise 2.33. Verify that $f^*E \to N$ admits a smooth vector bundle structure if the map $f : N \to M$ is smooth.

Exercise 2.34. For a smooth manifold M, define the *diagonal map* $\Delta : M \to M \times M : x \mapsto (x, x)$. Show that there is a natural isomorphism $\Delta^*T(M \times M) = TM \oplus TM$.

Example 2.35 (**Vector fields along a map**)**.** Pullback bundles appear naturally in many situations, e.g. in Exercise 2.34. For another example, consider a smooth 1-parameter family of smooth maps

$$f_t : N \to M$$

for $t \in (-1, 1)$. Differentiating with respect to t at $t = 0$ gives a smooth map $\xi : N \to TM$,

$$\xi(x) = \left.\frac{\partial}{\partial t} f_t(x)\right|_{t=0}.$$

Clearly $\xi(x) \in T_{f_0(x)}M$, thus ξ is a section of the pullback bundle $f_0^*TM \to N$. Such a section is also known as a *vector field along* f_0.

2.4 Vector bundles with extra structure

In this section we survey various types of structure that can be added to a vector bundle, such as metrics, volume elements and symplectic forms. The general pattern is that each additional structure allows us to restrict our attention to a system of transition maps that take values in some proper *subgroup* of the general linear group. This observation will be important when we relate these structures to principal bundles in Section 2.8.

2.4.1 Complex structures

As has been mentioned already, a complex vector bundle can always be thought of as a real bundle with one additional piece of structure, namely, a definition of "scalar multiplication by i". To formalize this, consider first a real vector space V, with $\dim V = 2m$. A *complex structure* on V is a linear map $J \in \mathrm{End}(V)$ such that $J^2 = -\mathbb{1}$, where $\mathbb{1}$ denotes the identity transformation on V. Note that no such J can exist if V has odd dimension. (Prove it!) But there are many such structures on $\mathbb{R}^{2m}$, for example the matrix

$$J_0 = \begin{pmatrix} 0 & -\mathbb{1}_m \\ \mathbb{1}_m & 0 \end{pmatrix},$$

where $\mathbb{1}_m$ denotes the m-by-m identity matrix. We call J_0 the *standard* complex structure on $\mathbb{R}^{2m}$, for the following reason: if $\mathbb{R}^{2m}$ is identified with $\mathbb{C}^m$ via the correspondence

$$(x_1, \dots, x_m, y_1, \dots, y_m) \longleftrightarrow (x_1 + iy_1, \dots, x_m + iy_m),$$

then multiplication by i on $\mathbb{C}^m$ is equivalent to multiplication on the left by J_0 in $\mathbb{R}^{2m}$.

In the same manner, a choice of complex structure J on the space V of dimension $2m$ gives V the structure of an m-dimensional complex vector space, with scalar multiplication defined by

$$(a + ib)v := (a\mathbb{1} + bJ)v.$$

Exercise 2.36. Show that every complex structure J on $\mathbb{R}^{2m}$ is similar to the standard structure J_0, i.e. there exists $S \in \mathrm{GL}(2m, \mathbb{R})$ such that $SJS^{-1} = J_0$. *Hint:* construct a basis $(v_1, \dots, v_m, w_1, \dots, w_m)$ of $\mathbb{R}^{2m}$ with $w_k = Jv_k$ for each k.

Consider now a real vector bundle $E \to M$ of rank $2m$. A *complex structure* on $E \to M$ is defined to be any smooth section $J : M \to \mathrm{End}(E)$ such that for all $x \in M$, $[J(x)]^2 = -\mathrm{Id}_{E_x}$. This defines each fiber E_x as a complex vector space of dimension m.

Proposition 2.37. *The real bundle $E \to M$ with complex structure J admits the structure of a smooth complex vector bundle of rank m.*

Proof. It suffices if for every $x \in M$, we can find a neighborhood $x \in \mathcal{U} \subset M$ and a smooth trivialization $\Phi : E|_{\mathcal{U}} \to \mathcal{U} \times \mathbb{C}^m$ which is complex linear on each fiber, i.e. for $y \in \mathcal{U}$ and $v \in E_y$,

$$\Phi(Jv) = i\Phi(v).$$

This can be done by constructing a smooth family of complex bases for the fibers in a neighborhood of x, then applying Proposition 2.20. We leave the rest as an exercise—the idea is essentially to solve Exercise 2.36, not just for a single matrix J but for a smooth family of them. □

This result can be rephrased as follows: since $\pi : E \to M$ is a smooth real vector bundle, there is already a maximal system of local trivializations $\Phi_\alpha : E|_{\mathcal{U}_\alpha} \to \mathcal{U}_\alpha \times \mathbb{R}^{2m}$ with smooth transition maps

$$g_{\beta\alpha} : \mathcal{U}_\alpha \cap \mathcal{U}_\beta \to \mathrm{GL}(2m, \mathbb{R}).$$

Identifying $\mathbb{C}^m$ with $\mathbb{R}^{2m}$ such that $i = J_0$, we can identify $\mathrm{GL}(m, \mathbb{C})$ with the subgroup

$$\mathrm{GL}(m, \mathbb{C}) = \{A \in \mathrm{GL}(2m, \mathbb{R}) \mid AJ_0 = J_0 A\}. \tag{2.1}$$

Then Proposition 2.37 says that after throwing out enough of the trivializations in the system $\{\mathcal{U}_\alpha, \Phi_\alpha\}$, we obtain a system which still covers M and such that all transition maps take values in the subgroup $\mathrm{GL}(m, \mathbb{C}) \subset \mathrm{GL}(2m, \mathbb{R})$.

2.4.2 Bundle metrics

A metric on a vector bundle is a choice of smoothly varying inner products on the fibers. To be more precise, if $\pi : E \to M$ is a smooth vector bundle, then a bundle metric is a smooth map

$$\langle\ ,\ \rangle : E \oplus E \to \mathbb{F}$$

such that the restriction to each fiber $E_x \oplus E_x$ defines a (real or complex) inner product on E_x. In the real case, $\langle \cdot, \cdot \rangle|_{E_x \oplus E_x}$ is therefore a bilinear form which is symmetric and positive definite. This is called a *Euclidean structure*, making the pair $(E, \langle\ ,\ \rangle)$ into a *Euclidean vector bundle.*

In the complex case, the form $\langle \cdot, \cdot \rangle|_{E_x \oplus E_x}$ is positive definite and *sesquilinear*, which means that for all $v, w \in E_x$ and $c \in \mathbb{C}$,

(i) $\langle v, w \rangle = \overline{\langle w, v \rangle}$,

(ii) $\langle cv, w \rangle = \bar{c}\langle v, w \rangle$ and $\langle v, cw \rangle = c\langle v, w \rangle$,

(iii) $\langle v_1 + v_2, w \rangle = \langle v_1, w \rangle + \langle v_2, w \rangle$ and $\langle v, w_1 + w_2 \rangle = \langle v, w_1 \rangle + \langle v, w_2 \rangle$,

(iv) $\langle v, v \rangle > 0$ if $v \neq 0$.

The metric $\langle\ ,\ \rangle$ is then called a *Hermitian structure*, and $(E, \langle\ ,\ \rangle)$ is a *Hermitian vector bundle.*

Proposition 2.38. *Every smooth vector bundle $\pi : E \to M$ admits a bundle metric.*

We leave the proof as an exercise—the key is that since metrics are positive definite, any sum of metrics defines another metric. Thus we can define metrics in local trivializations and add them together via a partition of unity. This is a standard argument in Riemannian geometry, see [Spi99] for details.

Example 2.39 (Riemannian manifolds). The most important application of bundle metrics is to the tangent bundle $TM \to M$ over a smooth manifold. A metric on TM is usually denoted by a smooth symmetric tensor field $g \in \Gamma(T_2^0 M)$, and the pair (M, g) is called a *Riemannian manifold*. The metric g determines a notion of length for smooth paths $\gamma : [t_0, t_1] \to M$ by setting

$$\text{length}(\gamma) = \int_{t_0}^{t_1} |\dot{\gamma}(t)|_g \, dt,$$

where by definition, $|X|_g = \sqrt{g(X, X)}$. The study of smooth manifolds with metrics and their geometric implications falls under the heading of *Riemannian geometry*. It is a very large subject—we'll touch upon a few of the fundamental principles in our later study of connections and curvature.

Example 2.40 (Quantum wave functions). An example of a Hermitian vector bundle comes from quantum mechanics, where the probability distribution of a single particle (without spin) in three-dimensional space can be modeled by a *wave function*

$$\psi : \mathbb{R}^3 \to \mathbb{C}.$$

The probability of finding the particle within a region $\mathcal{U} \subset \mathbb{R}^3$ is given by

$$P_{\mathcal{U}}(\psi) = \int_{\mathcal{U}} \langle \psi(x), \psi(x) \rangle \, dx,$$

where $\langle v, w \rangle := \bar{v}w$ defines the standard inner product on $\mathbb{C}$. Though one wouldn't phrase things this way in an introductory class on quantum mechanics, one can think of ψ as a section of the trivial Hermitian line bundle $(\mathbb{R}^3 \times \mathbb{C}, \langle\ ,\ \rangle)$. This is not merely something a mathematician would say to intimidate a physicist—in quantum field theory it becomes quite important to consider fields as sections of (possibly nontrivial) Hermitian vector bundles.

Where general vector bundles are concerned, the most important theoretical result about bundle metrics is the following. Recall that for two inner product spaces $(V, \langle\ ,\ \rangle_1)$ and $(W, \langle\ ,\ \rangle_2)$, an *isometry* $A : V \to W$ is an isomorphism such that

$$\langle Av, Aw \rangle_2 = \langle v, w \rangle_1$$

for all $v, w \in V$.

Proposition 2.41. *Let $\pi : E \to M$ be a vector bundle with a metric $\langle\ ,\ \rangle$. Then for every $x \in M$, there is an open neighborhood $x \in \mathcal{U} \subset M$ and a trivialization*

$$\Phi : E|_{\mathcal{U}} \to \mathcal{U} \times \mathbb{F}^m$$

such that the resulting linear maps $E_x \to \{x\} \times \mathbb{F}^m$ are all isometries (with respect to $\langle\ ,\ \rangle$ and the standard inner product on $\mathbb{F}^m$).

Proof. Using Proposition 2.20, it suffices to construct a frame $(v_1, \ldots, v_m)$ on some neighborhood of x so that the resulting bases on each fiber are orthonormal. One can do this by starting from an arbitrary frame and applying the Gram-Schmidt orthogonalization process. □

A trivialization that defines isometries on the fibers is called an *orthogonal* trivialization, or *unitary* in the complex case. As a result of Proposition 2.41, we can assume without loss of generality that a system of local trivializations $\{\mathcal{U}_\alpha, \Phi_\alpha\}$ for a Euclidean vector bundle consists only of orthogonal trivializations, in which case the transition maps $g_{\beta\alpha} : \mathcal{U}_\alpha \cap \mathcal{U}_\beta \to \mathrm{GL}(m, \mathbb{R})$ actually take values in the orthogonal group

$$\mathrm{O}(m) = \{A \in \mathrm{GL}(m, \mathbb{R}) \mid A^{\mathrm{T}} A = \mathbb{1}\}.$$

For a Hermitian vector bundle, instead of $\mathrm{O}(m)$ we have the unitary group

$$\mathrm{U}(m) = \{A \in \mathrm{GL}(m, \mathbb{C}) \mid A^\dagger A = \mathbb{1}\},$$

where by definition $A^\dagger$ is the complex conjugate of A^{T}. Taking this discussion a step further, we could formulate an alternative *definition* of a Euclidean or Hermitian structure on a vector bundle as a maximal system of local trivializations such that all transition maps take values in the subgroups $\mathrm{O}(m) \subset \mathrm{GL}(m, \mathbb{R})$ or $\mathrm{U}(m) \subset \mathrm{GL}(m, \mathbb{C})$. It's easy to see that such a system always defines a unique bundle metric. These remarks will be of fundamental importance later when we study structure groups.

2.4.3 Volume forms and volume elements

On any real vector space V of dimension m, a *volume form* μ can be defined as a nonzero element of the one-dimensional vector space $\Lambda^m V$, called the "top-dimensional exterior product". For example, $\mathbb{R}^m$ has a natural volume form

$$\mu = e_{(1)} \wedge \ldots \wedge e_{(m)},$$

where $(e_{(1)}, \ldots, e_{(m)})$ is the standard basis of unit vectors in $\mathbb{R}^m$. A choice of volume form $\mu \in \Lambda^m V$ determines a notion of "signed volume" for subsets of V as follows: given any ordered m-tuple of vectors $(v_1, \ldots, v_m)$, we define the oriented parallelopiped spanned by these vectors to have signed volume $\mathrm{Vol}(v_1, \ldots, v_m) \in \mathbb{R}$, where

$$v_1 \wedge \ldots \wedge v_m = \mathrm{Vol}(v_1, \ldots, v_m) \cdot \mu.$$

Here we're using the fact that $\dim \Lambda^m V = 1$ and $\mu \neq 0$ to deduce that there always is such a number. If you're not used to thinking about wedge products of vectors but are familiar with differential forms, then you can think of volume forms alternatively as follows: $\Lambda^m V^*$ is also 1-dimensional, thus the choice $\mu \in \Lambda^m V$ defines a unique alternating m-form $\omega \in \Lambda^m V^*$ such that

$$\omega(v_1, \ldots, v_m) = \mathrm{Vol}(v_1, \ldots, v_m).$$

A volume form is therefore equivalent to a choice of a nonzero alternating m-form ω on V, and almost all statements in the following about $\mu \in \Lambda^m V$ can be rephrased as statements about $\omega \in \Lambda^m V^*$.

Exercise 2.42.

(a) Show that if $\pi : E \to M$ is any (real or complex) line bundle, then a choice of smooth nowhere zero section $s \in \Gamma(E)$ determines a bundle isomorphism $E = E^*$ in a natural way.

(b) Show that for any vector bundle $\pi : E \to M$ of rank m and any integer $k \le m$, there is a canonical bundle isomorphism $\Lambda^k(E^*) = (\Lambda^k E)^*$.

(c) Conclude, as in the discussion above, that a choice of volume form for a real bundle $\pi : E \to M$ of rank m defines naturally an isomorphism $\Lambda^m E = \Lambda^m(E^*)$, and thus a corresponding volume form for the dual bundle $E^* \to M$.

You should take a moment to convince yourself that this defines something consistent with the intuitive notion of volume: in particular, multiplying any of the vectors v_j by a positive number changes the volume by the same factor, and any linear dependence among the vectors $(v_1, \ldots, v_m)$ produces a "degenerate" parallelopiped, with zero volume. The standard volume form on $\mathbb{R}^m$ is defined so that the unit cube has volume one. Somewhat less intuitively perhaps, $\mathrm{Vol}(v_1, \ldots, v_m)$ is not always positive: it changes sign if we reverse the order of any two vectors, since this changes the orientation of the parallelopiped. Thus one must generally take the absolute value $|\mathrm{Vol}(v_1, \ldots, v_m)|$ to find the geometric volume.

These remarks extend easily to a complex vector space of dimension m, though in this case volume can be a complex number—we will forego any attempt to interpret this geometrically. In either the real or complex case, a linear isomorphism

$$A : (V, \mu_1) \to (W, \mu_2)$$

between m-dimensional vector spaces with volume forms is said to be *volume preserving* if every parallelopiped $P \subset V$ has the same (signed or complex) volume as $A(P) \subset W$. For example, a matrix $A \in \mathrm{GL}(m, \mathbb{F})$ is volume preserving with respect to the natural volume form on $\mathbb{F}^m$ if and only if $\det A = 1$.

Exercise 2.43. A linear map $A : V \to W$ defines a corresponding linear map

$$A_* : \Lambda^m V \to \Lambda^m W,$$

sending $v_1 \wedge \ldots \wedge v_m$ to $Av_1 \wedge \ldots \wedge Av_m$, as well as a pullback

$$A^* : \Lambda^m W^* \to \Lambda^m V^*,$$

such that $A^*\omega(v_1, \ldots, v_m) = \omega(Av_1, \ldots, Av_m)$. Show that $A : (V, \mu_1) \to (W, \mu_2)$ is volume preserving if and only if $A_*\mu_1 = \mu_2$. Equivalently, defining volume via m-forms $\omega_1 \in \Lambda^m V^*$ and $\omega_2 \in \Lambda^m W^*$, A is volume preserving if and only if $A^*\omega_2 = \omega_1$.

For any vector bundle $\pi : E \to M$ of rank m, the top-dimensional exterior bundles $\Lambda^m E \to M$ and $\Lambda^m E^* \to M$ are line bundles. In the real case, we saw in Exercise 2.32 that these line bundles are trivial if and only if $E \to M$ is orientable, in which case there exist smooth sections $\mu : M \to \Lambda^m E$ and $\omega : M \to \Lambda^m E^*$ that are *nowhere zero*. Such a section is very far from unique, since we can always obtain another via multiplication with a smooth positive function $f : M \to (0, \infty)$.

We shall call a nowhere zero section $\mu \in \Gamma(\Lambda^m E)$ or $\omega \in \Gamma(\Lambda^m E^*)$ a *volume form* on E, because it defines a smoothly varying notion of signed volume $\mathrm{Vol}(\cdot, \dots, \cdot)$ on each fiber. This should be mostly familiar from the special case where E is the tangent bundle of a smooth n-manifold M; then a volume form is defined to be a nowhere zero differential n-form (cf. [Spi65]), i.e. a section of $\Lambda^n T^*M$.

Even though orientation doesn't make sense in the complex case, the above remarks can still be extended almost verbatim to a complex bundle $\pi : E \to M$ of rank m if $\Lambda^m E$ is trivial. A volume form is again defined as a smooth nowhere zero section $\mu \in \Gamma(\Lambda^m E)$, and it now defines a "complex volume" $\mathrm{Vol}(v_1, \dots, v_m)$ for any ordered m-tuple of vectors in the same fiber.

Proposition 2.44. *Let $\pi : E \to M$ be a vector bundle of rank m with a volume form $\mu \in \Gamma(\Lambda^m E)$. Then for every $x \in M$, there is an open neighborhood $x \in \mathcal{U} \subset M$ and a trivialization*

$$\Phi : E|_{\mathcal{U}} \to \mathcal{U} \times \mathbb{F}^m$$

such that the resulting linear maps $E_x \to \{x\} \times \mathbb{F}^m$ are all volume preserving.

Exercise 2.45. Prove Proposition 2.44. *Hint:* given any frame $(v_1, \dots, v_m)$ over a subset $\mathcal{U} \subset M$, one can multiply v_1 by a smooth function such that $\mathrm{Vol}(v_1, \dots, v_m) \equiv 1$.

In particular, a maximal system of local trivializations $\{\mathcal{U}_\alpha, \Phi_\alpha\}$ can be reduced to a system of *volume preserving* trivializations, for which all transition maps take values in the *special linear group*

$$\mathrm{SL}(m, \mathbb{F}) = \{A \in \mathrm{GL}(m, \mathbb{F}) \mid \det A = 1\}.$$

A Euclidean structure $\langle \ , \ \rangle$ on an oriented real bundle $\pi : E \to M$ determines a unique preferred volume form μ such that the oriented cube spanned by any positively oriented orthonormal basis of E_x has volume one. That this is well defined follows from Exercise 2.46 below. Now orientation preserving orthogonal transformations are also volume preserving, so we can combine Propositions 2.41 and 2.44 to assume that all transition maps take values in the *special orthogonal group*

$$\mathrm{SO}(m) = \mathrm{O}(m) \cap \mathrm{SL}(m, \mathbb{R}).$$

Note that $\mathrm{SO}(m)$ can alternatively be defined as the subgroup

$$\mathrm{SO}(m) = \{A \in \mathrm{O}(m) \mid \det A = 1\}.$$

This is equivalent to the fact that every transformation of $\mathbb{R}^m$ that preserves both the inner product and the orientation also preserves signed volume, i.e. a metric and orientation together determine a preferred volume form.

Exercise 2.46. Consider $\mathbb{F}^m$ with its standard inner product, and denote the standard unit basis vectors by $e_{(1)}, \ldots, e_{(m)}$.

(a) Show that if $(v_{(1)}, \ldots, v_{(m)})$ is any other ordered basis of $\mathbb{F}^m$ then

$$v_{(1)} \wedge \ldots \wedge v_{(m)} = \det(V) \cdot (e_{(1)} \wedge \ldots \wedge e_{(m)})$$

where $V \in \mathrm{GL}(m, \mathbb{F})$ is the matrix whose jth column is $v_{(j)} \in \mathbb{F}^m$. *Hint:* since $\Lambda^m \mathbb{F}^m$ is 1-dimensional, the expression

$$(v_{(1)}, \ldots, v_{(m)}) \mapsto \frac{v_{(1)} \wedge \ldots \wedge v_{(m)}}{e_{(1)} \wedge \ldots \wedge e_{(m)}}$$

defines an antisymmetric m-linear form on $\mathbb{F}^m$. How many such forms are there?

(b) If $\mathbb{F} = \mathbb{R}$ and we give $\mathbb{R}^m$ the standard orientation, show that for every positively oriented orthonormal basis $(v_{(1)}, \ldots, v_{(m)})$,

$$v_{(1)} \wedge \ldots \wedge v_{(m)} = e_{(1)} \wedge \ldots \wedge e_{(m)}.$$

(c) Conclude that every m-dimensional real vector space with an orientation and an inner product admits a volume form for which all positively oriented unit cubes have signed volume one.

The situation for a complex bundle $\pi : E \to M$ with Hermitian structure $\langle\ ,\ \rangle$ is slightly different. Now a volume form $\mu \in \Gamma(\Lambda^m E)$ can be called *compatible* with $\langle\ ,\ \rangle$ if every fiber has an orthonormal basis with volume one. Notice however that if $\mathrm{Vol}(v_1, \ldots, v_m) = 1$, then $(e^{i\theta} v_1, v_2, \ldots, v_m)$ is another orthonormal basis and $\mathrm{Vol}(e^{i\theta} v_1, v_2, \ldots, v_m) = e^{i\theta}$, so we cannot require that *all* orthonormal bases have volume one. Relatedly, there is no well defined notion of "orientation" that could help us choose a compatible volume form uniquely; given any compatible form μ, another such form is defined by $f\mu$ if $f : M \to \mathbb{C}$ is any smooth function with $|f| = 1$.

Given $\langle\ ,\ \rangle$ and a compatible volume form μ, an easy variation on Proposition 2.44 shows that we can assume all transition maps take values in the *special unitary group*

$$\mathrm{SU}(m) = \mathrm{U}(m) \cap \mathrm{SL}(m, \mathbb{C}).$$

The nonuniqueness of the volume form is related to the fact that $\mathrm{SU}(m)$ is a submanifold of *one dimension less* in $\mathrm{U}(m)$; by contrast, $\mathrm{SO}(m)$ and $\mathrm{O}(m)$ have the same dimension, the former being a connected component of the latter.

A non-orientable real bundle $\pi : E \to M$ of rank m does not admit any volume form, but one can still define a smoothly varying unsigned volume on the fibers. This is most easily seen in the case where there is a bundle metric $\langle\ ,\ \rangle$: then it is natural to say that all orthonormal bases span non-oriented cubes of volume one, and define a *positive* volume for all other bases and parallelopipeds accordingly. The resulting structure is called a *volume element* on the bundle $\pi : E \to M$. In particular, every Riemannian manifold (M, g), orientable or not, has a natural volume element on its tangent bundle, which one can use to define integration of real-valued functions on M.

More generally, one can define a *volume element* on any real bundle $\pi : E \to M$ of rank m to be a choice of two nonzero volume forms $\pm\mu_x \in \Lambda^m E_x$ on each fiber E_x, which vary smoothly in the following sense: every $x_0 \in M$ is contained in an open neighborhood $\mathcal{U} \subset M$ which admits a smooth volume form $\mu_{\mathcal{U}} : \mathcal{U} \to \Lambda^m E$ such that

$$\pm\mu_x = \pm\mu_{\mathcal{U}}(x)$$

for all $x \in \mathcal{U}$. The existence of volume elements in general follows from the existence of bundle metrics, with a little help from Exercise 2.46. Then the volume of parallelopipeds is defined by

$$v_1 \wedge \ldots \wedge v_m = \mathrm{Vol}(v_1, \ldots, v_m) \cdot (\pm\mu_x)$$

for any unordered basis $\{v_1, \ldots, v_m\}$ of E_x, with the plus or minus sign chosen so that volume is positive.

Exercise 2.47. Show that every real bundle with a volume element admits a system of local trivializations such that the transition maps take values in the group

$$\{A \in \mathrm{GL}(m, \mathbb{R}) \mid \det A = \pm 1\}.$$

Example 2.48 (Volume elements in coordinates). It's useful to have explicit coordinate expressions for volume elements on Riemannian manifolds, particularly if one wants to understand the physics literature, where coordinates are used in preference to geometrically invariant objects.[4] We begin by finding an expression for the volume form on a general real bundle of rank m with metric $\langle\ ,\ \rangle$, with respect to a local frame $(e_{(1)}, \ldots, e_{(m)})$

[4]For instance, physicists tend to define a *tensor* purely in terms of its coordinate expression, stipulating that it must "transform properly" with respect to coordinate changes.

on some open set $\mathcal{U} \subset M$. Since the question is purely local, we lose no generality in assuming that $E|_{\mathcal{U}} \to \mathcal{U}$ is an oriented bundle. The frame $(e_{(1)}, \ldots, e_{(m)})$ is then assumed to be positively oriented, but is otherwise arbitrary; in particular, we do not require it to be orthonormal. The metric now defines a real m-by-m symmetric matrix G with entries

$$G_{ij} = \langle e_{(i)}, e_{(j)} \rangle,$$

and using the frame to express vectors $v = v^i e_{(i)} \in E_x$ for $x \in \mathcal{U}$, we have

$$\langle v, w \rangle = G_{ij} v^i w^j.$$

Now choose an *orthonormal* frame $(\hat{e}_{(1)}, \ldots, \hat{e}_{(m)})$; the corresponding matrix $\widehat{G}$ is then the identity, with $\widehat{G}_{ij} = \delta_{ij}$. We can use this new frame to identify $E|_{\mathcal{U}} \to \mathcal{U}$ with the trivial Euclidean vector bundle $\mathcal{U} \times \mathbb{R}^m \to \mathcal{U}$, thus considering the other frame $(e_{(1)}, \ldots, e_{(m)})$ to consist of smooth $\mathbb{R}^m$-valued functions on $\mathcal{U}$. The vectors $(\hat{e}_{(1)}, \ldots, \hat{e}_{(m)})$ are now simply the standard unit basis vectors of $\mathbb{R}^m$. By Exercise 2.46, we have

$$e_{(1)} \wedge \ldots \wedge e_{(m)} = \det(V)\ \hat{e}_{(1)} \wedge \ldots \wedge \hat{e}_{(m)} = \det(V)\mu \tag{2.2}$$

where μ is the natural volume form determined by the metric and orientation, and V is the matrix whose columns are the vectors $e_{(1)}, \ldots, e_{(m)} \in \mathbb{R}^m$. We can relate the matrices V and G as follows. Computing in the orthonormal coordinates,

$$G_{ij} = \langle e_{(i)}, e_{(j)} \rangle = \widehat{G}_{k\ell} V^k_{\ i} V^\ell_{\ j} = (V^{\mathrm{T}})_i^{\ k} \delta_{k\ell} V^\ell_{\ i} = (V^{\mathrm{T}} \mathbb{1} V)_{ij} = (V^{\mathrm{T}} V)_{ij},$$

thus $G = V^{\mathrm{T}} V$, and $\det G = (\det V)^2$. Combining this with (2.2) gives a formula for the volume form in terms of the metric and the frame,

$$\mu = \frac{1}{\sqrt{\det G}}\ e_{(1)} \wedge \ldots \wedge e_{(m)}. \tag{2.3}$$

This assumes there is an orientation and $(e_{(1)}, \ldots, e_{(m)})$ is positively oriented, but it clearly applies to the non-oriented case as well if we insert $\pm$ signs in the appropriate places.

Let us apply this formula to the case of a Riemannian manifold (M, g) with coordinates $(x^1, \ldots, x^n)$ on some neighborhood $\mathcal{U} \subset M$. The coordinates determine a framing $(\partial_1, \ldots, \partial_n)$ of TM over $\mathcal{U}$, and one expresses the metric in these coordinates by the n-by-n symmetric matrix

$$g_{ij} = g(\partial_i, \partial_j),$$

so that if tangent vectors are written $X = X^i \partial_i$, then $g(X, Y) = g_{ij} X^i Y^j$. A volume element on (M, g) is actually a volume element on the *cotangent* bundle T^*M, and we want to express the natural volume element determined by g in terms of the frame $dx^1 \wedge \ldots \wedge dx^n$ on $\Lambda^n T^*M|_{\mathcal{U}}$. Assign to $\mathcal{U}$ the orientation determined by the coordinates, and let μ_g denote the corresponding volume form on $TM|_{\mathcal{U}} \to \mathcal{U}$. By (2.3) we have

$$\mu_g = \frac{1}{\sqrt{\det g}}\, \partial_1 \wedge \ldots \wedge \partial_n.$$

This is related to the differential n-form $\mathrm{Vol}(\cdot, \ldots, \cdot)$ by

$$X_1 \wedge \ldots \wedge X_n = \mathrm{Vol}(X_1, \ldots, X_n) \frac{1}{\sqrt{\det g}}\, \partial_1 \wedge \ldots \wedge \partial_n,$$

from which we compute

$$\mathrm{Vol}(X_1, \ldots, X_n) = \sqrt{\det g}\, \frac{X_1 \wedge \ldots \wedge X_n}{\partial_1 \wedge \ldots \wedge \partial_n}.$$

The fraction on the right defines an antisymmetric n-linear form on the vectors $(X_1, \ldots, X_n)$, which we see is identical to $dx^1 \wedge \ldots \wedge dx^n$. We are thus lead to the formula

$$\mathrm{Vol} = \sqrt{\det g}\; dx^1 \wedge \ldots \wedge dx^n. \tag{2.4}$$

Again, this assumes that $\mathcal{U} \subset M$ has an orientation and the coordinates are positively oriented, but we can easily remove this assumption. The proper expression for a volume *element*, expressed in arbitrary coordinates, is then

$$d\,\mathrm{vol}_g = \sqrt{\det g}\; dx^1 \ldots dx^n,$$

where the wedges have been removed to indicate that we are no longer keeping track of signs. This shows how to use local coordinate patches for computing integrals $\int_M f\; d\,\mathrm{vol}_g$ of smooth functions on Riemannian manifolds.

2.4.4 Indefinite metrics

It is sometimes useful to consider "metrics" that are not positive definite. On a real vector bundle $E \to M$, one can define an *indefinite metric* to be a smooth map $\langle\ ,\ \rangle : E \oplus E \to \mathbb{R}$ which restricts on each fiber to a bilinear form that is symmetric and *nondegenerate*; the latter condition means there is no nonzero $v \in E_x$ with $\langle v, w\rangle = 0$ for all $w \in E_x$. An example of such a bilinear form on $\mathbb{R}^4$ is the *Minkowski metric*, defined by $\langle v, w\rangle = v^{\mathrm{T}} \eta w$, where η is the symmetric matrix

$$\eta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

A form of this type, having three negative eigenvalues and one positive, is said to have *Lorentz* signature—these are important in Einstein's theory of relativity. In fact, general relativity deals exclusively with smooth 4-manifolds that have Lorentzian metrics.[5]

Given a real m-dimensional vector space V with an indefinite (i.e. symmetric and nondegenerate) inner product $\langle\ ,\ \rangle$, the *index* of $\langle\ ,\ \rangle$ is the largest dimension of any subspace $W \subset V$ on which $\langle\ ,\ \rangle|_W$ is negative definite. Thus the index is at most m, and is zero if and only if the inner product is positive definite. Choosing a basis to identify V with $\mathbb{R}^m$, the symmetric form $\langle\ ,\ \rangle$ can be written as

$$\langle v, w \rangle = v^{\mathrm{T}} \Sigma w$$

for some symmetric matrix Σ. Then by a standard result in linear algebra, $\Sigma = S^{\mathrm{T}} \Lambda S$ for some $S \in \mathrm{O}(m)$ and a diagonal matrix Λ, thus we can change the basis and assume Σ is diagonal without loss of generality. By the nondegeneracy condition, Σ must be invertible, so all its diagonal entries are nonzero. We can then rescale the basis vectors and assume finally that Σ takes the form

$$\eta_{m,k} := \begin{pmatrix} \mathbb{1}_{m-k} & \\ & -\mathbb{1}_k \end{pmatrix}$$

for some $k \in \{0, \dots, m\}$, where $\mathbb{1}_k$ denotes the k-by-k identity matrix. Clearly k is the index of $\langle\ ,\ \rangle$.

Similar remarks apply to any smooth indefinite metric $\langle\ ,\ \rangle$ on a real vector bundle $\pi : E \to M$ of rank m. On any fiber E_x, the diagonalization procedure outlined above implies that there is an *orthonormal* basis $(v_1, \dots, v_m)$ of E_x, which means in this case $\langle v_i, v_j \rangle = \pm\delta_{ij}$.

Exercise 2.49. Given an orthonormal basis on the fiber E_x, show that it can be extended to a smooth orthonormal frame over some open neighborhood of x. *Hint:* Gram-Schmidt!

We mention two important consequences of Exercise 2.49:

1. The index of $\langle\ ,\ \rangle$ is locally constant—it is thus uniquely defined over every connected component of the base M.

2. The orthonormal frames give rise to a system of local trivializations which are *orthogonal* in a generalized sense. We can then assume that the transition maps take values in the subgroup

$$\mathrm{O}(m-k, k) = \{A \in \mathrm{GL}(m, \mathbb{R}) \mid A^{\mathrm{T}} \eta_{m,k} A = \eta_{m,k}\}.$$

[5]The physical significance of Lorentz signature is that space-time is a 4-manifold in which one of the dimensions is not like the other three. Objects with mass are allowed to move through space-time only along so-called *time-like* paths $x(t)$, which satisfy $\langle \dot{x}(t), \dot{x}(t) \rangle > 0$. This means that in any coordinate system, the motion defined by $x(t)$ is slower than a certain limiting speed—the speed of light! One can also make a distinction between space-time directions that point "forward" or "backward" in time.

In the case where $E \to M$ is oriented, this can further be reduced to

$$\mathrm{SO}(m-k,k) = \mathrm{O}(m-k,k) \cap \mathrm{SL}(m,\mathbb{R}).$$

It should be noted that more refined notions of orientation are possible if k and $m-k$ are both nonzero. For instance, Minkowski space $(\mathbb{R}^4, \eta_{4,3})$ can be given separate orientations for the time-like directions and space-like hypersurfaces, and the associated group $\mathrm{O}(1,3)$ (also known as the *full Lorentz group*) has four components rather than two. In particular, the components of $\mathrm{O}(1,3)$ are distinguished by whether their elements preserve or reverse the direction of time and the orientation of space. The two components that either preserve or reverse both make up $\mathrm{SO}(1,3)$, and the subgroup $\mathrm{SO}(1,3)_+ \subset \mathrm{SO}(1,3)$ that preserves both is called the *proper orthochronous Lorentz group*. (See for example [SU01].)

For metrics on the tangent bundle of a smooth manifold, most of the basic theory of Riemannian geometry (connections, curvature, geodesics etc.) applies just as well to the indefinite case. On the other hand the existence result, Proposition 2.38, does *not* apply to nonpositive metrics in general. For example one can use covering spaces to show that S^2 does not admit any metric of index 1 (see [Spi99, Problem 9.7]).

Exercise 2.50. Adapt the arguments of Example 2.48 to write down the natural volume element determined by an indefinite metric. One thing to beware of here is that $\det g$ may be negative; but this can be fixed by inserting the appropriate sign changes.

2.4.5 Symplectic structures

On a real vector space V of even dimension $2m$, a symplectic structure is an antisymmetric bilinear form

$$\omega : V \oplus V \to \mathbb{R}$$

which is *nondegenerate*, in the sense that $\omega(v,w) = 0$ for all $w \in V$ only if $v = 0$. Note that by a simple argument in linear algebra, there is no symplectic structure on an odd-dimensional vector space (Prove it!). The standard symplectic structure on $\mathbb{R}^{2m}$ is defined as

$$\omega_0(v,w) = v^{\mathrm{T}} J_0 w,$$

where J_0 is the standard complex structure. For two symplectic vector spaces (V,ω_1) and (W,ω_2), a linear map $A : V \to W$ is called *symplectic* if

$$\omega_2(Av, Aw) = \omega_1(v,w)$$

for all $v, w \in V$. By constructing an appropriate basis (a *symplectic basis*), one can show that every symplectic vector space (V, ω) of dimension $2m$ is symplectically isomorphic to $(\mathbb{R}^{2m}, \omega_0)$. We denote by $\mathrm{Sp}(m) \subset \mathrm{GL}(2m, \mathbb{R})$ the subgroup consisting of invertible symplectic linear maps.

One can now define a *symplectic structure* on a vector bundle $\pi : E \to M$ of rank $2m$ as a smooth family of symplectic structures on the fibers, i.e. a smooth map

$$\omega : E \oplus E \to \mathbb{R}$$

that defines a symplectic structure on each fiber. The pair (E, ω) is then called a *symplectic vector bundle*, and we have the following statement about local trivializations:

Proposition 2.51. *If (E, ω) is a symplectic vector bundle over M, then for every $x \in M$, there is an open neighborhood $x \in \mathcal{U} \subset M$ and a* symplectic *trivialization, i.e. a trivialization*

$$\Phi : E|_{\mathcal{U}} \to \mathcal{U} \times \mathbb{R}^{2m}$$

such that the resulting linear maps $E_x \to \{x\} \times \mathbb{R}^{2m}$ are all symplectic (with respect to ω and the standard symplectic structure ω_0 on $\mathbb{R}^{2m}$).

The proof follows the usual recipe: one needs to construct a *symplectic frame* in a neighborhood of x, which is no more difficult in principle than constructing a symplectic basis on a single vector space. We leave the details as an exercise.

Adapting the remarks following Proposition 2.41 to this setting, a symplectic vector bundle can alternatively be defined as a bundle with a system of local trivializations for which the transition maps take values in $\mathrm{Sp}(m)$.

Note that every symplectic bundle (E, ω) of rank m inherits a natural volume form, defined on the dual bundle E^* by

$$\omega^m = \omega \wedge \ldots \wedge \omega.$$

That this is nonzero follows from the nondegeneracy of ω. Thus symplectic bundles have natural orientations and volume elements; this is related to the fact that every symplectic matrix $A \in \mathrm{Sp}(m)$ has determinant one, i.e. $\mathrm{Sp}(m) \subset \mathrm{SL}(2m, \mathbb{R})$.

Example 2.52 (**Hermitian bundles as symplectic bundles**)**.** A Hermitian bundle $(E, \langle\ ,\ \rangle)$ of rank m can be regarded as a real bundle of rank $2m$, with a complex structure J and a complex-valued bilinear form

$$\langle v, w \rangle = g(v, w) + i\omega(v, w),$$

where g and ω are real bilinear forms. It is easy to check that g defines a Euclidean structure on the real bundle E, while ω defines a symplectic structure. Thus a unitary trivialization on some subset $\mathcal{U} \subset M$ is also both an orthogonal trivialization and a symplectic trivialization. This is related to the convenient fact that if $\mathrm{GL}(m, \mathbb{C})$ is regarded as the subgroup of matrices in $\mathrm{GL}(2m, \mathbb{R})$ that commute with the standard complex structure J_0, then

$$\mathrm{U}(m) = \mathrm{O}(2m) \cap \mathrm{Sp}(m).$$

Conversely, every symplectic bundle (E, ω) can be given a *compatible* complex structure J, such that $g := \omega(\cdot, J\cdot)$ defines a Euclidean metric and $\langle \cdot, \cdot \rangle := g + i\omega$ becomes a Hermitian metric. (See [HZ94] for a proof of this fact.) Thus there is a close correspondence between symplectic and complex bundles.

Example 2.53 (**Symplectic manifolds**)**.** Just as in Riemannian geometry, the most important examples of symplectic vector bundles are tangent bundles, though here it is appropriate to add one extra condition. If ω is a symplectic structure on TM for some smooth $2n$-dimensional manifold M, then the pair (M, ω) is called an *almost symplectic manifold*. The condition for removing the word "almost" comes from observing that ω is now a differential 2-form on M; we call it a *symplectic form*, and the pair (M, ω) a *symplectic manifold*, if ω satisfies the "integrability condition"

$$d\omega = 0.$$

The importance of this condition lies largely in the following result, which shows that all symplectic manifolds are *locally* the same.

Proposition 2.54 (Darboux's theorem)**.** *If (M, ω) is a symplectic manifold and $x \in M$, then some neighborhood of x admits a coordinate system $(q_1, \ldots, q_n, p_1, \ldots, p_n)$ in which ω takes the canonical form*

$$\omega = \sum_{k=1}^{n} dp_k \wedge dq_k.$$

See [Arn89] or [HZ94] for a proof. There is no such statement for Riemannian manifolds, where the *curvature* defined by a metric g provides an obstruction for different manifolds to be locally isometric. Thus symplectic geometry is quite a different subject, and all differences between symplectic manifolds manifest themselves on the *global* rather than local level—this is the subject of the field known as *symplectic topology*.

Some motivation for the study of symplectic manifolds comes from the fact that they are the natural geometric setting for studying Hamiltonian systems. Given a symplectic manifold (M, ω) and a smooth function $H : M \to \mathbb{R}$, there is a unique smooth vector field, the *Hamiltonian vector field* $X_H \in \mathrm{Vec}(M)$, defined by the condition

$$\omega(X_H, \cdot) = -dH.$$

One then considers solutions $x : \mathbb{R} \to M$ to the differential equation $\dot{x}(t) = X_H(x(t))$. This may seem rather arbitrary at first, but any physicist will recognize the form that $\dot{x} = X_H(x)$ takes in the special coordinates provided by Darboux's theorem: writing

$$x(t) = (q_1(t), \dots, q_n(t), p_1(t), \dots, p_n(t)),$$

we find that for $k \in \{1, \dots, n\}$,

$$\dot{q}_k = \frac{\partial H}{\partial p_k} \quad \text{and} \quad \dot{p}_k = -\frac{\partial H}{\partial q_k}.$$

These are Hamilton's equations from classical mechanics, which determine the motion of a system with n degrees of freedom in phase space. Thus the symplectic form ω is precisely the structure needed for defining Hamilton's equations in a geometrically invariant way on M.

We refer to the book by Arnold [Arn89] for further details.

2.4.6 Arbitrary G-structures

One thing all of the examples above have in common is that each structure defines a system of preferred local trivializations (e.g. the orthogonal, or symplectic trivializations), with transition maps that take values in some proper subgroup $G \subset \mathrm{GL}(m, \mathbb{F})$. We now formalize this idea.

Definition 2.55. Let $\pi : E \to M$ be a vector bundle of rank m, and suppose G is a subgroup of $\mathrm{GL}(m, \mathbb{F})$. A *$G$-structure* on E is a maximal system of smooth local trivializations $\Phi_\alpha : E|_{\mathcal{U}_\alpha} \to \mathcal{U}_\alpha \times \mathbb{F}^m$ which cover M and have smooth G-valued transition maps:

$$g_{\beta\alpha} : \mathcal{U}_\alpha \cap \mathcal{U}_\beta \to G.$$

The trivializations $(\mathcal{U}_\alpha, \Phi_\alpha)$ are called *G-compatible.*

This definition is rather hard to use in practice, because one seldom wants to construct a system of local trivializations explicitly. But what it lacks in practicality, it makes up for in striking generality: all of the structures we've considered so far are special cases of G-structures. On a real vector bundle for instance, bundle metrics are equivalent to $\mathrm{O}(m)$-structures, volume forms are $\mathrm{SL}(m, \mathbb{R})$-structures and symplectic structures are $\mathrm{Sp}(m)$-structures. These particular examples can all be expressed in terms of some chosen *tensor field* on E, such as a metric or symplectic form—but in general one can define G-structures that have nothing directly to do with any tensor field. Indeed, the notion makes sense for any subgroup $G \subset \mathrm{GL}(m, \mathbb{F})$. One should of course be aware that given G and a bundle E, it cannot be assumed that G-structures always exist—this isn't even true for most of the examples above, e.g. a volume form exists only if E is orientable.

We can generalize one step further by considering not just subgroups of $\mathrm{GL}(m, \mathbb{F})$ but arbitrary groups that admit m-dimensional *representations*. By definition, an m-dimensional (real or complex) representation of G is a group homomorphism

$$\rho : G \to \mathrm{GL}(m, \mathbb{F}),$$

thus associating with each element $g \in G$ an m-by-m matrix $\rho(g)$. The representation is called *faithful* if ρ is an injective homomorphism, in which case it identifies G with a subgroup of $\mathrm{GL}(m, \mathbb{F})$. More generally, we can now make sense of the term *G-valued transition map* whenever an m-dimensional representation of G is given, since $g \in G$ then acts on vectors $v \in \mathbb{F}^m$ via the matrix $\rho(g)$. The representation need not be faithful, and G need not be a subgroup of $\mathrm{GL}(m, \mathbb{F})$.

2.5 Infinite dimensional vector bundles

With a little care, the theory of smooth manifolds and vector bundles can be extended to infinite dimensional settings, with quite powerful consequences in the study of differential equations. We won't go very deeply into this subject because the analysis is rather involved—but we can at least present the main ideas, and an example that should be sufficient motivation for further study.

Infinite dimensional spaces are typically function spaces, e.g. $C^\infty(M, \mathbb{R}^n)$, the space of all smooth maps from a smooth compact manifold M to $\mathbb{R}^n$. One of the most remarkable facts about differential calculus is that most of it can be extended in an elegant way to infinite dimensional vector spaces—however, the spaces for which this works are not arbitrary. We are most interested in the following special class of objects.

Definition 2.56. A *Banach space* is a vector space $\mathbf{X}$ with a norm $\| \; \|$ such that every Cauchy sequence converges.

Recall that a Cauchy sequence $x_k \in \mathbf{X}$ is any sequence for which the distances $\|x_k - x_j\|$ approach zero as both k and j go (independently) to infinity. The nontrivial aspect of a Banach space is that given any such sequence, there exists an element $x \in \mathbf{X}$ such that $\|x - x_k\| \to 0$, i.e. $x_k \to x$. Metric spaces with this property are called *complete*.

The requirement seems like nothing at first glance because, in fact, *all* finite dimensional normed vector spaces (real or complex) are Banach spaces. The spaces $C^k(M, \mathbb{R}^n)$ with $k < \infty$ are also Banach spaces when given suitable norms, e.g. the standard norm on $C^1([0,1], \mathbb{R}^n)$ is

$$\|f\| = \sup_{x \in [0,1]} |f(x)| + \sup_{x \in [0,1]} |f'(x)|.$$

Similarly, on any smooth vector bundle $\pi : E \to M$ over a compact base M, the spaces $C^k(E)$ of k-times differentiable sections are Banach spaces.[6] On the other hand, spaces consisting of smooth maps, e.g. $C^\infty(M, \mathbb{R}^n)$ and $\Gamma(E)$, generally *are not*. This presents a bit of a technical annoyance, because one would generally prefer to work only with spaces of smooth maps. We'll use a standard trick (Exercise 2.64) to get around this problem in our example below.

One thing to beware of in Banach spaces is that not all linear maps are continuous. For two Banach spaces $\mathbf{X}$ and $\mathbf{Y}$, we denote the space of *continuous* linear maps $\mathbf{X} \to \mathbf{Y}$ by $\mathcal{L}(\mathbf{X}, \mathbf{Y})$, with $\mathcal{L}(\mathbf{X}) := \mathcal{L}(\mathbf{X}, \mathbf{X})$. An invertible map $\mathbf{A} : \mathbf{X} \to \mathbf{Y}$ is now called an *isomorphism* if both $\mathbf{A}$ and $\mathbf{A}^{-1}$ are continuous. The space $\mathcal{L}(\mathbf{X}, \mathbf{Y})$ admits a natural norm, defined by

$$\|\mathbf{A}\|_{\mathcal{L}(\mathbf{X},\mathbf{Y})} = \sup_{x \in \mathbf{X} \setminus \{0\}} \frac{\|\mathbf{A}x\|_{\mathbf{Y}}}{\|x\|_{\mathbf{X}}},$$

and $\mathcal{L}(\mathbf{X}, \mathbf{Y})$ is then also a Banach space.

We now summarize some facts about calculus in Banach spaces. This is not the place to explore the subject in any detail, but the interested reader may consult the book by Lang [Lan93] to fill in the gaps. To start with, the following definition should look very familiar.

Definition 2.57. Let $\mathbf{X}$ and $\mathbf{Y}$ be real Banach spaces, with $\mathbf{U} \subset \mathbf{X}$ an open subset. Then a function $\mathbf{F} : \mathbf{U} \to \mathbf{Y}$ is *differentiable* at $x \in \mathbf{U}$ if there exists a continuous linear map $\mathbf{dF}(x) \in \mathcal{L}(\mathbf{X}, \mathbf{Y})$ such that for all sufficiently small $h \in \mathbf{X}$,

$$\mathbf{F}(x + h) = \mathbf{F}(x) + \mathbf{dF}(x)h + \|h\|_{\mathbf{X}} \cdot \boldsymbol{\eta}(h),$$

where $\boldsymbol{\eta}$ is a map from a neighborhood of $0 \in \mathbf{X}$ into $\mathbf{Y}$ with $\lim_{h\to 0} \boldsymbol{\eta}(h) = 0$. The linear operator $\mathbf{dF}(x)$ is called the *derivative* (or also *linearization*) of $\mathbf{F}$ at x.

The map $\mathbf{F} : \mathbf{U} \to \mathbf{Y}$ is *continuously differentiable* on $\mathbf{U}$ if $\mathbf{dF}(x)$ exists for all $x \in \mathbf{U}$ and defines a continuous function $\mathbf{dF} : \mathbf{U} \to \mathcal{L}(\mathbf{X}, \mathbf{Y})$.

Note that $\mathbf{F}$ is automatically continuous at $x \in \mathbf{U}$ if it is differentiable there.

The definition can be applied to complex Banach spaces as well by treating them as real Banach spaces. The derivative is then required to be only a *real* linear map.[7]

[6] Technically, one should not call these Banach spaces but rather *Banachable*, because although they do become Banach spaces when given suitable norms, these norms are not canonical. One must then show that all suitable choices of norms are *equivalent*, meaning that they all define the same notion of convergence.

[7] A map with a complex linear derivative is called *complex analytic*—the study of these is a different subject altogether.

The derivative of a map from an open set into a Banach space has now been defined as another map from an open set into a Banach space. Thus the process can be iterated, defining higher derivatives and the notion of *smooth* maps $\mathbf{U} \to \mathbf{Y}$. Standard features of differential calculus such as the sum rule, product rule and chain rule all have elegant generalizations to this setting.

The definitions so far make sense for all normed vector spaces, not just Banach spaces. On the other hand, the completeness condition plays a crucial role in proving the following generalization of a standard result from advanced calculus.

Theorem 2.58 (Inverse Function Theorem)**.** *Let* $\mathbf{X}$ *and* $\mathbf{Y}$ *be Banach spaces,* $\mathbf{U} \subset \mathbf{X}$ *an open subset, and* $\mathbf{F} : \mathbf{U} \to \mathbf{Y}$ *a smooth map such that for some* $x \in \mathbf{U}$ *with* $\mathbf{F}(x) = y \in \mathbf{Y}$*, the derivative* $\mathbf{dF}(x) : \mathbf{X} \to \mathbf{Y}$ *is an isomorphism. Then there are open neighborhoods* $x \in \mathcal{U} \subset \mathbf{U}$ *and* $y \in \mathcal{V} \subset \mathbf{Y}$ *such that the restriction of* $\mathbf{F}$ *to* $\mathcal{U}$ *is an invertible map* $\mathcal{U} \to \mathcal{V}$*. Its inverse* $\mathbf{F}^{-1}$ *is also smooth, and*

$$\mathbf{d}(\mathbf{F}^{-1})(y) = (\mathbf{dF}(x))^{-1} : \mathbf{Y} \to \mathbf{X}.$$

There is similarly an infinite dimensional version of the implicit function theorem, which can be treated as a corollary of the inverse function theorem. We refer to [Lan93] for a precise statement, and the proofs of both results.

We can now define smooth manifolds and vector bundles modeled on Banach spaces. Here we will present only a brief discussion of the main definitions—the recommended reference for this material is Lang's geometry book [Lan99].

Definition 2.59. A *smooth Banach manifold* is a Hausdorff topological space $\boldsymbol{M}$ with an open covering $\boldsymbol{M} = \bigcup_\alpha \boldsymbol{\mathcal{U}}_\alpha$ and a system of *charts*

$$\boldsymbol{\varphi}_\alpha : \boldsymbol{\mathcal{U}}_\alpha \to \boldsymbol{\Omega}_\alpha,$$

where each $\boldsymbol{\Omega}_\alpha$ is an open subset of some Banach space $\mathbf{X}_\alpha$, and the *transition maps* $\boldsymbol{\varphi}_\beta \circ \boldsymbol{\varphi}_\alpha^{-1}$ are smooth wherever they are defined.

Definition 2.60. Given a smooth Banach manifold $\boldsymbol{M}$, a *Banach space bundle* $\boldsymbol{\pi} : \boldsymbol{E} \to \boldsymbol{M}$ over $\boldsymbol{M}$ is a smooth Banach manifold $\boldsymbol{E}$ with a smooth surjective map $\boldsymbol{\pi}$, such that each *fiber* $\boldsymbol{E}_x = \boldsymbol{\pi}^{-1}(x)$ for $x \in \boldsymbol{M}$ is a Banach space, and there is a *system of local trivializations* $\{(\boldsymbol{\mathcal{U}}_\alpha, \boldsymbol{\Phi}_\alpha)\}$, where the open sets $\{\boldsymbol{\mathcal{U}}_\alpha\}$ cover $\boldsymbol{M}$, and $\{\boldsymbol{\Phi}_\alpha\}$ is a set of homeomorphisms

$$\boldsymbol{\Phi}_\alpha : \boldsymbol{\pi}^{-1}(\boldsymbol{\mathcal{U}}_\alpha) \to \boldsymbol{\mathcal{U}}_\alpha \times \mathbf{Y}_\alpha.$$

Here $\mathbf{Y}_\alpha$ is a Banach space, and the restriction of $\mathbf{\Phi}_\alpha$ to each fiber is a Banach space isomorphism. Furthermore, these trivializations must be *smoothly compatible*, in that for $\alpha \neq \beta$, $\mathbf{\Phi}_\beta \circ \mathbf{\Phi}_\alpha^{-1}$ takes the form

$$(x, v) \mapsto (x, \mathbf{g}_{\beta\alpha}(x)v)$$

wherever it is defined, and the *transition maps*

$$\mathbf{g}_{\beta\alpha} : \boldsymbol{\mathcal{U}}_\alpha \cap \boldsymbol{\mathcal{U}}_\beta \to \mathcal{L}(\mathbf{Y}_\alpha, \mathbf{Y}_\beta)$$

are all smooth.

This last point about the transition maps $\mathbf{g}_{\beta\alpha}$ contains some very serious analysis in the infinite dimensional case, for it is a stronger condition than just requiring that

$$(\boldsymbol{\mathcal{U}}_\alpha \cap \boldsymbol{\mathcal{U}}_\beta) \times \mathbf{Y}_\alpha \to \mathbf{Y}_\beta : (x, v) \mapsto \mathbf{g}_{\beta\alpha}(x)v$$

be a smooth map. Unlike in finite dimensional calculus, where maps are usually presumed smooth until proven nonsmooth, it typically takes considerable effort to prove that a map between Banach spaces is smooth. Thus the construction of a smooth Banach manifold or bundle structure on any given object can be a rather involved process, even if one can see intuitively that certain spaces should be Banach manifolds. A very general procedure for these constructions on so-called "manifolds of maps" was presented in a paper by H. Eliasson [Elĭ67], whose ideas can be applied to the example below.

Example 2.61 (**Stability of periodic orbits**)**.** Consider a smooth n-dimensional manifold M with a smooth time-dependent vector field X_t which is 1-periodic in time, i.e. X_t is a smooth family of smooth vector fields such that $X_{t+1} \equiv X_t$. We wish to consider the orbits $x : \mathbb{R} \to M$ of the dynamical system

$$\dot{x}(t) = X_t(x(t)), \tag{2.5}$$

and particularly those that are 1-*periodic*, satisfying $x(t+1) = x(t)$. Such a solution can be regarded as a map $x : S^1 \to M$, where the circle S^1 is defined as $\mathbb{R}/\mathbb{Z}$. Denote by $\Phi^t : M \to M$ the flow of this system from time 0 to time t, i.e. Φ^t is a smooth family of diffeomorphisms such that for any $x_0 \in M$, $x(t) := \Phi^t(x_0)$ is the unique orbit of (2.5) with $x(0) = x_0$.[8]

Given any 1-periodic solution $x(t)$ with $x(0) = x_0$, we call the orbit *nondegenerate* if the linear isomorphism

$$d\Phi^1(x_0) : T_{x_0}M \to T_{x_0}M$$

[8] Φ^t may not be globally well defined if M is noncompact, but this will cause no problem in our discussion.

does not have 1 as an eigenvalue. This implies that the orbit is *isolated*, i.e. there is no other 1-periodic orbit in some neighborhood of this one (a proof is outlined in Exercise 2.63).

Nondegeneracy has an even stronger consequence: that the orbit is "stable" under small perturbations of the system. To state this precisely, add an extra parameter to the vector field X_t^ϵ, producing a smooth 2-parameter family of vector fields, 1-periodic in t, with the real number ϵ varying over some neighborhood of 0 such that

$$X_t^0 \equiv X_t.$$

This will be called a *smooth* 1*-parameter perturbation* of the time dependent vector field X_t. We aim to prove the following:

Theorem 2.62. *Given any nondegenerate* 1*-periodic orbit* $x(t)$ *of the system* (2.5) *and a smooth* 1*-parameter perturbation* X_t^ϵ *of* X_t*, there is a unique smooth* 1*-parameter family of loops*

$$x^\epsilon : S^1 \to M$$

for ϵ *in a sufficiently small neighborhood of* 0*, such that* $\dot{x}^\epsilon = X_t^\epsilon(x^\epsilon)$ *and* $x^0(t) \equiv x(t)$*.*

Exercise 2.63. The following argument proves both Theorem 2.62 and the previous statement that a nondegenerate periodic orbit is isolated. Let $\Delta \subset M \times M$ denote the *diagonal* submanifold $\Delta = \{(x, x) \mid x \in M\}$, and consider the embedding

$$F : M \to M \times M : x \mapsto (x, \Phi^1(x)).$$

Show that F intersects Δ *transversely* at $F(x_0) \in \Delta$ if and only if the orbit $x(t)$ is nondegenerate. The result follows from this via the finite dimensional implicit function theorem—fill in the gaps. (See [Hir94] for more on transversality and its implications.)

Alternative proof of Theorem 2.62. Leaving aside Exercise 2.63 for the moment, the result can also be proved by setting up the system (2.5) as a smooth section of a Banach space bundle. Assume for simplicity that M is orientable (this restriction can be removed with a little cleverness). We first ask the reader to believe that the space of maps

$$\mathcal{B} := C^1(S^1, M)$$

admits a natural smooth Banach manifold structure, such that for each $x \in C^1(S^1, M)$, the tangent space $T_x\mathcal{B}$ is canonically isomorphic to a Banach space of sections on the pullback bundle (cf. Example 2.35),

$$T_x\mathcal{B} = C^1(x^*TM).$$

Thus $T\mathcal{B} \to \mathcal{B}$ is a smooth Banach space bundle. It is contained in another smooth Banach space bundle $\mathcal{E} \to \mathcal{B}$, with fibers

$$\mathcal{E}_x = C^0(x^*TM).$$

All of these statements can be proved rigorously using the formalism of Eliasson [Elĭ67]. We now construct a smooth section $\mathbf{s} : \mathcal{B} \to \mathcal{E}$ as follows. For a C^1-loop $x : S^1 \to M$, $\mathbf{s}(x)$ will be the C^0-vector field along x given by

$$\mathbf{s}(x)(t) = \dot{x}(t) - X_t(x(t)).$$

The fact that this section is smooth also follows from [Elĭ67]. Clearly, the C^1-solutions of the system (2.5) are precisely the zeros of the section $\mathbf{s} : \mathcal{B} \to \mathcal{E}$.

Exercise 2.64. Show that any C^k-solution to (2.5) is actually of class C^{k+1}. By induction then, all C^1-solutions are smooth! (This is the simplest example of an *elliptic bootstrapping* argument.)

We can now attack Theorem 2.62 via the infinite dimensional implicit function theorem. Let $x \in \mathcal{B}$ be an orbit of (2.5), hence $\mathbf{s}(x) = 0$. By choosing appropriate charts and local trivializations, we can identify the section $\mathbf{s}$ in a neighborhood of $x \in \mathcal{B}$ with some smooth map

$$\mathbf{F} : \mathbf{U} \to \mathbf{Y},$$

where $\mathbf{X}$ and $\mathbf{Y}$ are Banach spaces, $\mathbf{U} \subset \mathbf{X}$ is an open neighborhood of 0 and $\mathbf{F}(0) = 0$. We can also assume without loss of generality that the zero set $\mathbf{s}^{-1}(0)$ in a neighborhood of x corresponds to the set $\mathbf{F}^{-1}(0)$.

Lemma 2.65. *The linearization* $\mathbf{dF}(\mathbf{0}) : \mathbf{X} \to \mathbf{Y}$ *is an isomorphism.*

We postpone the proof of this until Chapter 3, where we will develop a convenient means of computing $\mathbf{dF}(0)$ via covariant derivatives—this is where the assumption that M is orientable will be used. To see the implications for stability of periodic orbits, consider now a smooth 1-parameter perturbation X_t^ϵ of the vector field X_t. This can be viewed as a 1-parameter perturbation $\mathbf{s}^\epsilon$ of the section $\mathbf{s}$, or equivalently, a smooth map

$$\widetilde{\mathbf{F}} : (-\delta, \delta) \times \mathbf{U} \to \mathbf{Y} : (\epsilon, u) \mapsto \mathbf{F}^\epsilon(u)$$

where $(-\delta, \delta) \subset \mathbb{R}$ is a sufficiently small interval and $\mathbf{F}^0 \equiv \mathbf{F}$. The linearization $\mathbf{d}\widetilde{\mathbf{F}}(0,0) : \mathbb{R} \oplus \mathbf{X} \to \mathbf{Y}$ takes the form

$$\mathbf{d}\widetilde{\mathbf{F}}(0,0)(h,v) = h\frac{\partial \widetilde{\mathbf{F}}}{\partial \epsilon}(0,0) + \mathbf{dF}(0)v,$$

and it follows easily from Lemma 2.65 that this operator is surjective, with a one-dimensional kernel. Thus by the implicit function theorem, some neighborhood of 0 in the set $\widetilde{\mathbf{F}}^{-1}(0)$ is a smooth one-dimensional submanifold of $(-\delta, \delta) \times \mathbf{U}$. It follows that the same is true for the set

$$\{(\epsilon, y) \in (-\delta, \delta) \times \mathcal{B} \mid \mathbf{s}^{\epsilon}(y) = 0\},$$

and modulo a few analytical details, this implies Theorem 2.62. $\square$

The preceding may seem like an unwarranted amount of effort given that we already saw a more elementary proof in Exercise 2.63. The real reason to consider this approach is that a similar argument can be applied to the study of certain systems of *partial* differential equations, where no such finite dimensional method is available. We now look briefly at an example that is important in symplectic topology.

Example 2.66 (**Pseudoholomorphic curves**). Given a manifold W of dimension $2n$, an *almost complex structure* on W is a complex structure $J \in \Gamma(\mathrm{End}(TW))$ on its tangent bundle, i.e. for each $x \in W$, J defines a linear map $T_xW \to T_xW$ with $J^2 = -\mathrm{Id}$. This gives the tangent bundle $TW \to W$ the structure of a complex vector bundle with rank n (cf. Section 2.4), and the pair (W, J) is called an *almost complex manifold*. For $n = 1$, an almost complex manifold (Σ, j) is also called a *Riemann surface* Given a Riemann surface (Σ, j) and an almost complex manifold (W, J) of dimension $2n$, a map $u : \Sigma \to W$ is called a *pseudoholomorphic* (or *J-holomorphic*) *curve* if it satisfies

$$Tu \circ j = J \circ Tu. \tag{2.6}$$

Working in local coordinates on both manifolds, (2.6) becomes a system of $2n$ first order partial differential equations in two variables (since $\dim \Sigma = 2$), called the *nonlinear Cauchy-Riemann equations*. The next exercise explains the reason for this terminology.

Exercise 2.67. If $(\Sigma, j) = (W, J) = (\mathbb{C}, i)$, show that a map $f : \mathbb{C} \to \mathbb{C}$ satisfies (2.6) if and only if it is an analytic function.

The infinite dimensional implicit function theorem can be applied to the study of J-holomorphic curves just as in Example 2.61.

Exercise 2.68. Without worrying about the analytical details, write down a Banach space bundle $\mathcal{E}$ over the Banach manifold $\mathcal{B} := C^1(\Sigma, W)$ and a section $\mathcal{B} \to \mathcal{E}$ whose zeros are precisely the J-holomorphic curves $u : (\Sigma, j) \to (W, J)$ of class C^1. (As in Exercise 2.64, it turns out that any such solution is actually a smooth map, due to *elliptic regularity theory*. Note that technically, Σ must be compact in order to apply Eliasson's formalism, but there are various ways to work around this for certain noncompact domains.)

Much more on J-holomorphic curves can be found in the book by McDuff and Salamon [MS04].

2.6 General fiber bundles

By now it should be apparent that vector bundles are worthy of study. We next examine some objects that clearly deserve to be called "bundles," though their fibers are not vector spaces.

Example 2.69 (Frame bundles). Recall that if $\pi : E \to M$ is a smooth vector bundle of rank m, a *frame* over some subset $\mathcal{U} \subset M$ is an ordered set $(v_1, \ldots, v_m)$ of smooth sections $v_j \in \Gamma(E|_{\mathcal{U}})$ which span each fiber E_x for $x \in \mathcal{U}$. Restricting to a single fiber, a frame on E_x is simply an ordered basis of E_x. Denote by FE_x the set of all frames on E_x, and let

$$FE = \bigcup_{x \in M} FE_x.$$

This space has a natural topology and smooth manifold structure such that any frame over an open subset $\mathcal{U} \subset M$ defines a smooth map $\mathcal{U} \to FE$. We can also define a smooth projection map $\pi_F : FE \to M$ such that $\pi_F^{-1}(x) = FE_x$ for all $x \in M$. It seems natural to call the sets $\pi_F^{-1}(x)$ *fibers*, and $\pi_F : FE \to M$ is called the *frame bundle* of E. The fibers FE_x are not vector spaces, but rather smooth manifolds.

Exercise 2.70. Show that for each $x \in M$, FE_x can be identified with the general linear group $\mathrm{GL}(m, \mathbb{F})$, though not in a canonical way.

A *section* of the bundle $FE \to M$ over some subset $\mathcal{U} \subset M$ is defined—predictably—as a map $s : \mathcal{U} \to FE$ such that $\pi \circ s = \mathrm{Id}_{\mathcal{U}}$. Thus a *smooth* section $s : \mathcal{U} \to FE$ is precisely a frame over $\mathcal{U}$ for the vector bundle E, and we denote the space of such sections by $\Gamma(FE|_{\mathcal{U}})$. Two things must be observed:

(i) $\Gamma(FE|_{\mathcal{U}})$ is not a vector space, though it does have a natural topology, i.e. one can define what it means for a sequence of smooth sections $s_k \in \Gamma(FE|_{\mathcal{U}})$ to converge (uniformly with all derivatives) to another smooth section $s \in \Gamma(FE|_{\mathcal{U}})$.

(ii) Sections $s : \mathcal{U} \to FE$ can always be defined over sufficiently small open subsets $\mathcal{U} \subset M$, but there might not be any *global* sections $s : M \to FE$. Indeed, such a section exists if and only if the vector bundle $E \to M$ is trivializable.

Example 2.71 (Unit sphere bundles and disk bundles). Given a real vector bundle $\pi : E \to M$ of rank m with a smooth bundle metric $\langle\ ,\ \rangle$, define

$$S^{m-1}E = \{v \in E \mid \langle v, v \rangle = 1\}.$$

Then the restriction of π to $S^{m-1}E$ is another surjective map $p : S^{m-1}E \to M$, whose fibers $p^{-1}(x)$ are each diffeomorphic to the sphere S^{m-1}.

A section $s : M \to S^{m-1}E$ is simply a section of the vector bundle E whose values are unit vectors. Unit sphere bundles can sometimes be useful as alternative descriptions of vector bundles: they carry all the same information, but have the nice property that if M is compact, so is the total space $S^{m-1}E$. In particular, if (M, g) is a compact Riemannian n-manifold, then S^1TM is a compact $(2n-1)$-manifold, and the geodesic equation on M defines a smooth vector field on S^1TM.

A closely related object is the *disk bundle*

$$D^mE = \{v \in E \mid \langle v, v\rangle \le 1\},$$

with fibers that are smooth compact manifolds with boundary. Such objects arise naturally in the construction of *tubular neighborhoods*: given a manifold M and a closed submanifold $N \subset M$, it turns out that a neighborhood of N in M can always be identified with a disk bundle over N, with fibers of dimension $\dim M - \dim N$ (cf. [Bre93]).

Clearly it is useful to consider bundles whose fibers are all diffeomorphic manifolds, varying over the base in some smooth way. Each of the examples so far has been defined in terms of a given vector bundle, but this needn't be the case in general, as the next example shows.

Example 2.72 (Mapping tori). Suppose N is a smooth n-manifold and $\varphi : N \to N$ is a diffeomorphism. Then we define the *mapping torus* of φ to be the smooth $(n+1)$-manifold

$$M_\varphi = (\mathbb{R} \times N)/\sim,$$

with the equivalence relation defined by $(t, x) \sim (t + 1, \varphi(x))$. The projection $\mathbb{R} \times N \to \mathbb{R}$ then descends to a smooth surjective map

$$\pi : M_\varphi \to S^1 := \mathbb{R}/\mathbb{Z},$$

with the "fibers" $(M_\varphi)_{[t]} := \pi^{-1}([t])$ all diffeomorphic to N. To see that the fibers vary "smoothly" with respect to $t \in S^1$, we can construct "local trivializations" $\Phi : \pi^{-1}(\mathcal{U}) \to \mathcal{U} \times N$ over open subsets $\mathcal{U} \subset S^1$. For example, regarding $\mathcal{U}_1 := (0, 1)$ as an open subset of S^1, there is the obvious diffeomorphism $\Phi_1 : \pi^{-1}(\mathcal{U}_1) \to \mathcal{U}_1 \times N$ defined by

$$\Phi_1([(t, x)]) = (t, x) \qquad \text{for } t \in (0, 1).$$

The same expression with $t \in \left(\frac{1}{2}, \frac{3}{2}\right)$ gives a well defined diffeomorphism $\Phi_2 : \pi^{-1}(\mathcal{U}_2) \to \mathcal{U}_2 \times N$ for the open set $\mathcal{U}_2 := \left(\frac{1}{2}, \frac{3}{2}\right) \subset S^1$. Observe that $\mathcal{U}_1 \cup \mathcal{U}_2$ covers S^1, thus the two trivializations $\{\mathcal{U}_j, \Phi_j\}_{j\in\{1,2\}}$ determine what we will momentarily define as a "smooth fiber bundle structure" for $\pi : M_\varphi \to S^1$.

Notice that the coordinate vector field $\widetilde{V}(t,x) = \frac{\partial}{\partial t}$ on $\mathbb{R} \times N$ descends to a well defined smooth vector field V on the total space M_φ, whose 1-periodic orbits can be identified with the fixed points of φ. In this way the mapping torus can be used to analyze fixed points by changing a *discrete* dynamical system into a *continuous* dynamical system.

This next example similarly has no direct relation to any vector bundle.

Example 2.73 (The Hopf fibration). Regard S^3 as the unit sphere in $\mathbb{C}^2$, and define a map with values in the extended complex plane by

$$\pi : S^3 \to \mathbb{C} \cup \{\infty\} : (z_1, z_2) \mapsto \frac{z_1}{z_2}.$$

Using the stereographic projection to identify $\mathbb{C} \cup \{\infty\}$ with S^2, the map $\pi : S^3 \to S^2$ turns out to be smooth, and its fibers $\pi^{-1}(z)$ are sets of the form $\{e^{i\theta}(z_1, z_2) \mid \theta \in \mathbb{R}\}$ for fixed $(z_1, z_2) \in S^3 \subset \mathbb{C}^2$. Thus we have a fiber bundle $\pi : S^3 \to S^2$ with fibers diffeomorphic to S^1. Such objects are called *circle bundles*, and this one in particular is known as the *Hopf fibration.* It plays an important role in homotopy theory—in particular, π is the simplest (and in some sense the archetypal) example of a map from S^3 to S^2 that cannot be deformed continuously to the identity. The proof of this fact uses a general result that relates the higher homotopy groups of the total space, fiber and base of any fiber bundle. For a topologist, this gives more than sufficient motivation for the study of fiber bundles.

Here now are the definitions—we once again begin in a purely topological context, and subsequently incorporate the notion of smoothness.

Definition 2.74. A *fiber bundle* consists of three topological spaces E, M and F, called the *total space*, *base* and *standard fiber* respectively, together with a continuous surjective map $\pi : E \to M$ such that for every point $x \in M$ there exists an open neighborhood $x \in \mathcal{U} \subset M$ and a *local trivialization*

$$\Phi : \pi^{-1}(\mathcal{U}) \to \mathcal{U} \times F.$$

Here Φ is a homeomorphism that takes $\pi^{-1}(x)$ to $\{x\} \times F$ for each $x \in \mathcal{U}$.

The *fibers* $E_x := \pi^{-1}(x)$ of a fiber bundle are thus homeomorphic to some fixed space F, by homeomorphisms that vary continuously over the base.

Remark 2.75. Geometers often use the terms *fibration* and *fiber bundle* synonymously, though the former has a slightly more general meaning in topology, cf. [Hat02].

Definition 2.76. Suppose $\pi : E \to M$ is a fiber bundle with standard fiber F, such that M and F are both smooth manifolds. A *smooth structure* on $\pi : E \to M$ is a maximal set of local trivializations $\{\mathcal{U}_\alpha, \Phi_\alpha\}$ such that the open sets $\{\mathcal{U}_\alpha\}$ cover M and the maps $\{\Phi_\alpha\}$ are diffeomorphisms.

We see from this definition that a smooth fiber bundle has each fiber *diffeomorphic* to the standard fiber F, by diffeomorphisms that vary *smoothly* over M. It's not hard to show from this that the total space E is then a smooth manifold of dimension $\dim M + \dim F$.

The terms *section* and *fiber preserving map* have the same definition in this context as for a vector bundle.

2.7 Structure groups

The version of a fiber bundle defined above is somewhat weaker than the definition given in some texts, notably the classic book by Steenrod [Ste51]. The stricter definition includes an extra piece of structure, called the *structure group* of a bundle. To explain this, we will need to assume some knowledge of topological groups and Lie groups, which are discussed in Appendix B.

The most important Lie groups are of course the linear groups: $\mathrm{GL}(n, \mathbb{F})$, $\mathrm{SL}(n, \mathbb{F})$, $\mathrm{O}(n)$, $\mathrm{SU}(n)$, $\mathrm{Sp}(n)$ and so forth, all of which arise naturally in the study of vector bundles with extra structure such as bundle metrics or volume forms. In the study of general fiber bundles, one also encounters various "infinite dimensional groups," such as the following examples. For any topological space F, there is the group

$$\mathrm{Homeo}(F) = \{\varphi : F \to F \mid \varphi \text{ is bijective, } \varphi \text{ and } \varphi^{-1} \text{ both continuous}\}.$$

On an oriented topological manifold, this has an important subgroup

$$\mathrm{Homeo}^+(F) = \{\varphi \in \mathrm{Homeo}(F) \mid \varphi \text{ preserves orientation}\};$$

we refer to [Hat02] for the definition of "orientation preserving" on a topological manifold. If F is a smooth manifold, we have the corresponding subgroups

$$\mathrm{Diff}(F) = \{\varphi \in \mathrm{Homeo}(F) \mid \varphi \text{ and } \varphi^{-1} \text{ are smooth}\},$$

and in the oriented case,

$$\mathrm{Diff}^+(F) = \{\varphi \in \mathrm{Diff}(F) \mid \varphi \text{ preserves orientation}\}.$$

There are still other groups corresponding to extra structures on a smooth manifold F, such as a Riemannian metric g or a symplectic form ω. These two in particular give rise to the groups of *isometries* and *symplectomorphisms* respectively,

$$\mathrm{Isom}(F, g) = \{\varphi \in \mathrm{Diff}(F) \mid \varphi^* g \equiv g\},$$
$$\mathrm{Symp}(F, \omega) = \{\varphi \in \mathrm{Diff}(F) \mid \varphi^* \omega \equiv \omega\}.$$

Most of these examples are infinite dimensional,[9] which introduces some complications from an analytical point of view. They cannot be considered smooth Lie groups, though they clearly are topological groups.

Given a topological group G and a topological space F, we say that G *acts continuously from the left* on F if there is a continuous map

$$G \times F \to F : (a, x) \mapsto ax$$

satisfying

$$ex = x \qquad \text{and} \qquad (ab)x = a(bx)$$

for all $a, b \in G$ and $x \in F$, where $e \in G$ is the identity element. The simplest example is the natural action of any subgroup $G \subset \mathrm{Homeo}(F)$ on F, defined by $(\varphi, x) \mapsto \varphi(x)$.

Similarly, a *right action*

$$F \times G \to F : (x, a) \mapsto xa$$

must satisfy

$$xe = x \qquad \text{and} \qquad x(ab) = (xa)b.$$

For example, a subgroup $G \subset \mathrm{Homeo}(F)$ defines a right action on F by $(x, \varphi) \mapsto \varphi^{-1}(x)$. For any left or right action, the operation of any single group element $a \in G$ on F defines a homeomorphism.

You should take a moment to convince yourself that the definitions of "left action" and "right action" are *not* fully equivalent: the key is the associativity condition. For instance, the map $(x, \varphi) \mapsto \varphi^{-1}(x)$ mentioned above defines only a right action, not a left action.

On the other hand, any left action $(a, x) \mapsto ax$ defines a corresponding right action by $(x, a) \mapsto a^{-1}x$, and vice versa. In this way one can switch back and forth between left and right actions for the sake of notational convenience; the distinction is essentially a matter of bookkeeping.

Remark 2.77. The following example shows why it can be convenient to consider both left and right actions in the same discussion. For some topological group G, let $F = G$ and consider the natural actions $G \times F \to F$ and $F \times G \to F$ defined by group multiplication. These define homeomorphisms of F by $L_a(h) = ah$ and $R_b(h) = hb$ for every $a, b \in G$, and due to the associative law, $L_a \circ R_b = R_b \circ L_a$. This will be useful when we define the group action on frame bundles in Section 2.8.

The terms *smooth left/right action* now have obvious definitions if G is a Lie group and F is a smooth manifold. In this case, each element $a \in G$ defines a diffeomorphism on F.

[9] The exception is $\mathrm{Isom}(F, g)$, which is generated infinitessimally by the finite dimensional space of *Killing vector fields*; see e.g. [GHL04].

We are now ready to relate all of this to fiber bundles. For a general bundle $\pi : E \to M$, it's clear that any two local trivializations $\Phi_\alpha : \pi^{-1}(\mathcal{U}_\alpha) \to \mathcal{U}_\alpha \times F$ and $\Phi_\beta : \pi^{-1}(\mathcal{U}_\beta) \to \mathcal{U}_\beta \times F$ can be related by

$$\Phi_\beta \circ \Phi_\alpha^{-1}(x, p) = (x, g_{\beta\alpha}(x)p) \tag{2.7}$$

where $g_{\beta\alpha} : \mathcal{U}_\alpha \cap \mathcal{U}_\beta \to \mathrm{Homeo}(F)$ is a continuous "transition map," defined in terms of the natural left action of $\mathrm{Homeo}(F)$ on F. At the most basic level then, we can say that the bundle $\pi : E \to M$ "has structure group $\mathrm{Homeo}(F)$".

Definition 2.78. Suppose $\pi : E \to M$ is a fiber bundle, with standard fiber F, and G is a topological group. A *reduction of the structure group* of this bundle to G is a system of local trivializations $\{\mathcal{U}_\alpha, \Phi_\alpha\}$ such that $\{\mathcal{U}_\alpha\}$ is an open covering of M, and the maps Φ_α are related to each other as in (2.7) by a family of continuous *transition maps*

$$g_{\beta\alpha} : \mathcal{U}_\alpha \cap \mathcal{U}_\beta \to G,$$

acting on the trivializations via some continuous left action $G \times F \to F$.

When $\pi : E \to M$ is a smooth fiber bundle and G is a Lie group acting smoothly on the standard fiber F, we additionally require the transition maps to be smooth.

If such a reduction exists, we say that the bundle $\pi : E \to M$ *has structure group* G. A fiber bundle with structure group G is called a *G-bundle*.

For a fiber bundle with extra structure, it is appropriate to introduce a somewhat restricted notion of bundle isomorphism.

Definition 2.79. Suppose $\pi_1 : E_1 \to M$ and $\pi_2 : E_2 \to M$ are two G-bundles with the same standard fiber F and the same G-action on F. Then a *G-bundle isomorphism* is a fiber preserving homeomorphism (or diffeomorphism) $f : E_1 \to E_2$ such that for every pair of local trivializations $\Phi_\alpha : \pi_1^{-1}(\mathcal{U}_\alpha) \to \mathcal{U}_\alpha \times F$ and $\Psi_\beta : \pi_2^{-1}(\mathcal{U}_\beta) \to \mathcal{U}_\beta \times F$, there is a continuous (or smooth) map $f_{\beta\alpha} : \mathcal{U}_\alpha \cap \mathcal{U}_\beta \to G$ with

$$\Psi_\beta \circ f \circ \Phi_\alpha^{-1}(x, p) = (x, f_{\beta\alpha}(x)p).$$

Steenrod [Ste51] regards the structure group as part of the intrinsic structure of a fiber bundle, included in the definition. In principle, our definition is equivalent to his simply by naming $\mathrm{Homeo}(F)$ as the structure group—but this is beside the point, because many of the most interesting cases are those where the structure group can be reduced to a simpler group G, particularly if G is a finite dimensional Lie group.

It's time for some examples.

Example 2.80 (**Vector bundles with structure**)**.** A smooth vector bundle $\pi : E \to M$ of rank m can now be *defined* as a smooth fiber bundle with standard fiber $\mathbb{F}^m$ and structure group $\mathrm{GL}(m, \mathbb{F})$, which acts on $\mathbb{F}^m$ in the natural way. The notion of a $\mathrm{GL}(m, \mathbb{F})$-bundle isomorphism is then equivalent to our previous definition of vector bundle isomorphism.

The group reduces further if the bundle is given extra structure: for instance, a Euclidean or Hermitian bundle metric reduces $\mathrm{GL}(m, \mathbb{F})$ to $\mathrm{O}(m)$ or $\mathrm{U}(m)$ respectively, while a volume form yields $\mathrm{SL}(m, \mathbb{F})$. The combination of a Hermitian metric and compatible complex volume form reduces the structure to $\mathrm{SU}(m)$. There are corresponding statements about symplectic structures ($\mathrm{Sp}(m)$), indefinite metrics ($\mathrm{O}(m-k, k)$) and so forth. Indeed, a vector bundle with a G-structure for any Lie group G can be defined as a fiber bundle with standard fiber $\mathbb{F}^m$ and structure group G, where G acts on $\mathbb{F}^m$ via some given group representation. Each such structure comes with its own definition of a bundle isomorphism; for instance, a smooth vector bundle isomorphism on a Euclidean bundle is an $\mathrm{O}(m)$-bundle isomorphism if and only if its restriction to each pair of fibers is an isometry.

Example 2.81 (**Frame bundles with structure**)**.** In addition to reducing the structure group to a subgroup of $\mathrm{GL}(m, \mathbb{F})$, the aforementioned structures on vector bundles each define special frame bundles that have the same structure group. For example, given a Euclidean vector bundle $(E, \langle\ ,\ \rangle)$ over M, we can define the *orthogonal frame bundle* $F^{\mathrm{O}(m)}E \to M$, a subbundle of FE which contains only orthonormal frames. The standard fiber of $F^{\mathrm{O}(m)}E$ is the Lie group $\mathrm{O}(m)$, which admits a natural action of $\mathrm{O}(m)$, defined by the group multiplication. Clearly any two trivializations can then be related by a transition map with values in $\mathrm{O}(m)$.

The same remarks apply to any reduction of the structure group of a vector bundle: there is always a corresponding frame bundle, consisting of the frames that are "preferred" by the structure in question, e.g. symplectic frames, unitary frames, frames of unit volume, etc.. The frame bundle always has the same structure group as the vector bundle, and its fibers are also diffeomorphic to this group. We'll have more to say about this in the next section, on principal bundles.

Example 2.82 (**Unit sphere bundles**)**.** For the bundle $S^{m-1}E \to M$ defined in Example 2.71 with respect to a Euclidean vector bundle $(E, \langle\ ,\ \rangle)$, there is a natural reduction of the structure group to $\mathrm{O}(m)$. Namely, we restrict to orthonormal frames on E and use these to define local trivializations of $S^{m-1}E$, so that transition maps in $\mathrm{O}(m)$ act on the unit sphere $S^{m-1} \subset \mathbb{R}^m$.

Example 2.83 (**Symplectic fibrations**)**.** We now give an example where the structure group is an infinite dimensional subgroup of $\mathrm{Homeo}(F)$. A *symplectic fibration* is a smooth fiber bundle $\pi : E \to M$ whose standard fiber is a symplectic manifold (F, ω), with structure group $\mathrm{Symp}(F, \omega)$. This last condition gives each fiber E_x the structure of a symplectic manifold, such that the local trivializations restrict to symplectomorphisms $E_x \to \{x\} \times F$. Thus E can be thought of as a bundle of symplectic manifolds that are all related by smoothly varying symplectomorphisms. A fiber preserving diffeomorphism $f : E^1 \to E^2$ between two such bundles is a $\mathrm{Symp}(F, \omega)$-bundle isomorphism if and only if its restriction to $E^1_x \to E^2_x$ for each $x \in M$ is a symplectomorphism.

For a very simple example, one can take any symplectic manifold (F, ω) with a symplectomorphism $\varphi \in \mathrm{Symp}(F, \omega)$ and define the mapping torus M_φ as in Example 2.72. This is now a symplectic fibration over S^1.

We can now give the simplest possible definition of "orientation" for a fiber bundle. To simplify matters, we restrict to the smooth case.

Definition 2.84. For a smooth fiber bundle $\pi : E \to M$ with standard fiber F, an *orientation* of the bundle is a reduction of its structure group from $\mathrm{Diff}(F)$ to $\mathrm{Diff}^+(F)$.

Take a moment to convince yourself that this matches the old definition in the case of a real vector bundle. The next step is of course to give an example of a non-orientable fiber bundle—it just so happens that the most popular example is also the best known non-orientable manifold in the world.

Example 2.85 (The Möbius strip). Recall the non-orientable line bundle $\ell \to S^1$ from Example 2.12, where S^1 is identified with the unit circle in $\mathbb{C}$ and the fibers are lines in $\mathbb{R}^2$, specifically

$$\ell_{e^{i\theta}} = \mathbb{R}\begin{pmatrix}\cos(\theta/2)\\ \sin(\theta/2)\end{pmatrix} \subset \mathbb{R}^2.$$

We can define a Euclidean structure on this bundle using the natural inner product of $\mathbb{R}^2$, and then consider the corresponding "unit disk bundle"

$$D^1E = \{v \in \ell \mid \langle v, v\rangle \le 1\},$$

a smooth fiber bundle over S^1 with standard fiber $[-1, 1]$. The total space of D^1E is the Möbius strip.

2.8 Principal bundles

We've already seen some examples of principal fiber bundles, namely the *frame* bundles $F^G E \to M$ corresponding to any vector bundle $E \to M$

with structure group G. These bundles have the special property that their standard fiber and structure group are the same Lie group G, which acts from the left on itself by group multiplication. There is also a natural fiber preserving action of G on $F^G E$: to see this, consider the example of the orthonormal frame bundle $F^{\mathrm{O}(m)}E$ for a real vector bundle E with Euclidean metric $\langle\ ,\ \rangle$. Given any orthonormal frame $p = (v_1, \ldots, v_m) \in (F^{\mathrm{O}(m)}E)_x$ for E_x and a matrix $A \in \mathrm{O}(m)$ with entries $A^i{}_j$, another such frame $p \cdot A \in (F^{\mathrm{O}(m)}E)_x$ is obtained by right multiplication,

$$p \cdot A = \begin{pmatrix} v_1 & \cdots & v_m \end{pmatrix} \begin{pmatrix} A^1{}_1 & \cdots & A^1{}_m \\ \vdots & \ddots & \vdots \\ A^m{}_1 & \cdots & A^m{}_m \end{pmatrix} = (v_i A^i{}_1, \ldots, v_i A^i{}_m).$$

Similar remarks apply to bundles of unitary frames, oriented frames and so forth.

Since frame bundles play a helpful role in understanding vector bundles with extra structure, we give a special name to the class of fiber bundles that share the same special properties. The following is the first of two equivalent definitions we will give—it applies to smooth bundles, though it should be clear how to generalize it to topological bundles and groups.

Definition 2.86. A smooth *principal fiber bundle* $\pi : E \to M$ is a smooth fiber bundle whose standard fiber is a Lie group G, such that the structure group reduces to G, acting on itself by left multiplication.

These are sometimes simply called *principal bundles*, or *principal G-bundles* if we want to specify the structure group.

The definition above is analogous to the notion of a vector bundle as a fiber bundle with standard fiber $\mathbb{F}^m$ and structure group $\mathrm{GL}(m, \mathbb{F})$ (see Example 2.80). This is, of course, not the most intuitive definition possible, which is why we originally characterized vector bundles in terms of the intrinsic linear structure on the fibers. An analogous definition is possible for principal bundles $\pi : E \to M$, as soon as we understand what intrinsic structure the fibers possess. For each $x \in M$, E_x is diffeomorphic to a Lie group G, but *not in a canonical way*—in particular, there is no special element $e \in E_x$ that we can call the identity. On the other hand, having reduced the structure group to G, we've chosen a special set of local trivializations $\Phi_\alpha : \pi^{-1}(\mathcal{U}_\alpha) \to \mathcal{U}_\alpha \times G$ that are related to each other on each fiber by the left action of G. This has the effect of inducing a natural fiber preserving *right action*

$$E \times G \to E : (p, g) \mapsto pg,$$

defined in terms of any local trivialization $\Phi_\alpha(p) = (\pi(p), f_\alpha(p)) \in \mathcal{U}_\alpha \times G$ by $f_\alpha(pg) = f_\alpha(p)g$. This is independent of the choice because the *right* action of G on itself commutes with the *left* action of the transition functions (cf. Remark 2.77); indeed if $p \in E_x$ for $x \in \mathcal{U}_\alpha \cap \mathcal{U}_\beta$,

$$f_\beta(pg) = g_{\beta\alpha}(x) f_\alpha(pg) = g_{\beta\alpha}(x) f_\alpha(p) g = f_\beta(p) g.$$

Clearly the action restricts to a right action of G on each fiber E_x, and one easily sees that this is identical to the group action on frames that we defined above for the special case when E is a frame bundle. Two other important properties are worth noting:

(i) G acts *freely*, i.e. without fixed points, meaning $pg = p$ for some $p \in E$ and $g \in G$ if and only if g is the identity. Equivalently, G acts freely on each fiber E_x.

(ii) G acts *transitively* on each fiber: for any $p, q \in E_x$, there exists $g \in G$ such that $q = pg$.

Conversely, suppose we have a smooth fiber bundle $\pi : E \to M$ and a Lie group G with a fiber preserving smooth right action $E \times G \to E$ that restricts to a free and transitive action on each fiber. These two properties of the action imply that each fiber is diffeomorphic to G. In fact, for any $x \in M$, we can choose an open neighborhood $x \in \mathcal{U} \subset M$ and a smooth section $s : \mathcal{U} \to E$, and use it to define the inverse of a local trivialization,

$$\Phi^{-1} : \mathcal{U} \times G \to \pi^{-1}(\mathcal{U}) : (x, g) \mapsto s(x) g.$$

Now choose an open cover $M = \bigcup_\alpha \mathcal{U}_\alpha$ with smooth sections $s_\alpha : \mathcal{U}_\alpha \to E$, and use these to define a system of local trivializations $\{\Phi_\alpha, \mathcal{U}_\alpha\}$ in the same manner. For $x \in \mathcal{U}_\alpha \cap \mathcal{U}_\beta$ there is a unique element $g_{\beta\alpha}(x) \in G$ such that $s_\alpha(x) = s_\beta(x) g_{\beta\alpha}(x)$, and one could use an implicit function theorem argument to show that the map $g_{\beta\alpha} : \mathcal{U}_\alpha \cap \mathcal{U}_\beta \to G$ is smooth. Then we have

$$\Phi_\beta \circ \Phi_\alpha^{-1}(x, g) = \Phi_\beta(s_\alpha(x) g) = \Phi_\beta(s_\beta(x) g_{\beta\alpha}(x) g) = (x, g_{\beta\alpha}(x) g),$$

so $g_{\beta\alpha}$ are indeed the transition maps for the system $\{\mathcal{U}_\alpha, \Phi_\alpha\}$. We've proved that the following is equivalent to Definition 2.86.

Definition 2.87. A smooth *principal fiber bundle* is a smooth fiber bundle $\pi : E \to M$ together with a Lie group G and a fiber preserving right action $E \times G \to E$ which restricts to each fiber freely and transitively.

Thus each fiber has the intrinsic structure of a smooth manifold with a free and transitive G-action; as a result, the fiber must be diffeomorphic to G. This is the closest we can come to giving the fibers an actual group structure without requiring the bundle to be trivial—indeed, by the same trick we used to construct local trivializations above, we have:

Proposition 2.88. *A smooth principal bundle $\pi : E \to M$ is trivial if and only if it admits a global smooth section $s : M \to E$.*

This is of course quite different from the situation in a vector bundle, which always admits an infinite dimensional vector space of global smooth sections. For principal bundles there is a one-to-one correspondence between sections and trivializations. A consequence is that any smoothly varying group structure on the fibers would trivialize the bundle, since there would then be a global section $s : M \to E$, taking every point to the identity element in the corresponding fiber.

Definition 2.89. Given two principal G-bundles E_1 and E_2 over the same base M, a *principal bundle isomorphism* is a smooth map $\Psi : E_1 \to E_2$ which is fiber preserving and G-equivariant, i.e. $\Psi(pg) = \Psi(p)g$.

Exercise 2.90. Notice that the definition above does not explicitly require Ψ to be an invertible map. Show however that every principal bundle isomorphism is *automatically* invertible, and its inverse is another principal bundle isomorphism.

Exercise 2.91. Show that a smooth map between principal G-bundles is a principal bundle isomorphism if and only if it is a G-bundle isomorphism (see Definition 2.79).

Clearly, a principal bundle is trivializable if and only if it is isomorphic to the *trivial* principal bundle

$$\pi : M \times G \to M,$$

where the right action is $(x, h)g := (x, hg)$.

As we've seen already, any vector bundle $\pi : E \to M$ with structure group G yields a principal G-bundle $F^G E \to M$, the frame bundle. Principal bundles arise naturally in other contexts as well.

Example 2.92 (The Hopf fibration). Recall the bundle $\pi : S^3 \to S^2$ from Example 2.73, with standard fiber S^1, where S^3 is regarded as the unit sphere in $\mathbb{C}^2$. The right action

$$S^3 \times \mathrm{U}(1) \to S^3 : ((z_1, z_2), e^{i\theta}) \mapsto e^{i\theta}(z_1, z_2)$$

gives the Hopf fibration the structure of a principal $\mathrm{U}(1)$-bundle.

Example 2.93 (An $\mathrm{SO}(n)$-bundle over S^n). Regarding S^n as the unit sphere in $\mathbb{R}^{n+1}$, the natural action of $\mathrm{SO}(n+1)$ on $\mathbb{R}^{n+1}$ restricts to a transitive action on S^n. Let $x_0 = (1, 0, \ldots, 0) \in S^n$, and define the surjective map

$$\pi : \mathrm{SO}(n+1) \to S^n : A \mapsto Ax_0.$$

Using the injective homomorphism

$$\mathrm{SO}(n) \hookrightarrow \mathrm{SO}(n+1) : B \mapsto \begin{pmatrix} 1 & 0 \\ 0 & B \end{pmatrix},$$

we can regard $\mathrm{SO}(n)$ as a subgroup of $\mathrm{SO}(n+1)$ and define the natural right action

$$\mathrm{SO}(n+1) \times \mathrm{SO}(n) \to \mathrm{SO}(n+1) : (A, B) \mapsto AB.$$

Now observe that $\mathrm{SO}(n)$ is precisely the subgroup of matrices in $\mathrm{SO}(n+1)$ that fix the point x_0. Thus if $A \in \pi^{-1}(x)$ for any $x \in S^n$, we have $Ax_0 = x$ and $ABx_0 = Ax_0 = x$, so the right action is fiber preserving. It is now simple to verify that the action is free and transitive on each fiber, thus $\pi : \mathrm{SO}(n+1) \to S^n$ with this action is a principal $\mathrm{SO}(n)$-bundle. As with the Hopf fibration, this construction is useful for understanding the topology of the spaces involved, particularly the special orthogonal groups. Similar constructions can be made for the unitary and special unitary groups.

Exercise 2.94. Construct a principal $\mathrm{U}(n)$-bundle over S^{2n+1} with total space $\mathrm{U}(n+1)$.

References

[Arn89] V. I. Arnol′d, *Mathematical methods of classical mechanics*, Springer-Verlag, New York, 1989.

[Bre93] G. E. Bredon, *Topology and geometry*, Springer-Verlag, New York, 1993.

[Elĭ67] H. I. Elĭasson, *Geometry of manifolds of maps*, J. Differential Geometry **1** (1967), 169–194.

[GHL04] S. Gallot, D. Hulin, and J. Lafontaine, *Riemannian geometry*, 3rd ed., Springer-Verlag, Berlin, 2004.

[Hat02] A. Hatcher, *Algebraic topology*, Cambridge University Press, Cambridge, 2002.

[Hir94] M. W. Hirsch, *Differential topology*, Springer-Verlag, New York, 1994.

[HZ94] H. Hofer and E. Zehnder, *Symplectic invariants and Hamiltonian dynamics*, Birkhäuser Verlag, Basel, 1994.

[Lan93] S. Lang, *Real and functional analysis*, 3rd ed., Springer-Verlag, New York, 1993.

[Lan99] ———, *Fundamentals of differential geometry*, Springer-Verlag, New York, 1999.

[MS04] D. McDuff and D. Salamon, *J-holomorphic curves and symplectic topology*, American Mathematical Society, Providence, RI, 2004.

[SU01] R. U. Sexl and H. K. Urbantke, *Relativity, groups, particles: Special relativity and relativistic symmetry in field and particle physics*, Springer-Verlag, Vienna, 2001.

[Spi65] M. Spivak, *Calculus on manifolds: A modern approach to classical theorems of advanced calculus*, W. A. Benjamin, Inc., New York-Amsterdam, 1965.

[Spi99] ———, *A comprehensive introduction to differential geometry*, 3rd ed., Vol. 1, Publish or Perish Inc., Houston, TX, 1999.

[Ste51] N. Steenrod, *The Topology of Fibre Bundles*, Princeton University Press, Princeton, N. J., 1951.