STRONGLY FILLABLE CONTACT MANIFOLDS AND J–HOLOMORPHIC FOLIATIONS

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Abstract. We prove that every strong symplectic filling of a planar contact manifold admits a symplectic Lefschetz fibration over the disk, and every strong filling of T^3 similarly admits a Lefschetz fibration over the annulus. It follows that strongly fillable planar contact structures are also Stein fillable, and all strong fillings of T^3 are equivalent up to symplectic deformation and blowup. These constructions result from a compactness theorem for punctured J–holomorphic curves that foliate a convex symplectic manifold. We use it also to show that the compactly supported symplectomorphism group on T^*T^2 is contractible, and to define an obstruction to strong fillability that yields a non-gauge-theoretic proof of Gay's recent nonfillability result [Gay06] for contact manifolds with positive Giroux torsion.

Contents

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1. Introduction

Let M be a closed, connected and oriented 3–manifold. A (positive, cooriented) *contact structure* on M is a 2–plane distribution of the form $\xi = \ker \lambda$, where the *contact form* $\lambda \in \Omega^1(M)$ satisfies $\lambda \wedge d\lambda > 0$. It is a natural question in contact geometry to ask whether a given contact manifold (M, ξ) is symplectically fillable, meaning the following: we say that a compact and connected symplectic manifold (W, ω) with boundary $\partial W = M$ is a *weak filling* of (M, ξ) if $\omega|_\xi > 0$, and it is a *strong filling* if $\xi = \ker \iota_Y \omega$ for some vector field Y defined near ∂W which points transversely outward at the boundary and satisfies $L_Y \omega = \omega$. If Y extends globally over W, then $\iota_Y \omega$ defines a global primitive of ω and thus makes (W, ω) an *exact filling*. A still stronger notion is a *Stein filling* (W, ω), which comes with an integrable complex structure J and admits a proper plurisubharmonic function $\varphi : W \to [0, \infty)$ for which ∂W is a level set, Y is the gradient and $\omega = -dd^{\mathbb{C}} \varphi$. We refer to [Etn98, OS04] for more details on these notions.

The vector field Y near the boundary of a strong filling is called a *Liouville vector field*, and it induces a contact form $\lambda := \iota_Y \omega|_M$. As we'll review shortly, the existence of Y is then equivalent to the condition that one can smoothly glue the *positive symplectization* $([0, \infty) \times M, d(e^a \lambda))$ to (W, ω) along $\partial W = \{0\} \times M$; in the language of symplectic field theory (cf. [BEH+03]), this produces a symplectic cobordism with a positive cylindrical end. One can also replace λ by a positive multiple of any other contact form defining ξ after attaching to (W, ω) a trivial symplectic cobordism (see (2.1) below). In either case, the enlarged symplectic manifold is exact if (W, ω) is an exact filling.

In this paper we examine some of the consequences for strong symplectic fillings and Stein fillings when a subset of the contact manifold (or rather its symplectization) admits foliations by J–holomorphic curves. It turns out that whenever a foliation with certain properties exists, it can be extended from $[0, \infty) \times M$ to fill the entirety of W with embedded J–holomorphic curves, forming a symplectic Lefschetz fibration (Theorems 1 and 2), and this decomposition is stable under deformations of the symplectic structure (Theorem 3). The existence of such a fibration has consequences for the topology of the filling, e.g. for planar contact structures, it implies that the notions "strongly fillable" and "Stein fillable" are equivalent (Corollary 1). For the 3–torus, our arguments establish a conjecture of Stipsicz [Sti02] by showing that all minimal strong fillings are symplectically deformation equivalent, and exact fillings in particular are symplectomorphic to star shaped domains in T^*T^2 (Theorem 4); moreover, the group of compactly supported symplectomorphisms on T^*T^2 is contractible (Theorem 5). In other situations, one finds that the foliation on W produces an obvious contradiction, thus implying that the contact manifold cannot be strongly fillable (Theorem 6)—this is the case in particular for any contact manifold with positive Giroux torsion (Example 2.11).

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2. Main results

2.1. **Existence of Lefschetz fibrations and Stein structures.** Recall that a contact manifold (M, ξ) is called *planar* if it admits an open book decomposition that supports ξ and has pages of genus zero. We refer to [Etn06] or [OS04] for the precise definitions; for our purposes in the statement of the theorem below, an open book decomposition is a fibration $\pi : M \setminus B \to S^1$ where the *binding* B is a link in M. Then the *pages* are the preimages $\pi^{-1}(t)$ and the condition "supports ξ" means essentially that $\xi = \ker \lambda$ for some contact form (a so-called *Giroux form*) such that $d\lambda$ is symplectic on the pages and λ is positive on the binding. One can always "fatten" an open book decomposition by expanding B to a tubular neighborhood $\mathcal{N}(B)$ and slightly shrinking the pages, thus deforming π to a nearby map

$$\hat{\pi} : M \setminus \mathcal{N}(B) \to S^1.$$

We will use this notation consistently in the following.

Suppose W and Σ are compact oriented manifolds of real dimension 4 and 2 respectively, possibly with boundary. A *Lefschetz fibration* $\Pi : W \to \Sigma$ is then a smooth surjective map which is a locally trivial fibration outside of finitely many critical values $q \in \operatorname{int} \Sigma$, where each *singular fiber* $\Pi^{-1}(q)$ has a unique critical point, at which Π can be modeled in some choice of complex coordinates by $\Pi(z_1, z_2) = z_1^2 + z_2^2$. For (W, ω) a symplectic manifold, we call the Lefschetz fibration *symplectic* if the fibers are symplectic submanifolds. If $q' \in \Sigma$ is close to a critical value q, then there is a special circle $C \subset \Pi^{-1}(q')$, called a *vanishing cycle*, such that the singular fiber $\Pi^{-1}(q)$ can be identified with $\Pi^{-1}(q')$ after collapsing C to a point. (Again, see [OS04] for precise definitions.) One says that the Lefschetz fibration is *allowable* if all vanishing cycles are homologically nontrivial in their fibers.

Denote by $\mathbb{D} \subset \mathbb{C}$ the closed unit disk, whose boundary $\partial\mathbb{D}$ is naturally identified with $S^1 = \mathbb{R}/\mathbb{Z}$. For any symplectic manifold (W, ω) with contact boundary (M, ξ), the restriction of a symplectic Lefschetz fibration $\Pi : W \to \mathbb{D}$ over $\partial\mathbb{D}$ defines an open book decomposition supporting ξ (see [OS04, §10.2]). One can see in particular that for any Liouville vector field

Y near ∂W, the induced contact form $\lambda := \iota_Y \omega$ satisfies $d\lambda > 0$ on each fiber over $\partial\mathbb{D}$. One can now ask whether the converse holds: given an open book $\hat{\pi} : M \setminus \mathcal{N}(B) \to S^1$ supporting ξ and a strong filling W, does W admit a Lefschetz fibration over $\mathbb{D}$ that restricts to $\hat{\pi}$ on $\partial W \setminus \mathcal{N}(B)$? This would be too ambitious as stated, as one cannot expect that the contact form induced on ∂W will define positive area on the pages of $\hat{\pi}$: this cannot be true in particular if $\ker \omega|_{\partial W}$ is ever tangent to a page.

This problem can be avoided by enlarging the filling so as to induce different contact forms (but the same contact structure) on the boundary: if $\iota_Y \omega|_{\partial W} = e^f \lambda$ for some contact form λ and smooth function $f : M \to \mathbb{R}$, then for any other function $g : M \to \mathbb{R}$ with $g > f$ one can define the domain

$$\mathcal{S}_f^g = \{ (a, m) \in \mathbb{R} \times M \mid f(m) \le a \le g(m) \}. \tag{2.1}$$

This yields a symplectic cobordism $(\mathcal{S}_f^g, d(e^a \lambda))$ with Liouville vector field ∂_a, inducing the contact forms $\iota_{\partial_a} d(e^a \lambda) = e^f \lambda$ and $e^g \lambda$ on its negative and positive boundaries respectively. We shall refer to such domains as *trivial symplectic cobordisms*, and will sometimes also consider noncompact versions for which $f = -\infty$ or $g = +\infty$. The following is proved by a routine computation.

Lemma 2.1. *Assume (W, ω) is a strong filling of (M, ξ) with Liouville vector field Y near ∂W, and $\iota_Y \omega = \lambda'$. Suppose further that λ is a contact form on M and $f : M \to \mathbb{R}$ is a smooth function such that $\lambda'|_M = e^f \lambda$. Then if φ_Y^t denotes the flow of Y for time t, for sufficiently small $\epsilon > 0$, there is a symplectic embedding*

$$\psi : \left(\mathcal{S}_{f-\epsilon}^f, d(e^a \lambda) \right) \hookrightarrow (W, \omega) : (a, m) \mapsto \varphi_Y^{a - f(m)}(m)$$

that maps $\partial \mathcal{S}_{-\infty}^f$ to ∂W and is a diffeomorphism onto a closed neighborhood of ∂W in W. Moreover $\psi^* \lambda' = e^a \lambda$ and $\psi_* \partial_a = Y$.

In light of this, one can smoothly glue any trivial symplectic cobordism of the form $(\mathcal{S}_f^g, d(e^a \lambda))$ to (W, ω), and the enlarged filling is exact if (W, ω) is an exact filling. An important simple example is the case where $f \equiv 0$ and $g = \infty$: then we are simply attaching the positive symplectization $([0, \infty) \times M, d(e^a \lambda))$ where $\lambda = \iota_Y \omega|_{\partial W}$. It will often be convenient however to take nonconstant f, so that the contact form appearing in $d(e^a \lambda)$ may be chosen at will.

Recall that an *exceptional sphere* in a symplectic 4–manifold (W, ω) is a symplectically embedded 2–sphere with self-intersection number -1, and (W, ω) is called *minimal* if it contains no exceptional spheres. We can now state the first main result.

Theorem 1. *Suppose (W,ω) is a strong symplectic filling of a planar contact manifold (M,ξ), and $\pi : M \setminus B \to S^1$ is a planar open book supporting ξ. Then there is an enlarged filling (W',ω) obtained by attaching a trivial symplectic cobordism to W, such that W' admits a symplectic Lefschetz fibration $\Pi : W' \to \mathbb{D}$ for which $\Pi|_{\partial W' \setminus \mathcal{N}(B)} = \hat{\pi}$. Moreover, $\Pi : W' \to \mathbb{D}$ is allowable if W is minimal.*

The following corollary was pointed out to me by John Etnyre:

Corollary 1. *Every strongly fillable planar contact manifold is also Stein fillable.*

Proof. Suppose (W,ω) is a strong filling of (M,ξ) and the latter is planar. By blowing down as in [McD90] and then attaching a trivial symplectic cobordism, we can modify W to a minimal filling $(\widehat{W},\hat{\omega})$ that admits an allowable symplectic Lefschetz fibration due to Theorem 1. It then follows from Eliashberg's topological characterization of Stein manifolds [Eli90b] (see also [GS99, AO01]) that $(\widehat{W},\hat{\omega})$ is symplectically deformation equivalent to a Stein domain. □

Recall that by a result of Giroux [Gir], a contact 3–manifold is Stein fillable if and only if it admits a supporting open book whose monodromy is a product of positive Dehn twists. One can understand this in the context of Lefschetz fibrations as follows: if (W,ω) is a Stein filling of (M,ξ), then it admits a Lefschetz fibration over the disk by a result of Loi-Piergallini [LP01] or Akbulut-Ozbagci [AO01]. The monodromy of the resulting open book decomposition of M can then be obtained by composing positive Dehn twists along the vanishing cycles of each singular fiber (see for example [OS04]). Conversely, any open book with this property can be realized as the boundary of some Lefschetz fibration, which admits a Stein structure due to Eliashberg [Eli90b]. Giroux asked whether it might in fact be true that *every* open book of (M,ξ) must have this property when (M,ξ) is Stein fillable. Theorem 1 implies an affirmative answer at least for the planar open books:

Corollary 2. *If (M,ξ) is a planar contact manifold, then it is strongly (and thus Stein) fillable if and only if every supporting planar open book has monodromy isotopic to a product of positive Dehn twists.*

As an immediate consequence of Corollary 1, we also obtain a new obstruction to the existence of planar open books:

Corollary 3. *If (M,ξ) is a contact manifold which is strongly fillable but not Stein fillable, then it is not planar.*

Remark 2.2. It was not known until recently whether strong and Stein fillability are equivalent notions: a negative answer was provided by a construction due to P. Ghiggini [Ghi05] of strongly fillable contact manifolds

that are not Stein fillable. It follows then from the above results that Ghiggini's contact structures are not planar.

The reason here for the restriction to planar contact structures is that a planar open book can always be presented as the projection of a 2–dimensional $\mathbb{R}$–invariant family of J–holomorphic curves in the symplectization $\mathbb{R}\times M$. This is a special case of a construction due to C. Abbas [Abb] that relates open book decompositions on general contact manifolds to solutions of a nonlinear elliptic problem, which specifically in the planar case gives J–holomorphic curves. (An existence proof for the planar case is also given in [Wenc].) For analytical reasons, J–holomorphic curves with the desired properties and higher genus generically cannot exist.[1] Nonetheless, one can sometimes derive interesting results for non-planar contact manifolds using other kinds of decompositions with genus zero fibers, of which the following is an example.

Let $T^3 = S^1 \times S^1 \times S^1 = T^2 \times S^1$ with coordinates (q_1, q_2, θ), and write the standard contact structure on T^3 as $\xi_0 = \ker \lambda_0$ where

$$\lambda_0 = \cos(2\pi\theta)\ dq_1 + \sin(2\pi\theta)\ dq_2.$$

This can be identified with the canonical contact form on the unit cotangent bundle $S^*T^2 \subset T^*T^2$ as follows: writing points in T^2 as (q_1, q_2), we use the natural identification of T^*T^2 with $T^2 \times \mathbb{R}^2 \ni (q_1, q_2, p_1, p_2)$ and write the canonical 1–form as $p_1\ dq_1 + p_2\ dq_2$. The 3–torus is then $S^*T^2 = T^2 \times \partial\mathbb{D}$, with the θ–coordinate corresponding to the point $(p_1, p_2) = (\cos(2\pi\theta), \sin(2\pi\theta)) \in \partial\mathbb{D}$, and λ_0 is the restriction of $p_1\ dq_1 + p_2\ dq_2$ to this submanifold. The canonical symplectic form $\omega_0 := dp_1 \wedge dq_1 + dp_2 \wedge dq_2$ on $T^*T^2 = T^2 \times \mathbb{R}^2$ can then be written as $-dd^{\mathbb{C}} f$ for the proper plurisubharmonic function $f(q,p) = \frac{1}{2}|p|^2$, thus $T^2 \times \mathbb{D}$ is a Stein domain; we shall refer to it as the *standard* Stein filling of (T^3, ξ_0). More generally, one has the following construction:

Definition 2.3. A *star shaped domain* $\mathcal{S} \subset T^*T^2$ is a subset of the form $\{(q, tf(q,p)\cdot p) \in T^*T^2 \mid t \in [0,1],\ (q,p) \in S^*T^2\}$ for some smooth function $f : S^*T^2 \to (0,\infty)$.

Observe that the boundary $\partial\mathcal{S}$ of a star shaped domain is always transverse to the radial Liouville vector field $p_1\partial_{p_1} + p_2\partial_{p_2}$, thus $(\mathcal{S}, \omega_0)$ is clearly an exact filling of T^3.

Eliashberg showed in [Eli96] that ξ_0 is the *only* strongly fillable contact structure on T^3. It is not planar due to [Etn04, Theorem 4.1], as the standard filling has $b_2^0(T^2 \times \mathbb{D}) \neq 0$, though Van Horn-Morris [VHM07] has shown that it does admit a genus 1 open book. It also admits the following decomposition, which one might think of as a generalization of an open

[1] Hofer pointed out this trouble in [Hof00] and suggested the aforementioned elliptic problem as a potential remedy, but its compactness properties are not yet fully understood.

book with planar pages. Let $Z = \{\theta \in \{0, 1/2\}\} \subset T^3$, a union of two disjoint pre-Lagrangian 2–tori, and define

$$\pi : T^3 \setminus Z \to \{0,1\} \times S^1$$
$$(q_1, q_2, \theta) \mapsto \begin{cases} (0, q_2) & \text{if } \theta \in (0, 1/2), \\ (1, q_2) & \text{if } \theta \in (1/2, 1). \end{cases} \tag{2.2}$$

This is a smooth fibration, and we can think of it intuitively as a union of two open book decompositions with cylindrical pages, and the subset Z playing the role of the binding. It supports the contact structure in the sense that $d\lambda_0$ is positive on each fiber, and the fibers have natural compactifications with boundary in Z such that λ_0 is positive on these boundaries. As with an open book, one can "fatten" Z to a neighborhood $\mathcal{N}(Z)$ and deform π to a nearby fibration

$$\hat{\pi} : T^3 \setminus \mathcal{N}(Z) \to \{0,1\} \times S^1,$$

whose fibers are compact annuli.

Theorem 2. *Suppose (W, ω) is any strong symplectic filling of (T^3, ξ_0). Then one can attach to W a trivial symplectic cobordism, producing an enlarged filling W' that admits a symplectic Lefschetz fibration $\Pi : W' \to [0,1] \times S^1$ for which $\Pi|_{\partial W' \setminus \mathcal{N}(Z)} = \hat{\pi}$. Moreover, every singular fiber is the union of an annulus with an exceptional sphere; in particular, there are no singular fibers if (W, ω) is minimal.*

There is also a stability result for the Lefschetz fibrations considered thus far. Note that in the following, we don't assume the symplectic forms ω_t are cohomologous. This result is applied in [Wene] to classify strong fillings of various contact manifolds up to symplectic deformation equivalence.

Theorem 3. *If (W, ω_t) for $t \in [0,1]$ is a smooth 1–parameter family of strong fillings of either a planar contact manifold (M, ξ) or (T^3, ξ_0), then by attaching a smooth family of trivial symplectic cobordisms, one can construct a smooth family of strong fillings (W', ω_t') for which ω_t' is independent of t near $\partial W'$, and there exists a smooth family of ω_t'–symplectic Lefschetz fibrations $\Pi_t : W' \to \Sigma$ as in Theorems 1 and 2, such that the critical points vary smoothly with t.*

2.2. **Classifying strong fillings of T^3.** Stipsicz showed using a gauge theory argument [Sti02] that all Stein fillings of T^3 are homeomorphic to $T^2 \times \mathbb{D}$, and conjectured that this result can be strengthened to a diffeomorphism. In fact, more turns out to be true: by Theorem 2, every minimal strong filling W of T^3 admits a symplectic fibration over the annulus with cylindrical fibers. One can now repeat this construction starting from a different decomposition of T^3 (corresponding to a change in the (q_1, q_2)–coordinates), and thus show that W admits two symplectic fibrations over the annulus, with cylindrical fibers such that any two fibers from each fibration intersect each other once transversely. This provides a diffeomorphism from W with an attached cylindrical end to T^*T^2, and in §5 we will use Moser isotopy arguments to show:

Theorem 4. *All minimal strong fillings of T^3 are symplectically deformation equivalent, and every exact filling of T^3 is symplectomorphic to a star shaped domain in (T^*T^2, ω_0).*

Corollary 4. *Every minimal strong filling of T^3, and in particular every Stein filling, is diffeomorphic to $T^2 \times \mathbb{D}$.*

The first uniqueness result of this type was obtained by Eliashberg [Eli90a], who showed that all Stein fillings of S^3 are diffeomorphic to the 4–ball. Shortly afterwards, McDuff [McD90] classified Stein fillings of the Lens spaces $L(p, 1)$ with their standard contact structures up to diffeomorphism, showing in particular that they are unique for all $p \neq 4$. McDuff argued by compactification in order to apply her classification results for rational and ruled symplectic 4–manifolds, and several other uniqueness and finiteness results have since been obtained using similar ideas, e.g. [Lis08, OO05]. Many of these uniqueness results can be recovered, and some of them strengthened or generalized, using the punctured holomorphic curve techniques introduced here (cf. [Wene]). By contrast, there are also contact manifolds that admit infinitely many non-diffeomorphic or non-homeomorphic Stein fillings: see [AEMS] and the references mentioned therein.

The aforementioned result of McDuff for $L(p, 1)$ was strengthened to uniqueness up to Stein deformation equivalence by R. Hind [Hin03], using a construction similar to ours, though the technical arguments are somewhat different. Hind uses a foliation by J–holomorphic planes asymptotic to a multiply covered orbit; since planes cannot undergo nodal degenerations unless there are closed curves involved, singular fibers are ruled out and the result is a smooth symplectic fibration outside of the asymptotic orbit. This fibration can then be used to construct a plurisubharmonic function with control over the critical points, thus leading to a uniqueness result up to Stein homotopy. It is plausible that one could apply Hind's idea to our construction and further sharpen our classification of Stein fillings for T^3, though we will not pursue this here.

Another consequence of Theorem 4 (and also a step in its proof) is that every exact filling of T^3 becomes symplectomorphic to (T^*T^2, ω_0) after attaching a positive cylindrical end. It is then natural to ask about the topology of the compactly supported symplectomorphism group. In §5 we will prove:

Theorem 5. *The group $\mathrm{Symp}_c(T^*T^2, \omega_0)$ of symplectomorphisms with compact support is contractible.*

2.3. **Obstructions to fillability.** The results stated so far all start with the assumption that a filling exists, and then use the existence of some J–holomorphic curves to deduce properties of the filling. In other situations, the same argument can sometimes lead to a contradiction, thus defining an obstruction to filling—to understand this, we must first recall some general notions about holomorphic curves in symplectizations and finite energy foliations.

If λ is a contact form on M, then the *Reeb vector field* $X_\lambda \in \mathrm{Vec}(M)$ is defined by the conditions

$$d\lambda(X_\lambda, \ ) \equiv 0, \qquad \lambda(X_\lambda) \equiv 1.$$

The *symplectization* $\mathbb{R} \times M$ then admits a natural splitting of its tangent bundle $T(\mathbb{R}\times M) = \mathbb{R}\oplus\mathbb{R}X_\lambda \oplus \xi$; let us denote the $\mathbb{R}$–coordinate on $\mathbb{R}\times M$ by a and let ∂_a denote the corresponding unit vector field. There is now a nonempty and contractible space $\mathcal{J}_\lambda(M)$ of almost complex structures J on $\mathbb{R} \times M$ having the following properties:

- J is invariant under the $\mathbb{R}$–action by translation on $\mathbb{R} \times M$
- $J\partial_a = X_\lambda$
- $J\xi = \xi$ and $J|_\xi$ is compatible with the symplectic structure $d\lambda|_\xi$

Given $J \in \mathcal{J}_\lambda(M)$, we will consider J–holomorphic curves

$$u : (\dot{\Sigma}, j) \to (\mathbb{R} \times M, J)$$

where (Σ, j) is a closed Riemann surface, $\dot{\Sigma} = \Sigma \setminus \Gamma$ is the punctured surface determined by some finite subset $\Gamma \subset \Sigma$, and u has *finite energy* in the sense defined in [Hof93]. The simplest examples of such curves are the so-called *orbit cylinders*

$$\tilde{x} : \mathbb{R} \times S^1 \to \mathbb{R} \times M : (s, t) \mapsto (Ts, x(Tt)),$$

for any T–periodic orbit $x : \mathbb{R} \to M$ of X_λ. We will not need to recall the precise definition of the energy here, only that its finiteness constrains the behavior of u at the punctures: each puncture is either removable or represents a positive/negative *cylindrical end*, at which u approximates an orbit cylinder, asymptotically approaching a (perhaps multiply covered) periodic orbit in $\{\pm\infty\} \times M$.

Recall that a T–periodic orbit is called *nondegenerate* if the transversal restriction of the linearized time T flow along the orbit does not have 1 as an eigenvalue. More generally, a *Morse-Bott submanifold* of T–periodic orbits is a submanifold $N \subset M$ consisting of T–periodic orbits such that the 1–eigenspace of the linearized flow is always precisely the tangent space to N. We say that λ is *Morse-Bott* if every periodic orbit belongs to a Morse-Bott submanifold; this will be a standing assumption throughout. Note that a nondegenerate orbit is itself a (1–dimensional) Morse-Bott submanifold.

Now consider a compact 3–dimensional submanifold $M_0 \subset M$, possibly with boundary, such that ∂M_0 is a Morse-Bott submanifold. The following objects were originally considered in [HWZ03]:

Definition 2.4. A *finite energy foliation* $\mathcal{F}$ on (M_0, λ, J) is a foliation of $\mathbb{R} \times M_0$ with the following properties:

- For any leaf $u \in \mathcal{F}$, the $\mathbb{R}$–translation of u by any real number is also a leaf in $\mathcal{F}$.
- Every $u \in \mathcal{F}$ is the image of an embedded finite energy J–holomorphic curve satisfying a uniform energy bound.

In light of the second requirement, we shall often blur the distinction between leaves and the J–holomorphic curves that parametrize them. The definition has several immediate consequences: most notably, let $\mathcal{P}_\mathcal{F}$ denote the set of all simple periodic orbits that have covers occurring as asymptotic orbits for leaves of $\mathcal{F}$. Then an easy positivity of intersections argument (see e.g. [Wen05]) implies that for each $\gamma \in \mathcal{P}_\mathcal{F}$, the orbit cylinder $\mathbb{R} \times \gamma$ is a leaf in $\mathcal{F}$, and every leaf that isn't one of these remains embedded under the natural projection

$$\pi : \mathbb{R} \times M \to M.$$

In fact, abusing notation to regard $\mathcal{P}_\mathcal{F}$ as a subset of M, the quotient $\mathcal{F}/\mathbb{R}$ defines a smooth foliation of $M_0 \setminus \mathcal{P}_\mathcal{F}$ by embedded surfaces transverse to X_λ. These projected leaves are noncompact and have closures with boundary in $\mathcal{P}_\mathcal{F}$. It is easy to see from this that $\partial M_0 \subset \mathcal{P}_\mathcal{F}$.

As we will see in Example 2.11, it is relatively easy to construct finite energy foliations in various simple local models of contact manifolds, and this will suffice for the obstruction to fillability that we have in mind. Global constructions are harder but do exist, for instance on the tight 3–sphere [HWZ03], on overtwisted contact manifolds [Wen08] and more generally on planar contact manifolds [Abb, Wenc].

Definition 2.5. We will say that a finite energy foliation $\mathcal{F}$ on (M_0, λ, J) is *positive* if every leaf that isn't an orbit cylinder has only positive ends.

Definition 2.6. A leaf $u \in \mathcal{F}$ will be called an *interior* leaf if it is not an orbit cylinder and all its ends belong to Morse-Bott submanifolds that lie in the interior of M_0.

Definition 2.7. A leaf $u \in \mathcal{F}$ will be called *stable* if it has genus 0, all its punctures are *odd* and $\mathrm{ind}(u) = 2$ (see the appendix for the relevant technical definitions).

This notion of a *stable* leaf is meant to ensure that u behaves well in the deformation and intersection theory of J–holomorphic curves. In practice, these conditions are easy to achieve for leaves of genus zero.

Definition 2.8. A leaf $u \in \mathcal{F}$ will be called *asymptotically simple* if all its asymptotic orbits are simply covered and belong to pairwise disjoint

Morse-Bott families; moreover every nontrivial Morse-Bott family among these is a circle of orbits foliating a torus.

Remark 2.9. This last condition can very likely be relaxed, but it's satisfied by most of the interesting examples I'm aware of so far and will simplify the compactness argument in §3 considerably, particularly in proving that limit curves are somewhere injective.

Theorem 6. *Suppose (M, ξ) has a Morse-Bott contact form λ, almost complex structure $J \in \mathcal{J}_\lambda(M)$ and compact 3–dimensional submanifold M_0 with Morse-Bott boundary, such that (M_0, λ, J) admits a positive finite energy foliation $\mathcal{F}$ containing an interior, stable and asymptotically simple leaf $u_0 \in \mathcal{F}$. Assume also that either of the following is true:*

1. *$M_0 \subsetneq M$.*
2. *There exists a leaf $u' \in \mathcal{F}$ which is not an orbit cylinder and is* different *from some interior stable leaf u_0 in the following sense: either u_0 and u' are not diffeomorphic, or if they are, then there is no bijection between the ends of u_0 and u' such that the asymptotic orbits of u_0 are all homotopic along Morse-Bott submanifolds to the corresponding asymptotic orbits of u'.*

Then (M, ξ) is not strongly fillable.

The idea behind this obstruction is that if (M, ξ) contains such a foliation and is fillable, one can extend the foliation into the filling and derive a contradiction by following the family of holomorphic curves along a path leading either outside of M_0 or to a "different" leaf $u' \in \mathcal{F}$. As we'll note in Remark 4.2, a similar argument leads to a proof of the Weinstein conjecture whenever a subset of M admits a finite energy foliation with an interior, stable and asymptotically simple leaf.

Example 2.10 (Overtwisted contact structures)**.** It was shown in [Wen08] that every overtwisted contact manifold globally admits a finite energy foliation satisfying the conditions of Theorem 6, so this implies a new (admittedly much harder) proof of the classic Eliashberg-Gromov result that all strongly fillable contact structures are tight (see also Remark 2.12). The foliation in question is produced by starting from a planar open book decomposition in S^3 and performing Dehn surgery and Lutz twists along a transverse link: each component of the link is surrounded by a torus which becomes a Morse-Bott submanifold in the foliation (see Figure 1). Note that an easier proof that strongly fillable manifolds are tight is possible using the result for Giroux torsion below; cf. [Gay06, Corollary 5].

Example 2.11 (Giroux torsion)**.** Let $T^2 = S^1 \times S^1$ and $T = T^2 \times [0, 1]$ with coordinates (q_1, q_2, θ). Given smooth functions $f, g : [0, 1] \to \mathbb{R}$, a 1–form

$$\lambda = f(\theta)\ dq_1 + g(\theta)\ dq_2$$

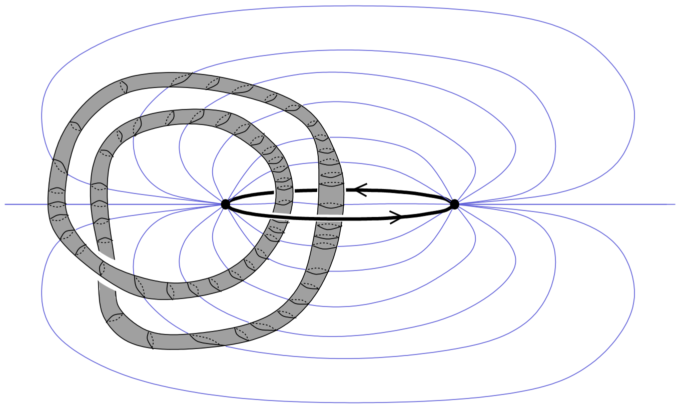

FIGURE 1. A global finite energy foliation produced from a planar open book decomposition on S^3 by surgery along a transverse link. Any overtwisted contact manifold can be foliated this way, giving a new proof that strongly fillable contact manifolds are tight.

is a positive contact form if and only if $D(\theta) := f(\theta)g'(\theta) - f'(\theta)g(\theta) > 0$, meaning the path $\theta \mapsto (f, g) \in \mathbb{R}^2$ winds counterclockwise around the origin. An important special case is the 1–form

$$\lambda_1 = \cos(2\pi\theta)\ dq_1 + \sin(2\pi\theta)\ dq_2,$$

with contact structure $\xi_1 := \ker \lambda_1$. A closed contact manifold (M, ξ) is said to have *positive Giroux torsion* if it admits a contact embedding of (T, ξ_1). Recently, D. Gay [Gay06] used gauge theory to show that contact manifolds with positive Giroux torsion are not strongly fillable, and another proof using the Ozsváth-Szabó contact invariant has been carried out by Ghiggini, Honda and Van Horn-Morris [GHVHM]. We shall now reprove this result by constructing an appropriate finite energy foliation in T; a pictorial representation of the proof is shown in Figure 2.

First note that one can always slightly expand the embedding of T and thus replace it with $T' := T^2 \times [-\epsilon, 1 + \epsilon]$ for some small $\epsilon > 0$, with the same contact form λ_1 as above. Now multiplying the contact form by a smooth positive function of θ, we can replace λ_1 by $\lambda = f(\theta)\ dq_1 + g(\theta)\ dq_2$ such that $g'(-\epsilon) = g'(1 + \epsilon) = 0$. Note that also $g'(1/4) = g'(3/4) = 0$. The result is that these four special values of θ all define Morse-Bott tori foliated by closed Reeb orbits in the $\pm\partial_{q_2}$ direction (with signs alternating).

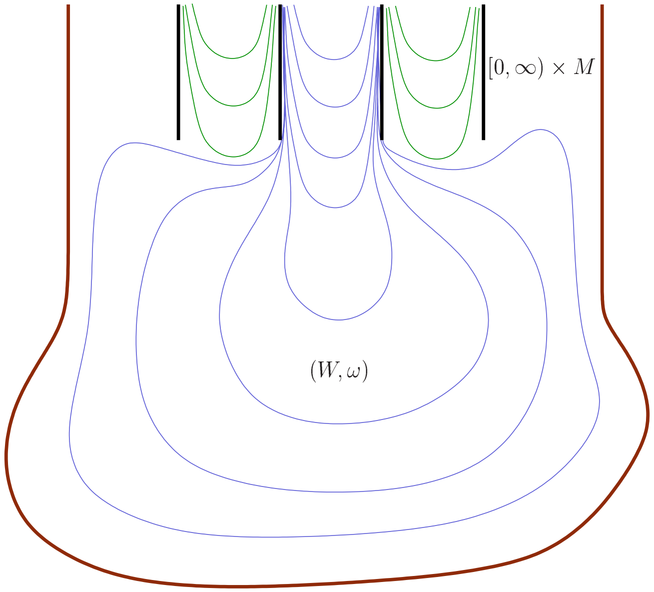

FIGURE 2. The reason why Giroux torsion contradicts strong fillability: one can construct a finite energy foliation consisting of three families of holomorphic cylinders with positive ends. The middle family contains interior stable leaves, which then spread to a foliation of any filling and must eventually run into the other families, giving a contradiction.

Indeed, it is easy to compute that the Reeb vector field takes the form

$$X_\lambda(q_1,q_2,\theta) = \frac{g'(\theta)}{D(\theta)}\partial_{q_1} - \frac{f'(\theta)}{D(\theta)}\partial_{q_2}.$$

Now choose J to be a complex structure on ξ_1 such that

$$J(C\partial_\theta) = -\frac{g(\theta)}{D(\theta)}\partial_{q_1} + \frac{f(\theta)}{D(\theta)}\partial_{q_2}$$

for some constant $C > 0$. As shown in [Wen08, §4.2], it is easy to construct a foliation by holomorphic cylinders in this setting: we simply suppose there exist cylinders $u : \mathbb{R} \times S^1 \to \mathbb{R} \times T'$ of the form

$$u(s,t) = (a(s), c, t, \theta(s)),$$

where $c \in S^1$ is a constant, and find that the nonlinear Cauchy-Riemann equations reduce to a pair of ODEs for $a(s)$ and $\theta(s)$; these have unique global solutions for any choice of $a_0 := a(0)$ and $\theta_0 := \theta(0)$. In particular, the solution $\theta(s)$ is monotone and maps $\mathbb{R}$ bijectively onto the largest interval $(\theta_-,\theta_+) \subset (-\epsilon, 1+\epsilon)$ containing θ_0 on which g' is nonvanishing. Likewise, $a(s) \to +\infty$ as $s \to \pm\infty$. As a result, in each of the subsets $\{\theta \in (-\epsilon, 1/4)\}$, $\{\theta \in (1/4, 3/4)\}$ and $\{\theta \in (3/4, 1+\epsilon)\}$, we obtain a smooth $(\mathbb{R}\times S^1)$–parametrized family of J–holomorphic curves that foliate the corresponding region; adding in the trivial cylinders for all four of the aforementioned Morse-Bott tori yields a positive finite energy foliation of T'. It is straightforward to verify that all curves in the foliation are stable in the sense defined here. Since the leaves in $\{\theta \in (1/4, 3/4)\}$ have their asymptotic orbits in the interior of T', and all other leaves have asymptotic orbits on different Morse-Bott submanifolds, Theorem 6 applies, giving a completely non-gauge-theoretic proof that no contact manifold containing (T', ξ_1) can be strongly fillable.

Remark 2.12. Giroux torsion is not generally an obstruction to *weak* fillability, e.g. this was demonstrated with examples on T^3 by Giroux [Gir94] and Eliashberg [Eli96]. Note also that overtwisted contact manifolds are not weakly fillable, but our method does not prove this, as Theorem 7 below requires the attachment of a positive cylindrical end to the boundary of the filling. This is an important difference between our technique and the "disk filling" methods used by Eliashberg in [Eli90a].

Remark 2.13. The setup used in Example 2.11 above for Giroux torsion is also suitable for (T^3, ξ_0), thus the same trick yields a positive stable finite energy foliation whose leaves project to the fibers of the fibration (2.2). We will make use of this foliation in the proof of Theorem 2.

Example 2.14. We've generally assumed the contact manifold (M,ξ) to be connected, but one can also drop this assumption. Theorem 6 then applies, for instance, to any disjoint union of contact manifolds containing a planar component. One recovers in this way a result of Etnyre [Etn04], that any strong symplectic filling with a planar boundary component must have connected boundary. This applies more generally if any boundary component admits a positive stable finite energy foliation, e.g. the standard T^3. A further generalization to *partially planar* contact manifolds is explained in [ABW], using similar ideas.

3. Holomorphic curves and compactness

The theorems of the previous section are consequences of the compactness properties of pseudoholomorphic curves belonging to a foliation in a symplectic 4–manifold with a positive cylindrical end. The setup for most of this section will be as follows: assume (M,ξ) has a Morse-Bott contact form λ and almost complex structure $J_+ \in \mathcal{J}_\lambda(M)$, a compact 3–dimensional submanifold $M_0 \subset M$ with Morse-Bott boundary and a positive finite energy foliation $\mathcal{F}_+$ of (M_0, λ, J_+) containing an interior stable leaf that is asymptotically simple. Assume further that (W^∞, ω) is a noncompact symplectic manifold admitting a decomposition

$$W^\infty = W \cup_{\partial W} ([R, \infty) \times M)$$

for some $R \in \mathbb{R}$, where W is a compact manifold with boundary $\partial W = M$ and $\omega|_{[R,\infty)\times M} = d(e^a \lambda)$, with a denoting the $\mathbb{R}$–coordinate on $\mathbb{R} \times M$. There is a natural compactification $\overline{W}^\infty$ of W^∞, defined by choosing any smooth structure on $[R, \infty]$ and replacing $[R, \infty) \times M$ in the above decomposition by $[R, \infty] \times M$; then $\overline{W}^\infty$ is a compact smooth manifold with boundary $\partial \overline{W}^\infty = M$.

The open manifold (W^∞, ω) is a natural setting for punctured pseudoholomorphic curves. Indeed, choose any number

$$a_0 \in [R, \infty)$$

and an almost complex structure J on W^∞ that is compatible with ω and satisfies $J|_{[a_0,\infty)\times M} = J_+$. Just as in the symplectization $\mathbb{R} \times M$, one then considers punctured J–holomorphic curves of finite energy in W^∞, such that each puncture is a positive end approaching a Reeb orbit at $\{+\infty\} \times M$.

Let $\mathcal{F}_0$ denote the collection of leaves in $\mathcal{F}_+$ that lie entirely within $[a_0, \infty) \times M$: observe that this includes some $\mathbb{R}$–translation of every leaf that isn't an orbit cylinder. Then each of these leaves embeds naturally into W^∞ as a finite energy J–holomorphic curve. After a generic perturbation of J compatible with ω in the region $W \cup ((R, a_0) \times M)$, standard transversality arguments as in [MS04] imply that every somewhere injective J–holomorphic curve $v : \dot{\Sigma} \to W^\infty$ not fully contained in $[a_0, \infty) \times M$ satisfies $\mathrm{ind}(v) \ge 0$. We will assume J satisfies this genericity condition unless otherwise noted.

Remark 3.1. Note that we are *not* assuming $J_+ \in \mathcal{J}_\lambda(M)$ is generic, which is important because we wish to apply the results below for foliations (M_0, λ, J_+) as constructed in Example 2.11, where J_+ is chosen to be as symmetric as possible. We can get away with this because of the distinctly 4–dimensional phenomenon of "automatic" transversality: in particular, Prop. A.1 guarantees transversality for stable leaves without any genericity assumption. We need genericity in the compactness argument of Theorem 7 only to ensure that nodal curves with components of negative index do not appear.

Denote by $\mathcal{M}$ the moduli space of finite energy J–holomorphic curves in W^∞, and let $\overline{\mathcal{M}}$ denote its natural compactification as in [BEH+03]: the latter consists of *nodal J–holomorphic buildings*, possibly with multiple

levels, including a *main* level in W^∞ and several *upper* levels, which are equivalence classes of nodal curves in $\mathbb{R} \times M$ up to $\mathbb{R}$–translation. There are no *lower* levels since W^∞ has no negative end.

Choose any interior stable leaf $u_0 \in \mathcal{F}_0$ that is asymptotically simple, let $\mathcal{M}_0 \subset \mathcal{M}$ be the connected component containing u_0 and $\overline{\mathcal{M}}_0 \subset \overline{\mathcal{M}}$ the closure of $\mathcal{M}_0$.

We will now prove two compactness results: one that gives the existence of a global foliation with isolated singularities on W^∞, and another that preserves this foliation under generic homotopies of the data.

Theorem 7. *If M contains a submanifold M_0 with finite energy foliation $\mathcal{F}_+$ as described above, then $M_0 = M$. Moreover, the moduli spaces $\mathcal{M}_0$ and $\overline{\mathcal{M}}_0$ have the following properties:*

1. *Every curve in $\mathcal{M}_0$ is embedded and unobstructed (i.e. the linearized Cauchy-Riemann operator is surjective), and no two curves in $\mathcal{M}_0$ intersect.*
2. *$\overline{\mathcal{M}}_0 \setminus \mathcal{M}_0$ consists of the following:*
   - (a) *A compact 1–dimensional manifold of buildings that each have an empty main level and one nontrivial upper level that is a leaf of $\mathcal{F}_+$ (see Remark 3.2 below),*
   - (b) *A finite set of 1–level nodal curves in W^∞, each consisting of two embedded index 0 components with self-intersection number -1 (see Remark 3.3 below), which intersect each other exactly once, transversely. These are all disjoint from each other and from the smooth embedded curves in $\mathcal{M}_0$.*
3. *The collection of curves in $\mathcal{M}_0$ plus the embedded curves in W^∞ that form components of nodal curves in $\overline{\mathcal{M}}_0$ forms a foliation of W^∞ outside of a finite set of "double points" where two leaves intersect transversely; these are the nodes of the isolated nodal curves in $\overline{\mathcal{M}}_0 \setminus \mathcal{M}_0$.*
4. *$\overline{\mathcal{M}}_0$ is a smooth manifold diffeomorphic to either $[0, 1] \times S^1$ or $\mathbb{D}$; it is the latter if and only if every asymptotic orbit of the interior stable leaf u_0 is nondegenerate.*

Remark 3.2. Note that the curves in the upper levels of a building are technically only *equivalence classes* of curves up to $\mathbb{R}$–translation, nonetheless it makes sense to speak of such a curve being a leaf of $\mathcal{F}_+$, since the latter is also an $\mathbb{R}$–invariant foliation.

Remark 3.3. The self-intersection number here is meant to be interpreted in the sense of Siefring's intersection theory for punctured holomorphic curves [Sie, SW]. This is reviewed briefly in the appendix, though it's most important to consider the case where the curve under consideration is closed: then the definition of "self-intersection number" reduces to the usual one.

Proof. As preparation, note that the stability condition for u_0 implies due to (A.2) that its normal Chern number $c_N(u_0)$ vanishes, hence $2 = \mathrm{ind}(u) > c_N(u) = 0$ for all $u \in \mathcal{M}_0$. The transversality criterion of Prop. A.1 thus guarantees that every $u \in \mathcal{M}_0$ is unobstructed once we prove that it is also embedded; we will do this in Step 7. The proof now proceeds in several steps.

Step 1: We claim that no curve $u \in \mathcal{M}_0$ can have an isolated intersection with any leaf $u_+ \in \mathcal{F}_0$. Clearly, for any given $u_+ \in \mathcal{F}_0$, positivity of intersections implies that the subset of curves $u \in \mathcal{M}_0$ that have no isolated intersection with u_+ is closed, and we must show that it's also open. There's a slightly subtle point here, as the noncompactness of the domain allows a theoretical possibility for intersections to "emerge from infinity" under perturbations of u. To rule this out, we use the intersection theory of punctured holomorphic curves defined in [Sie, SW] (a basic outline is given in the appendix). The point is that there exists a homotopy invariant intersection number $i(u; u_+) \in \mathbb{Z}$ that includes a count of "asymptotic intersections", and the condition $i(u; u_+) = 0$ is sufficient to guarantee that no curve homotopic to u ever has an isolated intersection with u_+. This number vanishes in the present case due to Lemma A.3.

Step 2: As an obvious consequence of Step 1, a similar statement is true for any component v of a building $u \in \overline{\mathcal{M}}_0$: v has no isolated intersection with any leaf $u_+ \in \mathcal{F}_+$ if v is in an upper level, or with any $u_+ \in \mathcal{F}_0$ if v is in the main level.

Step 3: If $u \in \overline{\mathcal{M}}_0 \setminus \mathcal{M}_0$, we claim that one of the following is true:

(1) u has only one nontrivial upper level, consisting of a leaf of $\mathcal{F}_+$ in $\mathbb{R} \times M$, and the main level is empty.
(2) u has no upper levels.

Indeed, suppose u has nontrivial upper levels and let v denote a nontrivial component of the topmost nontrivial level. Due to our assumptions on u_0, each positive end of v is then a simply covered orbit belonging to a distinct Morse-Bott submanifold in the interior of M_0, hence v is somewhere injective. The asymptotic formula of [HWZ96b] now implies that $\pi \circ v$ is an embedding into M near each end and is disjoint from the corresponding asymptotic orbit; hence it intersects some projected leaf of $\mathcal{F}_+$; we conclude that v intersects some leaf $u_+ \in \mathcal{F}_+$. By the result of Step 2, this intersection cannot be isolated, and since v is somewhere injective, we conclude $v \in \mathcal{F}_+$. As a result, v has no negative ends and its positive ends are in one-to-one correspondence with those of u_0, so u can have no other nonempty components.

Step 4: Suppose $u \in \overline{\mathcal{M}}_0 \setminus \mathcal{M}_0$ satisfies the second alternative in Step 3: u is then a nodal curve in the main level. We claim that any nonconstant component v of u either is a leaf in $\mathcal{F}_0$ or it is not contained in the subset $[a_0, \infty) \times M \subset W^\infty$. There are two cases to consider: if v has no ends then it cannot be in $[a_0, \infty) \times M$ because the symplectic form here is exact, so no nonconstant closed holomorphic curve can exist. If on the other hand v has positive ends and is contained in $[a_0, \infty) \times M$, where J is $\mathbb{R}$–invariant, then a similar argument as in Step 3 finds an illegal isolated intersection of v with a leaf of $\mathcal{F}_0$ unless v is such a leaf.

Step 5: Continuing with the assumptions of Step 4, we claim that one of the following holds:

(1) u is smooth (i.e. has no nodes).
(2) u has exactly two components, both somewhere injective and with index 0.

To see this, recall first that u_0 has genus 0, thus u has arithmetic genus 0. Now suppose u has multiple components connected by $N \geq 1$ nodes. Every component of u is then either a punctured sphere with positive ends (denoted here by v_i), a nonconstant closed sphere (denoted w_i) or a *ghost bubble*, i.e. a constant sphere (denoted g_i). For a sphere v_i with ends, the asymptotic behavior of u_0 guarantees that v_i is somewhere injective. Then by Step 4, it is either a leaf of $\mathcal{F}_0$ or it is not contained in $[a_0, \infty) \times M$, hence the genericity assumption for J implies $\mathrm{ind}(v_i) \geq 0$. Consider now a nonconstant closed component w_i, which we assume to be a k_i–fold cover of a somewhere injective sphere $\hat{w}_i$ for some $k_i \in \mathbb{N}$. Again, Step 4 and the genericity of J guarantee that $\mathrm{ind}(\hat{w}_i) = 2c_1([\hat{w}_i]) - 2 \geq 0$, hence

$$\mathrm{ind}(w_i) = 2c_1([w_i]) - 2 = 2k_i c_1([\hat{w}_i]) - 2 = k_i \cdot \mathrm{ind}(\hat{w}_i) + 2(k_i - 1) \geq 2(k_i - 1).$$

Ghost bubbles are now easy to rule out: we have $\mathrm{ind}(g_i) = 2c_1([g_i]) - 2 = -2$, and by the stability condition of Kontsevich (cf. [BEH+03]), g_i has at least three nodes, each contributing 2 to the total index of u. Since we already know that the nonconstant components contribute nonnegatively to the index, the existence of a ghost bubble thus implies the contradiction $\mathrm{ind}(u) \geq 4$. With this detail out of the way, we add up the indices of all components, counting an additional 2 for each node, and find

$$\begin{aligned} 2 = \mathrm{ind}(u) &= \sum_i \mathrm{ind}(v_i) + \sum_i \mathrm{ind}(w_i) + 2N \\ &\geq 2 \sum_i (k_i - 1) + 2N. \end{aligned}$$

Since $N \geq 1$ by assumption, this implies that each k_i is 1 and $N = 1$, hence u has exactly two components, both somewhere injective with index 0.

Step 6: By Step 5, the nodal curves in $\overline{\mathcal{M}}_0$ have components that are unobstructed and have index 0, hence they are isolated. By the compactness of $\overline{\mathcal{M}}_0$, this implies that the set of nodal curves in $\overline{\mathcal{M}}_0 \setminus \mathcal{M}_0$ is finite. A standard gluing argument as in [MS04] now identifies a neighborhood of any nodal curve u in $\overline{\mathcal{M}}_0$ with an open subset of $\mathbb{R}^2$, where every curve other than u is smooth. Similarly, since every $u \in \mathcal{M}_0$ is unobstructed, the usual implicit function theorem in Banach spaces defines smooth manifold charts everywhere on $\mathcal{M}_0$. Outside a compact subset, $\overline{\mathcal{M}}_0 \setminus \partial\overline{\mathcal{M}}_0$ can be identified with the set of leaves in $\mathcal{F}_0$, and is thus diffeomorphic to $[0,\infty) \times V$ for some compact 1–manifold V, so $\partial\overline{\mathcal{M}}_0$ is diffeomorphic to V itself. The space $\overline{\mathcal{M}}_0$ is therefore a compact surface with boundary, and is orientable due to arguments in [BM04].

Step 7: We now use the intersection theory from [Sie, SW] to show that $\overline{\mathcal{M}}_0$ foliates W^∞. We noted already in Step 1 that $i(u;u') = 0$ for any two curves $u, u' \in \overline{\mathcal{M}}_0$, which implies that no two of these curves can ever intersect. Since every $u \in \mathcal{M}_0$ is obviously somewhere injective due to its asymptotic behavior, the adjunction formula (A.6) implies $\operatorname{sing}(u) = 0$ and thus these curves are also embedded. Consider now a nodal curve $u \in \overline{\mathcal{M}}_0$, with its two components u_1 and u_2, and observe that (A.2) implies $c_N(u_1) = c_N(u_2) = -1$. Applying the adjunction formula again, we find

$$\begin{aligned}
0 = i(u;u) &= i(u_1;u_1) + i(u_2;u_2) + 2i(u_1;u_2) \\
&\geq 2\operatorname{sing}(u_1) + c_N(u_1) + 2\operatorname{sing}(u_2) + c_N(u_2) + 2i(u_1;u_2) \\
&= 2\operatorname{sing}(u_1) + 2\operatorname{sing}(u_2) + 2\left[i(u_1;u_2) - 1\right].
\end{aligned}$$

Thus $\operatorname{sing}(u_1) = \operatorname{sing}(u_2) = 0$, implying both components are embedded, and $i(u_1;u_2) = 1$, so the node is the only intersection, and is transverse. The adjunction formula for each of u_1 and u_2 individually now also implies $i(u_1;u_1) = i(u_2;u_2) = -1$. (Note that the $\operatorname{cov}_\infty(z)$ terms must all vanish, as this is manifestly true for u_0 and they depend only on the orbits). By the gluing argument mentioned in Step 6, a neighborhood of u in $\overline{\mathcal{M}}_0$ is a smooth 2–parameter family of embedded curves from $\mathcal{M}_0$; these foliate a neighborhood of the union of u_1 and u_2. Similarly, the implicit function theorem in [Wend] or [Wen05] implies that for any $u \in \mathcal{M}_0$, the nearby curves in $\mathcal{M}_0$ foliate a neighborhood of u. This shows that

$$\{p \in W^\infty \mid p \text{ is in the image of some } u \in \overline{\mathcal{M}}_0\}$$

is an open subset of W^∞. It is also clearly a closed subset since $\overline{\mathcal{M}}_0$ is compact. We conclude that all of W^∞ is filled by the curves in $\overline{\mathcal{M}}_0$.

Step 8: It follows easily now that $M_0 = M$, as one can take a sequence of curves in $\mathcal{M}_0$ whose images approach $(+\infty, p)$ for any $p \in M$; since a subsequence converges to a leaf of $\mathcal{F}_+$, we conclude that $\mathcal{F}_+$ fills all of M.

Step 9: Having shown already that $\overline{\mathcal{M}}_0$ is a compact orientable surface with boundary, we prove finally that it must be either $\mathbb{D}$ or $[0,1] \times S^1$. Define a smooth map

$$\Pi : W^\infty \to \overline{\mathcal{M}}_0 \tag{3.1}$$

by sending $p \in W^\infty$ to the unique curve in $\overline{\mathcal{M}}_0$ whose image contains p. We can extend Π over $\overline{W}^\infty \setminus \mathcal{P}_{\mathcal{F}_+}$ by sending $p \in M \setminus \mathcal{P}_{\mathcal{F}_+}$ to the unique leaf in $\mathcal{F}_+/\mathbb{R} = \partial\overline{\mathcal{M}}_0$ containing p.

Assume first that there are degenerate orbits among the asymptotic orbits of the interior stable leaf $u_0 \in \mathcal{F}_+$: such an orbit belongs to a Morse-Bott 2–torus $T_0 \subset M$ foliated by Reeb orbits that are asymptotic limits of leaves in $\mathcal{F}_+$. By the definition of $\mathcal{M}_0$, every curve $u \in \mathcal{M}_0$ and thus every leaf in $\mathcal{F}_+$ has a unique end asymptotic to some orbit in T_0. In this case $\partial\overline{\mathcal{M}}_0$ must have two connected components, and we can parametrize them as follows. Identify a neighborhood of T_0 in M with $(-1,1) \times S^1 \times S^1$ such that $\{0\} \times S^1 \times S^1 = T_0$ and the Reeb orbits are all of the form $\{0\} \times \{\text{const}\} \times S^1$. Then we can arrange that for sufficiently small $\epsilon > 0$, the loop $\gamma_+(t) = (+\infty, \epsilon, t, 0) \in \overline{W}^\infty$ passes through a different leaf of $\mathcal{F}_+$ for each t, thus without loss of generality, $\Pi \circ \gamma_+ : S^1 \to \partial\overline{\mathcal{M}}_0$ is an oriented parametrization of one boundary component of $\partial\overline{\mathcal{M}}_0$. The other boundary component can be given an oriented parametrization in the form $\Pi \circ \gamma_- : S^1 \to \partial\overline{\mathcal{M}}_0$ where $\gamma_-(t) = (+\infty, -\epsilon, -t, 0)$. Now moving both loops down slightly from ∞, we see that $[\gamma_-] = -[\gamma_+] \in \pi_1(\overline{W}^\infty \setminus \mathcal{P}_{\mathcal{F}_+})$, implying that the two boundary components of $\overline{\mathcal{M}}_0$ are homotopic, and therefore $\overline{\mathcal{M}}_0 \cong [0,1] \times S^1$.

If all orbits of u_0 are nondegenerate, then $\partial\overline{\mathcal{M}}_0$ must have only one component, which we can similarly parametrize by choosing a loop $\gamma : S^1 \to \{+\infty\} \times M$ that circles once around one of these orbits and passes once transversely through each leaf of $\mathcal{F}_+$. Moving γ again down from $+\infty$, it is contractible in $\overline{W}^\infty \setminus \mathcal{P}_{\mathcal{F}_+}$, implying $\partial\overline{\mathcal{M}}_0$ is contractible, thus $\overline{\mathcal{M}}_0 \cong \mathbb{D}$. □

To set up the second compactness result, assume that for $\tau \in [0,1]$, ω_τ is a smooth family of symplectic forms on W^∞ matching $d(e^a\lambda)$ on $[a_0,\infty) \times M$, and J_τ is a smooth family of almost complex structures compatible with ω_τ for each τ and matching $J_+ \in \mathcal{J}_\lambda(M)$ on $[a_0,\infty) \times M$. Assume also that the homotopy J_τ is generic on $W^\infty \setminus ([a_0,\infty) \times M)$ so that for any $\tau \in [0,1]$, every somewhere injective J_τ–holomorphic curve u not contained in $[a_0,\infty) \times M$ satisfies $\operatorname{ind}(u) \geq -1$. Then for each τ, let $\mathcal{M}_\tau$ denote the connected moduli space of J_τ–holomorphic curves containing an interior stable leaf in $\mathcal{F}_0$ that is asymptotically simple, and write its compactification as $\overline{\mathcal{M}}_\tau$.

Theorem 8. *The conclusions of Theorem 7 hold for the moduli spaces $\overline{\mathcal{M}}_\tau$ for each $\tau \in [0,1]$; in particular they are all smooth compact manifolds with boundary that form foliations of W^∞ with finitely many singularities, and their boundaries can be identified naturally with the set of leaves in the projected foliation $\mathcal{F}_+/\mathbb{R}$. Moreover, there exists a smooth 1–parameter family of diffeomorphisms $\overline{\mathcal{M}}_0 \to \overline{\mathcal{M}}_\tau$ that maps $\mathcal{M}_0$ to $\mathcal{M}_\tau$ and restricts to the natural identification $\partial\overline{\mathcal{M}}_0 \to \partial\overline{\mathcal{M}}_\tau$.*

Proof. For each $\tau \in [0,1]$, the proof of Theorem 7 requires only a small modification to work for the almost complex structure J_τ. The difference is that J_τ is now not necessarily generic, so we have a weaker lower bound on the indices of somewhere injective curves that are not contained in $[a_0,\infty) \times M$. The only place this makes a difference is in Step 5: we must now consider the possibility that u is a nodal curve in W^∞ with several components of possibly negative index. Since none of these components are contained in $[a_0,\infty) \times M$ and $\{J_\tau\}_{\tau \in [0,1]}$ is a generic homotopy, they all cover somewhere injective curves of index at least -1. We claim that this implies the somewhere injective curves have nonnegative index after all: for closed components the index is always even, so this is clear. The same turns out to be true for components with ends: since u_0 has only odd punctures, any punctured somewhere injective curve with a cover that forms a component of u has all its ends asymptotic to orbits that have odd covers, and must themselves therefore be odd. (See [Wenb, §4.2] for the proof that even orbits always have even covers; this statement applies equally well in the Morse-Bott setup described in the appendix.) It follows then from the index formula that the index of such a component must be even, and in this case therefore nonnegative. The rest of the compactness proof now follows just as before, with the added detail that all curves arising in the limit (including components of nodal curves) are unobstructed due to Prop. A.1, which does not require genericity.

By the above argument, we have moduli spaces $\overline{\mathcal{M}}_\tau$ that foliate W^∞ with J_τ–holomorphic curves outside of a finite set of nodes. Moreover, every curve in the foliation is unobstructed, so for any given $\tau_0 \in [0,1]$, the index 0 curves that are components of nodal curves in $\overline{\mathcal{M}}_{\tau_0}$ deform uniquely to J_τ–holomorphic curves for τ in some neighborhood of τ_0, and an intersecting pair of such curves forms a nodal curve. Since the curves in $\overline{\mathcal{M}}_{\tau_0}$ and $\overline{\mathcal{M}}_\tau$ near their respective boundaries are identical, a familiar intersection argument now shows that this nodal curve must belong to $\overline{\mathcal{M}}_\tau$. Similarly, index 2 curves in $\mathcal{M}_{\tau_0}$ deform to index 2 curves in $\mathcal{M}_\tau$, providing a local smooth 1–parameter family of diffeomorphisms

$$\overline{\mathcal{M}}_{\tau_0} \to \overline{\mathcal{M}}_\tau$$

for τ close to τ_0, which maps nodal curves to nodal curves and leaves in $\mathcal{F}_0$ and $\mathcal{F}_+$ to themselves. To extend this for all $\tau \in [0,1]$, it only remains to show that the "parametrized" moduli space

$$\overline{\mathcal{M}}_{[0,1]} := \{(\tau,u) \mid \tau \in [0,1],\, u \in \overline{\mathcal{M}}_\tau\}$$

is compact. This follows from the same arguments as above, after observing that the energies of $u \in \mathcal{M}_\tau$ depend only on the relative homology class defined by a leaf $u_0 \in \mathcal{F}_0$ and (continuously) on ω_τ, thus they are uniformly bounded. □

Remark 3.4. In some important situations, one can prove the two theorems above without any genericity assumption at all: the point is that genericity is usually needed to ensure a lower bound on the indices of components in

nodal curves, but is not required to show that the curves actually obtained in the limit are unobstructed. Thus if there are topological conditions preventing the appearance of nodal curves, then *any* compatible J or smooth family J_τ (also for τ varying in a higher-dimensional space) will suffice: this works in particular for exact fillings of T^3 and will play a crucial role in the proof of Theorem 5.

4. Lefschetz fibrations and obstructions to filling

We are now in a position to construct the Lefschetz fibrations that were promised in §2. It will be convenient to introduce the following notation.

Suppose (W,ω) is a strong filling of (M,ξ) and Y is a Liouville vector field near ∂W such that $\iota_Y\omega|_M = e^f\lambda$ for some contact form λ on M and smooth function $f : M \to \mathbb{R}$. Then for any constant $R > \max f$, we can use Lemma 2.1 to attach the trivial symplectic cobordism $(\mathcal{S}_f^R, d(e^a\lambda))$, producing an enlarged filling

$$(W^R,\omega) := (W,\omega) \cup_{\partial W} (\mathcal{S}_f^R, d(e^a\lambda)).$$

This has ∂_a as a Liouville vector field near ∂W^R, such that $\iota_{\partial_a}\omega|_{\partial W^R} = e^R\lambda$. One can now attach a cylindrical end,

$$(W^\infty,\omega) := (W^R,\omega) \cup_{\partial W^R} ([R,\infty) \times M, d(e^a\lambda)),$$

defining a noncompact symplectic cobordism which admits the compactification

$$\overline{W}^\infty = W^R \cup_{\partial W} ([R,\infty] \times M)\,.$$

We assign a smooth structure to $[R,\infty]$ so that $\overline{W}^\infty$ may be considered a smooth manifold with boundary, though its symplectic structure degenerates at $\partial\overline{W}^\infty$. It is sometimes useful however to define a new symplectic structure on W^∞ that does extend to infinity. Observe first that for any $\epsilon > 0$ with $R - \epsilon > \max f$, (W^∞,ω) contains the slightly extended cylindrical end $([R-\epsilon,\infty) \times M, d(e^a\lambda))$. Now choose $\delta \in (0,\epsilon)$ and a diffeomorphism

$$\varphi : [R-\epsilon,\infty] \to [e^{R-\epsilon}, e^R]$$

with the property that $\varphi(a) = e^a$ for $a \in [R-\epsilon, R-\delta]$. Then the symplectic form ω_φ on W^∞ defined by

$$\omega_\varphi = \begin{cases} d(\varphi\lambda) & \text{on } [R-\epsilon,\infty) \times M, \\ \omega & \text{everywhere else} \end{cases}$$

has a smooth extension to $\overline{W}^\infty$, such that the map

$$[R-\epsilon, R] \times M \to [R-\epsilon,\infty] \times M : (a,m) \mapsto (\varphi^{-1}(e^a), m)$$

extends to a symplectomorphism $(W^R,\omega) \to (\overline{W}^\infty, \omega_\varphi)$.

We will consider almost complex structures J on W^∞ that are compatible with ω, are generic in $W^\infty \setminus ([R-\delta,\infty) \times M)$ and match some fixed $J_+ \in$

$\mathcal{J}_\lambda(M)$ over $[R-\delta,\infty)\times M$. Observe that such a J is also compatible with the modified symplectic form ω_φ defined above, thus finite energy embedded J–holomorphic curves in W^∞ give rise to properly embedded symplectic submanifolds of $(\overline{W}^\infty,\omega_\varphi)\cong(W^R,\omega)$.

Lemma 4.1. *The almost complex structure J above can be chosen so that every closed, nonconstant J–holomorphic curve in (W^∞,J) is contained in the interior of W.*

Proof. It suffices to arrange that $W^\infty\setminus W$ is foliated by J–convex hypersurfaces. Choose $r<R-\delta$, let $h:[r,\infty)\times M\to\mathbb{R}$ denote any smooth function satisfying

(1) $\partial_a h>0$,
(2) $h(a,m)=a$ for $a\ge R-\delta$,
(3) $h(a,m)=a-r+f(m)$ for a near r,

and define a diffeomorphism

$$\psi:[r,\infty)\times M\to\mathcal{S}_f^\infty:(a,m)\mapsto(h(a,m),m).$$

This restricts to the identify on $[R-\delta,\infty)\times M$ and satisfies $\psi^*(e^a\lambda)=e^h\lambda$, thus it defines a symplectomorphism $([r,\infty)\times M,d(e^h\lambda))\to(\mathcal{S}_f^R,d(e^a\lambda))$. Now for $a\in[r,\infty)$, denote by $h_a:M\to(0,\infty)$ the smooth 1–parameter family of functions such that $e^{h(a,\cdot)}=e^a h_a$, and define the family of contact forms $\lambda_a:=h_a\lambda$ with corresponding Reeb vector fields X_a. Regarding λ_a in the natural way as a 1–form on $\mathbb{R}\times M$, we now have

$$d(e^h\lambda)=e^a\,da\wedge\lambda_a+e^a\,d\lambda_a,$$

and an almost complex structure $\hat{J}$ compatible with $d(e^h\lambda)$ can thus be constructed as follows. Given $J_+\in\mathcal{J}(\lambda)$, choose $\hat{J}$ on $[r,\infty)\times M$ so that it matches J_+ on $[R-\delta,\infty)\times M$, and at $\{a\}\times M$ satisfies

$$J\partial_a=X_a\qquad\text{and}\qquad J(\xi)=\xi,$$

where $J|_\xi$ is compatible with $d\lambda$ (and therefore also with $d\lambda_a$ for each a). Now the level sets $\{a\}\times M$ are $\hat{J}$–convex, thus an almost complex structure of the desired form on $\mathcal{S}_f^\infty$ is given by $J:=\psi_*\hat{J}$, and we can extend the latter to an ω–compatible almost complex structure on W^∞ for which the hypersurfaces $\psi(\{a\}\times M)$ for $a\ge r$ are J–convex. Since J–convexity is an open condition with respect to J, it is also safe to make a small perturbation on W^R so that J becomes generic outside of $[R-\delta,\infty)\times M$. □

Proof of Theorem 1. Assume (M,ξ) is a contact manifold supported by a planar open book $\pi:M\setminus B\to S^1$. Then using the construction in [Wenc], there is a nondegenerate contact form λ with $\ker\lambda=\xi$ and $J_+\in\mathcal{J}_\lambda(M)$ such that up to isotopy, the pages of π are projections to M of embedded J_+–holomorphic curves in $\mathbb{R}\times M$, with positive ends asymptotic to the

orbits in B. This defines a positive finite energy foliation $\mathcal{F}_+$ of (M,λ,J_+), with every leaf stable. Now if (W,ω) is a strong filling of (M,ξ), we define the enlarged fillings W^R and W^∞ with generic almost complex structure J as described above, and then Theorem 7 yields a moduli space $\overline{\mathcal{M}}_0$ of J–holomorphic curves that foliate W^∞ outside a finite set of transverse nodes, such that $\partial\overline{\mathcal{M}}_0$ is the space of leaves in $\mathcal{F}_+$ up to $\mathbb{R}$–translation. Since λ is nondegenerate, $\overline{\mathcal{M}}_0\cong\mathbb{D}$, and the map

$$\Pi:\overline{W}^\infty\setminus B\to\overline{\mathcal{M}}_0$$

defined as in (3.1) gives a symplectic Lefschetz fibration of $(\overline{W}^\infty\setminus B,\omega_\varphi)\cong(W^R\setminus B,\omega)$ over the disk. We can easily modify Π so that it extends over B: first fatten B to a tubular neighborhood $\mathcal{N}(B)\subset M$, then extend Π over this neighborhood by contracting the disk. We observe finally that if any singular fiber contains a closed component, this must be a holomorphic sphere $v:S^2\to W^\infty$ with $i(v;v)=-1$, thus an exceptional sphere, and for an appropriate choice of J it must be contained in W due to Lemma 4.1. Therefore if W is minimal, every component of a singular fiber has nonempty boundary, implying that the vanishing cycle is homologically nontrivial. □

Proof of Theorem 2. The argument is mostly the same as for Theorem 1, but using a specific Morse-Bott finite energy foliation constructed as in Example 2.11 (see Remark 2.13). In this case the space of leaves in T^3 is parametrized by two disjoint circles, thus the moduli space $\overline{\mathcal{M}}_0$ provided by Theorem 7 has two boundary components, and is therefore an annulus. The argument produces a Lefschetz fibration $\Pi:\overline{W}^\infty\setminus Z\to[0,1]\times S^1$, which one can extend over Z by fattening it to a neighborhood $\mathcal{N}(Z)$ and then filling in using the homotopy between components of $\partial\overline{\mathcal{M}}_0$.

It remains to show that all singular fibers consist of a union of a cylinder with an exceptional sphere. By Theorem 7, the only other option is a union of two transversely intersecting disks, which would give a vanishing cycle parallel to the boundary of the fiber. We can rule this out by looking at the monodromy maps of the fibrations at $\{0\}\times S^1$ and $\{1\}\times S^1$: these are the two connected components of the fibration in (2.2). Thus both monodromy maps are trivial, but they must also be related to each other by a product of positive Dehn twists, one for each nontrivial vanishing cycle. Since the mapping class group of the cylinder has only one generator, there is no product of positive Dehn twists that gives the identity, thus there can be no nontrivial vanishing cycles. □

Proof of Theorem 3. For a smooth 1–parameter family of strong fillings (W,ω_t) of (M,ξ) with $t\in[0,1]$ and a suitable Morse-Bott contact form λ, one can find a smooth family of functions $f_t:M\to\mathbb{R}$ such that for $R>\max\{f_t(m)\mid t\in[0,1],m\in M\}$, the trivial symplectic cobordism $(\mathcal{S}_{f_t}^R,d(e^a\lambda))$ can be attached to (W,ω_t), producing an enlarged filling

(W^R, ω_t) whose symplectic form is fixed near the boundary. Now attach the cylindrical end as usual and choose a generic smooth 1–parameter family J_t of ω_t–compatible almost complex structures that are identical on the end. If (M, ξ) is planar or is (T^3, ξ_0), then the result now follows by applying the same arguments as in the previous two proofs together with Theorem 8. □

Proof of Theorem 6. Suppose (M, ξ) is a contact manifold with a positive foliation $\mathcal{F}$ of (M_0, λ, J) containing an interior stable leaf $u \in \mathcal{F}$ that is asymptotically simple: then for any strong filling (W, ω), we can again fill W^∞ with J–holomorphic curves using Theorem 7, and we already have a contradiction if $M_0 \subsetneq M$. On the other hand if $M_0 = M$, we can find a point p that lies in some "different" leaf $u' \in \mathcal{F}$, and then consider for large n the sequence $u_n \in \mathcal{M}_0$, where u_n is the unique curve passing through $(n, p) \in [R, \infty) \times M \subset W^\infty$. As $n \to \infty$, a subsequence must converge to u', implying that u and u' are diffeomorphic and have ends in the same Morse-Bott manifolds, which is a contradiction. □

Remark 4.2. The *Weinstein conjecture* for a contact manifold (M, ξ) asserts that for any contact form λ with $\ker \lambda = \xi$, X_λ has a periodic orbit. The idea of using punctured holomorphic curves to prove this is originally due to Hofer [Hof93], and works so far under a variety of assumptions on (M, ξ) (see also [ACH05]). The conjecture for general contact 3–manifolds was proved recently by Taubes [Tau07], using Seiberg-Witten theory, but a general proof using only holomorphic curves is still lacking.

A minor modification of Theorem 7 yields a new proof of the Weinstein conjecture for any setting in which one can construct a positive foliation containing an interior stable leaf that is asymptotically simple, for instance on the standard 3–torus, or any contact manifold with positive Giroux torsion. The argument is a generalization of the one used by Abbas-Cieliebak-Hofer [ACH05] for planar contact structures: we replace the symplectic filling W by a cylindrical symplectic cobordism $\widehat{W}$, having $(M, c\lambda)$ for some large constant $c > 0$ at the positive end and $(M, f\lambda)$ for any smooth positive function $f : M \to \mathbb{R}$ with $f < c$ at the negative end. Then the same compactness argument works for any sequence of curves $u_n : \dot{\Sigma} \to \widehat{W}$ that is bounded away from the negative end. Just as in [ACH05], one can therefore produce a sequence u_n that runs to $-\infty$ in the negative end and breaks along a periodic orbit in $(M, f\lambda)$, proving the existence of such an orbit.[2]

[2]The compactness argument in [ACH05] contains a minor gap, as it ignores the possibility of nodal degenerations. Our argument fills the gap by showing that only embedded index 0 curves can appear in such degenerations, thus they are confined to a subset of codimension 2 and can be avoided by following a generic path to $-\infty$.

5. Fillings of T^3

We now proceed to the proofs of Theorems 4 and 5 on fillings of T^3. The key fact is that if a strong filling of (T^3, ξ_0) is minimal, then the Lefschetz fibration given by Theorem 2 is an honest symplectic fibration, i.e. it has no singular fibers. In fact, it is easy to construct two such fibrations, whose fibers intersect each other exactly once transversely; the situation is thus analogous to that of Gromov's characterization of split symplectic forms on $S^2 \times S^2$ ([Gro85], also subsequent related work by McDuff [McD90]). We can construct a simple model Stein manifold, which is symplectomorphic to T^*T^2 and carries an explicit decomposition by two fibrations for which the complex and symplectic structures both split. Matching this decomposition with the fibrations constructed for a general filling via Theorem 7 gives a symplectic deformation equivalence, which in the exact case yields a symplectomorphism via the Moser isotopy trick.

There is one subtle point here that doesn't arise in the closed case: since we intend to carry out the Moser isotopy on a noncompact manifold, it's important that our diffeomorphism be sufficiently well behaved near infinity, and this will not generally be the case without some effort. To see why not, observe that for any strong filling (W, ω) of (T^3, ξ_0), the asymptotics of the J–holomorphic curves in W^∞ given by Theorem 7 encode a homotopy invariant of the foliation. Indeed, suppose $\{\gamma_0^\eta\}_{\eta \in S^1}$ and $\{\gamma_1^\eta\}_{\eta \in S^1}$ are the two Morse-Bott families of Reeb orbits that serve as the asymptotic limits of the curves in the moduli space $\mathcal{M}$. Then we can choose a diffeomorphism

$$\mathbb{R} \times S^1 \to \mathcal{M} : (\rho, \eta) \mapsto u_{(\rho,\eta)}$$

such that $u_{(\rho,\eta)}$ has asymptotic orbits γ_0^η and $\gamma_1^{f(\rho,\eta)}$ for some continuous function

$$f : \mathbb{R} \times S^1 \to S^1,$$

which has the form $f(\rho, \eta) = \eta$ for $|\rho|$ large due to the fixed structure of $\mathcal{M}$ in the cylindrical end. The map $\rho \mapsto f(\rho, 0)$ thus defines a loop in S^1 whose homotopy class in $\pi_1(S^1) = \mathbb{Z}$ can be shown (using Theorem 8) to be an invariant determined by (W^∞, ω) and J up to compactly supported deformations. Now if (W_1, ω_1) and (W_2, ω_2) are two strong fillings that we wish to prove are symplectomorphic, we'd like to do so by choosing a diffeomorphism that both respects the structure of the holomorphic foliations and is "compactly supported" in the sense of respecting the natural identifications of W_1^∞ and W_2^∞ with $[R, \infty) \times T^3$ near infinity. It is easy enough to modify the foliations slightly so that an appropriate diffeomorphism can be constructed near infinity, but this will not be globally extendable unless the above construction gives the same class in $\pi_1(S^1)$ for both foliations.

The upshot is that it is not enough to take only T^*T^2 with its standard complex and symplectic structure as a model filling—rather, we will need a wider variety of models that come with holomorphic foliations attaining all possible values in $\pi_1(S^1)$. We'll construct such models in §5.1 by performing Luttinger surgery along the zero section in T^*T^2. Note that unlike the situation in a closed manifold, the manifolds obtained by surgery are all symplectomorphic, but the point is that their complex structures (and the resulting holomorphic foliations) behave differently at infinity. With these models in place, we'll carry out the Moser deformation argument in §5.2 to prove Theorem 4. Finally, §5.3 will use the stability of our fibrations under homotopies (Theorem 8) to prove Theorem 5.

5.1. **Model fillings and fibrations.** As usual, we identify T^*T^2 with $T^2\times\mathbb{R}^2$ and use coordinates (q_1,q_2,p_1,p_2), so that the standard symplectic structure is $\omega_0=d\lambda_0$, where $\lambda_0=p_1\,dq_1+p_2\,dq_2$. Each pair of coordinates (p_j,q_j) for $j=1,2$ defines a cylinder $Z_j=\mathbb{R}\times S^1$ so that we have a natural diffeomorphism

$$T^2\times\mathbb{R}^2=Z_1\times Z_2.$$

We define on each Z_j the standard complex structure $i\partial_{p_j}=\partial_{q_j}$ and symplectic structure $\omega_0=dp_j\wedge dq_j$, so that ω_0 on $Z_1\times Z_2$ is the direct sum $\omega_0\oplus\omega_0$, and we can similarly define a compatible complex structure i on $T^2\times\mathbb{R}^2$ as $i\oplus i$. This makes $(T^2\times\mathbb{R}^2,\omega_0,i)$ into a Stein manifold, with plurisubharmonic function $f:T^2\times\mathbb{R}^2\to[0,\infty):(q,p)\mapsto\frac{1}{2}|p|^2$ such that $-df\circ i=\lambda_0$, and the latter induces the Liouville vector field

$$\nabla f=p_1\partial_{p_1}+p_2\partial_{p_2},$$

whose flow is given by $\varphi^t_{\nabla f}(q,p)=(q,e^tp)$. The restriction of λ_0 to $\partial(T^2\times\mathbb{D})=T^3$ gives the standard contact form, which we'll denote in the following by α_0. We will use the coordinates (q,p) on T^3 with the assumption that $|p|=1$, and sometimes also write $(p_1,p_2)=(\cos 2\pi\theta,\sin 2\pi\theta)$ with $\theta\in S^1$.

We can use the flow of ∇f to embed the symplectization of T^3 into $(T^2\times\mathbb{R}^2,\omega_0)$: explicitly,

$$\Phi:(\mathbb{R}\times T^3,d(e^a\alpha_0))\hookrightarrow(T^2\times\mathbb{R}^2,\omega_0):(a,(q,p))\mapsto(q,e^ap)$$

satisfies $\Phi^*\lambda_0=e^a\alpha_0$. Using this to identify $(0,\infty)\times T^3$ with the complement of $T^2\times\mathbb{D}$, we can now choose a new almost complex structure J_0 with $J_0\partial_{p_j}=g(|p|)\partial_{q_j}$ for some function g, so that $J_0=i$ near the zero section and becomes $\mathbb{R}$–invariant on the end, in other words $J_0|_{[0,\infty)\times T^3}\in\mathcal{J}_{\alpha_0}(T^3)$. This choice of J_0 has precisely the form on $[0,\infty)\times T^3$ that was used in Example 2.11 (via Remark 2.13). In terms of the splitting $T^2\times\mathbb{R}^2=Z_1\times Z_2$, the cylinders $Z_1\times\{*\}$ and $\{*\}\times Z_2$ are now finite energy J_0–holomorphic curves, and those which lie entirely in $[0,\infty)\times T^3$ reproduce the foliations constructed in Example 2.11. In particular, each cylinder $Z_1\times\{*\}$ is asymptotic to a pair of Reeb orbits in the Morse-Bott tori $\{\theta=0,1/2\}$ with the same value of the coordinate $q_2\in S^1$ at both ends, and a corresponding statement is true for $\{*\}\times Z_2$ with the Morse-Bott tori $\{\theta=1/4,3/4\}$.

We shall now construct more holomorphically foliated model fillings using surgery along the zero section in $T^2\times\mathbb{R}^2$. The following is a special case of the surgery along a Lagrangian 2–torus in a symplectic 4–manifold introduced by Luttinger in [Lut95]; our formulation is borrowed from [ADK03].

For $r>0$, let $K_r=T^2\times[-r,r]\times[-r,r]$. Choose constants $\sigma:=(c,k_1,k_2)\in(0,\infty)\times\mathbb{Z}^2$ and a smooth cutoff function $\beta:\mathbb{R}\to[0,1]$ such that

- $\beta=0$ on a neighborhood of $(-\infty,-1]$,
- $\beta=1$ on a neighborhood of $[1,\infty)$,
- $\int_{-1}^1 t\beta'(t)\,dt=0$.

Define also the function $\chi:\mathbb{R}\to\mathbb{R}$ to equal 0 on $(-\infty,0)$ and 1 on $[0,\infty)$. Then there is a symplectomorphism $\psi_\sigma:(K_{2c}\setminus K_c,\omega_0)\to(K_{2c}\setminus K_c,\omega_0)$ given by

$$\psi_\sigma(q_1,q_2,p_1,p_2)=\left(q_1+k_1\chi(p_2)\beta\left(\frac{p_1}{c}\right),q_2+k_2\chi(p_1)\beta\left(\frac{p_2}{c}\right),p_1,p_2\right).$$

We construct a new symplectic manifold (W_σ,ω_σ) by deleting K_c from $T^2\times\mathbb{R}^2$ and gluing in K_{2c} via ψ_σ:

$$(W_\sigma,\omega_\sigma)=((T^2\times\mathbb{R}^2)\setminus K_c,\omega_0)\cup_{\psi_\sigma}(K_{2c},\omega_0).$$

In the following, we shall regard both $((T^2\times\mathbb{R}^2)\setminus K_c,\omega_0)$ and (K_{2c},ω_0) as symplectic subdomains of (W_σ,ω_σ), and fix local coordinates as follows. Let (q_1,q_2,p_1,p_2) denote the usual coordinates on $(T^2\times\mathbb{R}^2)\setminus K_c$, now viewed as a subset of W_σ, and on the glued in copy of $K_{2c}\subset W_\sigma$, denote the natural coordinates by (Q_1,Q_2,P_1,P_2). Thus on the region of overlap, $(q,p)=\psi_\sigma(Q,P)$ and

$$\omega_\sigma=dp_1\wedge dq_1+dp_2\wedge dq_2=dP_1\wedge dQ_1+dP_2\wedge dQ_2.$$

Observe that the (Q,P)–coordinates can be extended globally so that they define a symplectomorphism $(Q,P):(W_\sigma,\omega_\sigma)\to(T^2\times\mathbb{R}^2,\omega_0)$.

If $2c=e^R$, then the part of (W_σ,ω_σ) identified with $((T^2\times\mathbb{R}^2)\setminus K_c,\omega_0)$ naturally contains a symplectization end of the form $([R,\infty)\times T^3,d(e^a\alpha_0))$.

Lemma 5.1. *W_σ admits a 1–form λ_σ such that $d\lambda_\sigma=\omega_\sigma$ and $\lambda_\sigma|_{[R,\infty)\times T^3}=e^a\alpha_0$.*

Proof. The 1–form $e^a\alpha_0$ is the restriction to $[R,\infty)\times T^3$ of $\lambda_0:=p_1\,dq_1+p_2\,dq_2$, which is a well defined primitive of $\omega_0=\omega_\sigma$ on $(T^2\times\mathbb{R}^2)\setminus K_c$. Define $f(s)=\frac{1}{c}\int_{-c}^s t\beta'(t/c)\,dt$, a smooth function with support in $(-c,c)$ due to our assumptions on β. Then there is a smooth function $\Phi:(T^2\times\mathbb{R}^2)\setminus K_c\to\mathbb{R}$ defined by

$$\Phi(q_1,q_2,p_1,p_2)=\begin{cases}k_2f(p_2) & \text{if } p_1\geq c,\\ k_1f(p_1) & \text{if } p_2\geq c,\\ 0 & \text{otherwise,}\end{cases}$$

and a brief computation shows that on $(T^2 \times \mathbb{R}^2) \setminus K_c$, $\lambda_0 = P_1\, dQ_1 + P_2\, dQ_2 + d\Phi$. Now choosing a smooth function $\widehat{\Phi} : W_\sigma \to \mathbb{R}$ that matches Φ on $[R, \infty) \times T^3$ and vanishes in K_c, a suitable primitive is given by

$$\lambda_\sigma = P_1\, dQ_1 + P_2\, dQ_2 + d\widehat{\Phi}.$$

□

We wish to define an ω_σ–compatible almost complex structure J_σ on W_σ that matches J_0 on the end $[R, \infty) \times T^3$, i.e. for $|p| \ge e^R$, J_σ satisfies $-J_\sigma \partial_{q_j} = G(|p|)\partial_{p_j}$ for some positive smooth function G. Switching to (Q, P)–coordinates in K_{2c}, J_σ is now determined in $K_{2c} \cap ([R, \infty) \times T^3)$ by the conditions

$$-J_\sigma \partial_{Q_1} = \partial_{P_1} - G(|P|)\frac{k_1}{c}\chi(P_2)\beta'(P_1/c)\ \partial_{Q_1},$$
$$-J_\sigma \partial_{Q_2} = \partial_{P_2} - G(|P|)\frac{k_2}{c}\chi(P_1)\beta'(P_2/c)\ \partial_{Q_2}.$$

Thus if we replace χ in this expression by the cutoff function $t \mapsto \beta(t/c)$, which equals χ outside of $[-c, c]$, we obtain the desired extension of J_σ over K_{2c}. The following lemma is immediate.

Lemma 5.2. *For each constant $(\rho, \eta) \in \mathbb{R} \times S^1$, the surfaces $Z_1^{(\rho,\eta)} := \{(P_2, Q_2) = (\rho, \eta)\}$ and $Z_2^{(\rho,\eta)} := \{(P_1, Q_1) = (\rho, \eta)\}$ in W_σ are images of embedded finite energy J_σ–holomorphic cylinders. Moreover,*

1. *Each point in W_σ is the unique intersection point of a unique pair $Z_1^{(\rho,\eta)}$ and $Z_2^{(\rho',\eta')}$, whose tangent spaces at that point are symplectic complements.*
2. *For $|\rho| \ge c$, the cylinders $Z_1^{(\rho,\eta)}$ and $Z_2^{(\rho,\eta)}$ are identical to $Z_1 \times \{(\rho, \eta)\}$ and $\{(\rho, \eta)\} \times Z_2$ respectively in $T^2 \times \mathbb{R}^2 = Z_1 \times Z_2$. This collection therefore contains all of the curves in $[R, \infty) \times T^3$ constructed via Example 2.11 and Remark 2.13.*

The essential difference between $(W_\sigma, \omega_\sigma)$ and $(T^2 \times \mathbb{R}^2, \omega_0)$ is that they each come with holomorphic foliations that behave differently at infinity: the cylinder $Z_1^{(\rho,\eta)}$ for instance has one end asymptotic to the Reeb orbit at $\{\theta = 1/2, q_2 = \eta\}$, while its other end approaches the orbit at $\{\theta = 0, q_2 = \eta + k_2\beta(\rho/c)\}$. Thus the data $\sigma = (c, k_1, k_2)$ determine offsets within the respective families of Morse-Bott orbits at one end of each cylinder.

5.2. **Classification up to symplectomorphism.** Assume (W, ω) is a minimal strong filling of (T^3, ξ_0). Adopting the notation from §4, (W^R, ω) is the enlarged filling obtained by attaching a trivial symplectic cobordism such that the induced contact form at ∂W^R is $e^R\alpha_0$, and we can further attach a cylindrical end $([R, \infty) \times T^3, d(e^a\alpha_0))$ to construct (W^∞, ω). If (W, ω) is an exact filling with primitive λ, then we can also assume λ extends over W^∞ so that $\lambda|_{[R,\infty)\times T^3} = e^a\alpha_0$. Choosing an almost complex

structure J that is generic in W^R and has the standard form $J_0 \in \mathcal{J}_{\alpha_0}(T^3)$ on $[R, \infty) \times T^3$, we start from a finite energy foliation constructed as in Example 2.11 (via Remark 2.13), consisting of cylinders with ends asymptotic to orbits in the two Morse-Bott tori $Z = \{\theta \in \{0, 1/2\}\}$, then use Theorem 7 to produce a moduli space $\mathcal{M}_1$ of J–holomorphic cylinders foliating W^∞. Since (W, ω) is minimal, this produces a smooth fibration $\Pi_1 : W^\infty \to \mathcal{M}_1$, where both the fiber and the base are diffeomorphic to $\mathbb{R} \times S^1$.

We can now repeat the same trick starting from a different foliation of T^3: let $Z' = \{\theta \in \{1/4, 3/4\}\}$, a pair of Morse-Bott tori with Reeb orbits pointing in the direction orthogonal to those on Z. Then by a minor modification of the construction in Example 2.11, the fibration

$$T^3 \setminus Z' \to \{0, 1\} \times S^1$$
$$(q_1, q_2, \theta) \mapsto \begin{cases} (0, q_1) & \text{if } \theta \in (-1/4, 1/4), \\ (1, q_1) & \text{if } \theta \in (1/4, 3/4) \end{cases}$$

can also be presented as the projection to T^3 of a positive finite energy foliation on $\mathbb{R} \times T^3$, with the same contact form and almost complex structure as before. This yields a second moduli space $\mathcal{M}_2$ of J–holomorphic cylinders foliating W^∞, and a corresponding fibration $\Pi_2 : W^\infty \to \mathcal{M}_2 \cong \mathbb{R} \times S^1$.

Lemma 5.3. *Any $u_1 \in \mathcal{M}_1$ and $u_2 \in \mathcal{M}_2$ intersect each other exactly once, with intersection index $+1$.*

Proof. One can verify this explicitly from the foliations on $[R, \infty) \times T^3$ whenever both curves are near the boundaries of their respective moduli spaces, and since they have no asymptotic orbits in common, this implies $i(u_1; u_2) = 1$. The latter is a homotopy invariant condition, and the fact that the two curves have separate orbits guarantees that there is never any asymptotic contribution, hence there is always a unique intersection point $u_1(z_1) = u_2(z_2)$, contributing $+1$ to the intersection count. □

It follows that the map

$$\Pi_1 \times \Pi_2 : W^\infty \to \mathcal{M}_1 \times \mathcal{M}_2$$

is a diffeomorphism. Our goal is to use this to identify W^∞ with one of the model fillings constructed in §5.1.

For $\theta \in \{0, 1/4, 1/2, 3/4\}$, denote by $\mathcal{P}_\theta$ the 1–dimensional manifold of Morse-Bott orbits foliating the 2–torus whose θ–coordinate has the given value: each of these can be naturally identified with S^1 using either the q_1 or q_2–coordinate. Then as explained in the appendix, there exist real line bundles

$$E^\theta \to \mathcal{P}_\theta,$$

where the fibers E_x^θ are 1–dimensional eigenspaces of the asymptotic operators at $x \in \mathcal{P}_\theta$, and the asymptotic formula (A.3) defines "asymptotic evaluation maps"

$$\mathcal{M}_1 \xrightarrow{\mathrm{ev}_0} E^0 \qquad \mathcal{M}_1 \xrightarrow{\mathrm{ev}_{1/2}} E^{1/2}$$
$$\mathcal{M}_2 \xrightarrow{\mathrm{ev}_{1/4}} E^{1/4} \qquad \mathcal{M}_2 \xrightarrow{\mathrm{ev}_{3/4}} E^{3/4}.$$

For any $\sigma = (c, k_1, k_2) \in (0,\infty) \times \mathbb{Z}^2$, let $\mathcal{M}_1^\sigma$ and $\mathcal{M}_2^\sigma$ denote the moduli spaces of J_σ–holomorphic cylinders $Z_1^{(\rho,\eta)}$ and $Z_2^{(\rho,\eta)}$ respectively in $(W_\sigma, \omega_\sigma)$, constructed in the previous section: as a special case, $\mathcal{M}_1^0$ and $\mathcal{M}_2^0$ will denote the spaces of J_0–holomorphic cylinders $Z_1 \times \{*\}$, $\{*\} \times Z_2$ in $(T^2 \times \mathbb{R}^2, \omega_0)$. These last two moduli spaces are each canonically identified with $\mathbb{R} \times S^1$, and they also come with asymptotic evaluation maps ev_θ^0, defined as above. These are manifestly diffeomorphisms and have the property that the resulting maps

$$\begin{aligned} &(\mathrm{ev}_\theta^0)^{-1} \circ \mathrm{ev}_\theta : \mathcal{M}_1 \to \mathcal{M}_1^0 \text{ for } \theta = 0, 1/2, \\ &(\mathrm{ev}_\theta^0)^{-1} \circ \mathrm{ev}_\theta : \mathcal{M}_2 \to \mathcal{M}_2^0 \text{ for } \theta = 1/4, 3/4 \end{aligned} \tag{5.1}$$

are proper: indeed, for any $u \in \mathcal{M}_j$ outside of some compact subset, they define the natural identification between curves in $\mathcal{M}_j$ and $\mathcal{M}_j^0$ that are contained in the cylindrical end.

Lemma 5.4. *The maps defined in* (5.1) *are diffeomorphisms.*

Proof. They are local diffeomorphisms due to Lemma A.2. The claim thus reduces to the fact that any local diffeomorphism with compact support on a cylinder $\mathbb{R} \times S^1$ is a global diffeomorphism. □

By the lemma, we can compose (5.1) with the canonical identifications $\mathcal{M}_j^0 = \mathbb{R} \times S^1$ and define diffeomorphisms

$$\varphi_\theta : \mathcal{M}_1 \to \mathbb{R} \times S^1 \text{ for } \theta = 0, 1/2,$$
$$\varphi_\theta : \mathcal{M}_2 \to \mathbb{R} \times S^1 \text{ for } \theta = 1/4, 3/4,$$

so that the resulting compositions $\varphi_0 \circ \varphi_{1/2}^{-1}$ and $\varphi_{1/4} \circ \varphi_{3/4}^{-1}$ are diffeomorphisms of $\mathbb{R} \times S^1$ with compact support. Choose $c > 0$ sufficiently large so that both of these are supported in $[-c, c] \times S^1$ and (making R larger if necessary) $2c = e^R$. Now, recalling the cutoff function β from §5.1, set $\sigma = (c, k_1, k_2)$ where k_1, k_2 are the unique integers such that there is an isotopy $\{\psi_t^1 \in \mathrm{Diff}(\mathbb{R} \times S^1)\}_{t\in[0,1]}$ supported in $[-c, c] \times S^1$, with $\psi_0^1 = \varphi_0 \circ \varphi_{1/2}^{-1}$ and

$$\psi_1^1(\rho, \eta) = (\rho, \eta + k_2\beta(\rho/c)),$$

and similarly there is an isotopy ψ_t^2 from $\varphi_{1/4} \circ \varphi_{3/4}^{-1}$ to

$$\psi_1^2(\rho, \eta) = (\rho, \eta + k_1\beta(\rho/c)).$$

From now on, we will use the diffeomorphisms $\varphi_{1/2}$ and $\varphi_{3/4}$ to parametrize $\mathcal{M}_1$ and $\mathcal{M}_2$ respectively, denoting

$$u_1^{(\rho,\eta)} := \varphi_{1/2}^{-1}(\rho, \eta), \qquad u_2^{(\rho,\eta)} := \varphi_{3/4}^{-1}(\rho, \eta).$$

The point of this convention is that $u_1^{(\rho,\eta)} \in \mathcal{M}_1$ now approaches the Morse-Bott family $\{\theta = 1/2\}$ at the same orbit and along the same asymptotic eigenfunction as $Z_1^{(\rho,\eta)} \in \mathcal{M}_1^\sigma$, and a corresponding statement holds for $\mathcal{M}_2$ and $\mathcal{M}_2^\sigma$.

Lemma 5.5. *There exist constants $R_2 > R_1 > R$, an almost complex structure $\hat{J}$ on W^∞ tamed by ω, and moduli spaces $\widehat{\mathcal{M}}_1$ and $\widehat{\mathcal{M}}_2$ of embedded finite energy $\hat{J}$–holomorphic cylinders foliating W^∞, which have the following properties. For $j \in \{1, 2\}$, $\widehat{\mathcal{M}}_j$ can be parametrized by a cylinder*

$$\mathbb{R} \times S^1 \ni (\rho, \eta) \mapsto \hat{u}_j^{(\rho,\eta)} \in \widehat{\mathcal{M}}_j$$

such that

1. *In the region $W^R \cup ([R, R_1] \times T^3)$, $\hat{J} \equiv J$ and $\hat{u}_j^{(\rho,\eta)}$ is identical to $u_j^{(\rho,\eta)} \in \mathcal{M}_j$.*
2. *In $[R_2, \infty) \times T^3$, $\hat{J} \equiv J_\sigma$ and $\hat{u}_j^{(\rho,\eta)}$ is identical to $Z_j^{(\rho,\eta)} \in \mathcal{M}_j^\sigma$, where we use the natural identification of the ends of W^∞ and W_σ.*
3. *Lemma 5.3 holds also for the spaces $\widehat{\mathcal{M}}_1$ and $\widehat{\mathcal{M}}_2$.*

Proof. The curves $u_j^{(\rho,\eta)}$ already have the desired properties when $|\rho| \geq c$, so changes are needed only on compact subsets of $\mathcal{M}_j$, and only near the ends of these curves. The idea is simply to modify the foliation defined by $\{u_j^{(\rho,\eta)}\}_{(\rho,\eta)\in[-c,c]\times S^1}$ outside of a large compact subset to a new foliation of the same region such that the change to the tangent spaces is uniformly small. One can then make the new foliation $\hat{J}$–holomorphic for some $\hat{J}$ that is uniformly close to J and therefore also tamed by ω. Lemma 5.3 is trivial to verify for the modified foliations, because adjustments to $\mathcal{M}_1$ happen only in a region where $\mathcal{M}_2$ is unchanged, and vice versa. We proceed in two steps.

Choose $R_1 > 0$ sufficiently large so that for $|\rho| \leq c$, the tangent spaces of the curves $u_j^{(\rho,\eta)}$ in $[R_1, \infty) \times T^3$ are uniformly close to the tangent spaces of the asymptotic orbit cylinders. Then choosing R' much larger than R_1, a sufficiently gradual adjustment of the remainder term in the asymptotic formula (A.3) produces a new surface $\hat{u}_j^{(\rho,\eta)}$ in $[R_1, R'] \times T^3$ that looks like $u_j^{(\rho,\eta)}$ near $\{R_1\} \times T^3$ and $Z_j^{(\rho',\eta')} \in \mathcal{M}_j^0$ near $\{R'\} \times T^3$, where (ρ', η') is related to (ρ, η) via the diffeomorphism $\varphi_0 \circ \varphi_{1/2}^{-1}$ or $\varphi_{1/4} \circ \varphi_{3/4}^{-1}$.

It remains to adjust the parameters (ρ', η') so that in $[R_2, \infty) \times T^3$ for some $R_2 > R'$, $\hat{u}_j^{(\rho,\eta)}$ matches $Z_j^{(\rho,\eta)} \in \mathcal{M}_j^\sigma$. For this we use the isotopies ψ_t^j, defining the surface $\hat{u}_j^{(\rho,\eta)}$ so that its intersection with $\{s\} \times T^3$ for $s \in [R', R_2]$ matches $Z_j^{\psi^j_{f(t)}(\rho,\eta)} \in \mathcal{M}_j^0$ for some function $f : [R', R_2] \to [0, 1]$ with sufficiently small derivative. (Of course, R_2 must be large). □

We can now carry out the deformation argument.

Proposition 5.6. *There exists a diffeomorphism $\psi : W_\sigma \to W^\infty$ which restricts to the identity on $[R_2,\infty) \times T^3$, such that the 2–forms*

$$\omega_{(t)} := t\psi^*\omega + (1-t)\omega_\sigma$$

are symplectic for all $t \in [0,1]$.

Proof. Applying Lemma 5.3 to the spaces $\widehat{\mathcal{M}}_1$ and $\widehat{\mathcal{M}}_2$ and using the given identifications of both with $\mathbb{R} \times S^1$, we have a diffeomorphism

$$\widehat{\Pi}_1 \times \widehat{\Pi}_2 : W^\infty \to \widehat{\mathcal{M}}_1 \times \widehat{\mathcal{M}}_2 = (\mathbb{R} \times S^1) \times (\mathbb{R} \times S^1),$$

and there is a similar diffeomorphism

$$\Pi_1^\sigma \times \Pi_2^\sigma : W_\sigma \to \mathcal{M}_1^\sigma \times \mathcal{M}_2^\sigma = (\mathbb{R} \times S^1) \times (\mathbb{R} \times S^1).$$

Composing the second with the inverse of the first yields a diffeomorphism $\psi : W_\sigma \to W^\infty$ which equals the identity in $[R_2,\infty) \times T^3$. We claim that $\omega_{(t)} = t\psi^*\omega + (1-t)\omega_\sigma$ is nondegenerate, and thus symplectic for every $t \in [0,1]$. Indeed, the almost complex structure $\psi^*\hat{J}$ tames $\omega_{(1)} = \psi^*\omega$, and it also tames $\omega_{(0)} = \omega_\sigma$ since every tangent space now splits into a sum of ω_σ–symplectic complements that are also $\psi^*\hat{J}$–invariant. Thus $\psi^*\hat{J}$ is also tamed by $\omega_{(t)}$ for every $t \in [0,1]$, proving the claim. □

Proposition 5.7. *If (W,ω) is an exact filling, then one can arrange the diffeomorphism of Prop. 5.6 to be a symplectomorphism $(W_\sigma,\omega_\sigma) \to (W^\infty,\omega)$.*

Proof. Let $\psi : W_\sigma \to W^\infty$ be the diffeomorphism constructed in Prop. 5.6. By Lemma 5.1, there is a 1–form λ_σ on W_σ that satisfies $d\lambda_\sigma = \omega_\sigma$ globally and matches $\lambda = \psi^*\lambda = e^a\alpha_0$ on $[R_2,\infty) \times T^3$. Now $\omega_{(t)} = d\lambda_{(t)}$, where $\lambda_{(t)} = t\psi^*\lambda + (1-t)\lambda_\sigma$. Define a time-dependent vector field V_t on W_σ by

$$\omega_{(t)}(V_t,\cdot) = \lambda_\sigma - \psi^*\lambda.$$

Since $\lambda_\sigma - \psi^*\lambda$ vanishes in $[R_2,\infty) \times T^3$, the flow φ_V^t of V_t has compact support and is well defined for all t: the map

$$\psi \circ \varphi_V^1 : W_\sigma \to W^\infty$$

then gives the desired symplectomorphism $(W_\sigma,\omega_\sigma) \to (W^\infty,\omega)$. □

Proof of Theorem 4. By Prop. 5.6, (W,ω) is symplectically deformation equivalent to an exact filling, so let us assume from now on that it is exact. Then by Prop. 5.7, there is a symplectomorphism $\psi : (W^\infty,\omega) \to (W_\sigma,\omega_\sigma)$ which equals the identity in $[R,\infty) \times T^3$ for sufficiently large R, and we shall now use it to construct a symplectomorphism of (W,ω) to a star shaped domain in T^*T^2. Choose a global primitive λ of ω which matches $e^a\alpha_0$ on $[R,\infty) \times T^3$ and denote by Y and Y_σ the Liouville vector fields corresponding to λ and λ_σ respectively, so

$$\omega(Y,\cdot) = \lambda, \qquad \omega_\sigma(Y_\sigma,\cdot) = \lambda_\sigma.$$

Both of these match ∂_a on $[R,\infty) \times T^3$. There is also another Liouville vector field Y_0 on W_σ defined by $\omega_\sigma(Y_0,\cdot) = P_1\, dQ_1 + P_2\, dQ_2$, thus

$$Y_0 = P_1\, \partial_{P_1} + P_2\, \partial_{P_2},$$

and by the construction of λ_σ, $Y_0 = Y_\sigma$ on K_c. All of these have globally defined flows which dilate the respective symplectic forms, e.g. $(\varphi_Y^t)^*\omega = e^t\omega$ for all $t \in \mathbb{R}$.

By the construction of W^∞, there is a smooth function $f : T^3 \to \mathbb{R}$ such that the closure of $(W^\infty \setminus W,\omega)$ is the trivial symplectic cobordism $(\mathcal{S}_f^\infty, d(e^a\alpha_0))$, and $Y = \partial_a$ on this region. Now choose $T > 0$ sufficiently large so that

$$\varphi_Y^T(\partial W) \subset [R,\infty) \times T^3,$$

thus φ^T gives a symplectomorphism $(W,\omega) \to (\varphi_Y^T(W), e^{-T}\omega)$. Then $\psi\circ\varphi_Y^T$ maps (W,ω) symplectomorphically to the domain in $(W_\sigma, e^{-T}\omega_\sigma)$ bounded by $\partial\mathcal{S}_{-\infty}^{f+T} \subset [R,\infty) \times T^3$, which is transverse to Y_σ. The composition

$$\psi_T := \varphi_{Y_\sigma}^{-T} \circ \psi \circ \varphi_Y^T : (W^\infty,\omega) \to (W_\sigma,\omega_\sigma)$$

now maps W to a compact domain in W_σ with boundary transverse to Y_σ.

Recall next from the proof of Lemma 5.1 that $\lambda_\sigma = P_1\, dQ_1 + P_2\, dQ_2 + d\widehat{\Phi}$ for some smooth function $\widehat{\Phi} : W_\sigma \to \mathbb{R}$ that vanishes in K_c, and we can assume without loss of generality that $\Phi(Q_1,Q_2,P_1,P_2)$ depends only on P_1 and P_2. It follows that

$$Y_\sigma = Y_0 + \widehat{Y}$$

for some vector field $\widehat{Y}$ that vanishes in K_c and has components only in the Q_1 and Q_2–directions. We can therefore choose $\tau > 0$ sufficiently large so that $\varphi_{Y_\sigma}^{-\tau}$ maps $\psi_T(W)$ into K_c and then

$$\varphi_{Y_0}^\tau \circ \varphi_{Y_\sigma}^{-\tau} : (W_\sigma,\omega_\sigma) \to (W_\sigma,\omega_\sigma)$$

is a symplectomorphism that maps $\psi_T(W)$ to a compact domain with boundary transverse to Y_0. Under the symplectomorphism $(W_\sigma,\omega_\sigma) \to (T^2 \times \mathbb{R}^2,\omega_0)$ defined by the (Q,P)–coordinates, this becomes a star shaped domain. Since all such domains can be deformed symplectically to the standard filling $(T^2 \times \mathbb{D},\omega_0)$, the uniqueness of strong fillings follows. □

5.3. **Symplectomorphism groups.** We now prove Theorem 5: observe that by the Whitehead theorem, it suffices to prove that $\mathrm{Symp}_c(T^*T^2,\omega_0)$ is *weakly* contractible, i.e. $\pi_n(\mathrm{Symp}_c(T^*T^2,\omega_0)) = 0$ for every $n \ge 0$. The main idea of the argument goes back to Gromov [Gro85] in the closed case, and was also used by Hind [Hin03] in a situation analogous to ours (fillings of Lens spaces). The key is to construct a family of foliations by J–holomorphic cylinders for J varying in a ball whose boundary is determined by a given map $S^n \to \mathrm{Symp}_c(T^*T^2)$. Here it is crucial to note that since ω_0 is exact and the closed Reeb orbits in $T^3 = T^2 \times \partial\mathbb{D}$ are never contractible in $T^2 \times \mathbb{D}$, there cannot exist any closed or 1–punctured J–holomorphic spheres, hence the moduli spaces we construct have no nodal degenerations. In this situation, Theorems 7 and 8 go through without any genericity assumption for J (see Remark 3.4).

As in §5.1, choose an almost complex structure J_0 which matches the standard complex structure near the zero section and belongs to $\mathcal{J}_{\alpha_0}(T^3)$ on the cylindrical end $[0,\infty) \times T^3$, where it matches the form used in Example 2.11. Let λ_0 denote the canonical 1–form on T^*T^2, so $d\lambda_0 = \omega_0$.

Suppose now that

$$S^n \to \mathrm{Symp}_c(T^*T^2, \omega_0) : x \mapsto \psi_x$$

is a smooth family of symplectomorphisms which all equal the identity on $[R,\infty) \times T^3$ for some $R \geq 0$, and there is a fixed base point $x_0 \in S^n$ such that $\psi_{x_0} = \mathrm{Id}$. Let $J_x = \psi_x^* J_0$ for each $x \in S^n$: these are all ω_0–compatible almost complex structures that match J_0 on $[R,\infty)$. Now using the contractibility of the space of compatible almost complex structures, the family $\{J_x\}_{x\in S^n}$ can be filled in to a smooth family $\{J_x\}_{x\in B^{n+1}}$ that are all compatible with ω_0 and equal J_0 on $[R,\infty) \times T^3$, where B^n denotes the closed unit ball in $\mathbb{R}^n$.

Applying Theorem 8 (with Remark 3.4 in mind), there are now two unique smooth families of moduli spaces $\mathcal{M}_1^x$ and $\mathcal{M}_2^x$ for $x \in B^{n+1}$, each of which consists of embedded J_x–holomorphic cylinders foliating T^*T^2, such that each curve in $\mathcal{M}_1^x$ has one transverse intersection with each curve in $\mathcal{M}_2^x$. We have $J_{x_0} = J_0$, thus the curves in $\mathcal{M}_1^{x_0}$ and $\mathcal{M}_2^{x_0}$ are precisely the cylinders that make up the splitting

$$T^*T^2 = T^2 \times \mathbb{R}^2 = (\mathbb{R} \times S^1) \times (\mathbb{R} \times S^1),$$

as was explained in §5.1. More generally, for $x \in \partial B^{n+1}$ and $j \in \{1,2\}$, the curves in $\mathcal{M}_j^x$ can be obtained by composing curves in $\mathcal{M}_j^{x_0}$ with the symplectomorphism ψ_x^{-1}, and are thus identical on $[R,\infty) \times T^3$ to the curves in $\mathcal{M}_j^{x_0}$. As in the previous section, we can now use asymptotic evaluation maps to define diffeomorphisms

$$\mathbb{R} \times S^1 \to \mathcal{M}_j^x : (\rho,\eta) \mapsto u_{j,x}^{(\rho,\eta)}.$$

Arguing further as in Lemma 5.5, for $x \in B^{n+1} \setminus S^n$, change J_x on a region near infinity to a smooth family $\hat{J}_x$ tamed by ω_0 and matching J_0 on some region $[R_2,\infty) \times T^3$, such that for every fixed parameter (ρ,η), the curves $\hat{u}_{j,x}^{(\rho,\eta)}$ in the resulting moduli spaces $\widehat{\mathcal{M}}_j^x$ are identical on $[R_2,\infty) \times T^3$ for all $x \in B^{n+1}$. Then the intersection points define a smooth family of diffeomorphisms

$$\psi_x : T^*T^2 \to \widehat{\mathcal{M}}_1^x \times \widehat{\mathcal{M}}_2^x = (\mathbb{R} \times S^1) \times (\mathbb{R} \times S^1) = T^*T^2,$$

which match the original family $\psi_x \in \mathrm{Symp}_0(T^*T^2, \omega_0)$ for $x \in \partial B^{n+1}$ and all equal the identity on $[R_2,\infty) \times T^3$. We have now a smooth family of

symplectic forms $\omega_x := \psi_x^* \omega_0$ which are all standard on $[R_2,\infty) \times T^3$ and match ω_0 globally for $x \in \partial B^{n+1}$.

Lemma 5.8. *There exists a smooth family of* 1*–forms* $\{\lambda_x\}_{x\in B^{n+1}}$ *on* T^*T^2 *such that*

(1) $d\lambda_x = \omega_x$,
(2) $\lambda_x \equiv \lambda_0$ *for every* $x \in \partial B^{n+1}$,
(3) $\lambda_x = \lambda_0$ *on* $[R_2,\infty) \times T^3$ *for every* $x \in B^{n+1}$.

Proof. For each $x \in \partial B^{n+1}$, ψ_x is a symplectomorphism and thus $\lambda_0 - \psi_x^*\lambda_0$ is a closed 1–form with compact support. All such 1–forms are exact: indeed, any element of $H_1(T^*T^2)$ can be represented by a cycle γ lying outside the support of $\lambda_0 - \psi_x^*\lambda_0$, hence

$$\int_\gamma (\lambda_0 - \psi_x^*\lambda_0) = 0 \text{ for all } [\gamma] \in H_1(T^*T^2),$$

implying $[\lambda_0 - \psi_x^*\lambda_0] = 0 \in H^1_{DR}(T^*T^2)$. Then for $x \in \partial B^{n+1}$ there is a unique smooth family of compactly supported functions $f_x : T^*T^2 \to \mathbb{R}$ such that

$$\lambda_0 = \psi_x^*\lambda_0 + df_x.$$

Extending f_x to a smooth family of compactly supported functions for $x \in B^{n+1}$, the desired 1–forms can be defined by $\lambda_x = \psi_x^*\lambda_0 + df_x$. □

Now given the 1–forms λ_x from the lemma, define for $t \in [0,1]$,

$$\lambda_x^{(t)} := t\lambda_x + (1-t)\lambda_0, \qquad \omega_x^{(t)} := d\lambda_x^{(t)}.$$

The almost complex structure $\hat{J}_x$ is tamed by ω_0, and using the holomorphic foliations as in the proof of Theorem 4, we see that it is also tamed by $\omega_x = \psi_x^*\omega_0$, and thus by all $\omega_x^{(t)}$ for $t \in [0,1]$, proving that the latter are symplectic. Now define a smooth family of time-dependent vector fields V_x^t by

$$\omega_x^{(t)}(V_x^t, \cdot) = \lambda_0 - \lambda_x.$$

These vanish identically when $x \in \partial B^{n+1}$ and also vanish outside of a compact set for all x, thus the flows $\varphi_{V_x}^t$ are well defined and compactly supported for all t, and trivial if $x \in \partial B^{n+1}$. Moreover, $(\varphi_{V_x}^t)^*\omega_x^{(t)} = \omega_0$. We thus obtain a smooth family of compactly supported symplectomorphisms on (T^*T^2, ω_0) for $x \in B^{n+1}$ via the composition $\psi_x \circ \varphi_{V_x}^1$, which matches ψ_x for $x \in \partial B^{n+1}$. This shows that $\pi_n(\mathrm{Symp}_c(T^*T^2, \omega_0)) = 0$ for all n, and thus completes the proof of Theorem 5.

Appendix A. Fredholm and intersection theory

A.1. **Transversality.** In this appendix we recall some useful technical facts about finite energy J–holomorphic curves. Adopting the notation of §3, $(W^\infty, \omega) = (W,\omega) \cup_{\partial W} ([0,\infty) \times M, d(e^a\lambda))$ is the union of a compact symplectic manifold (W,ω) with contact boundary $\partial W = M$ attached smoothly to the positive cylindrical end $([0,\infty)\times M, d(e^a\lambda))$, where λ is a Morse-Bott contact form on M, defining the contact structure $\xi = \ker\lambda$. Let J denote an ω–compatible almost complex structure on W^∞ which is in $\mathcal{J}_\lambda(M)$ at the positive end. Then any nonconstant punctured J–holomorphic curve $u : (\dot{\Sigma}, j) \to (W^\infty, J)$ with finite energy is asymptotic at each puncture $z \in \Gamma$ to some periodic orbit of the Reeb vector field X_λ, for which we can choose a parametrization $x_z : S^1 \to M$ with $\lambda(\dot{x}_z)$ identically equal to the period $T_z > 0$. In order to describe the analytical invariants of u, it is convenient to introduce the *asymptotic operators*

$$\mathbf{A}_z : \Gamma(x_z^*\xi) \to \Gamma(x_z^*\xi) : v \mapsto -J(\nabla_t v - T_z \nabla_v X_\lambda),$$

where ∇ is any symmetric connection on M. Morally, this is the Hessian of the contact action functional on $C^\infty(S^1, M)$, whose critical points are periodic orbits; in particular one can show that $\mathbf{A}_z$ has trivial kernel if and only if the orbit x_z is nondegenerate. Choosing a unitary trivialization Φ for $x_z^*\xi$, $\mathbf{A}_z$ becomes identified with the operator

$$C^\infty(S^1, \mathbb{R}^2) \to C^\infty(S^1, \mathbb{R}^2) : v \mapsto -J_0\dot{v} - Sv$$

where $S(t)$ for $t \in S^1$ is a smooth loop of symmetric 2–by–2 matrices. Then there is a linear Hamiltonian flow $\Psi(t) \in \mathrm{Sp}(1)$ defined by solutions to the equation $-J_0\dot{v} - Sv = 0$, and 1 is in the spectrum of $\Psi(1)$ if and only if $\ker\mathbf{A}_z$ is nontrivial. When this is not the case, we define the *Conley-Zehnder index* $\mu_{\mathrm{CZ}}^\Phi(\mathbf{A}_z)$ in the standard way in terms of this path of symplectic matrices for $t \in [0,1]$ (cf. the discussion of the "μ–index" in [HWZ95, §3]). Note that the index depends on Φ up to an even integer, so its even/odd parity in particular is independent of Φ. In the Morse-Bott context, $\mathbf{A}_z$ may have nontrivial kernel, but one can generally pick a real number $\epsilon \neq 0$ and define $\mu_{\mathrm{CZ}}^\Phi(\mathbf{A}_z + \epsilon)$, which depends only on the sign of ϵ if the latter is sufficiently close to zero.

The *Fredholm index* of u can now be written as

$$\text{(A.1)} \qquad \mathrm{ind}(u) = -\chi(\dot{\Sigma}) + 2c_1^\Phi(u^*TW^\infty) + \sum_{z\in\Gamma} \mu_{\mathrm{CZ}}^\Phi(\mathbf{A}_z - \epsilon),$$

where $\epsilon > 0$ is an arbitrary small number, and $c_1^\Phi(u^*TW^\infty)$ is the *relative first Chern number* of the complex vector bundle (u^*TW^∞, J) with respect to the trivialization at the ends defined by combining Φ on ξ with the obvious trivialization of $\mathbb{R} \oplus \mathbb{R}X_\lambda$. It is straightforward to show from properties of the Conley-Zehnder index and relative Chern number that this sum doesn't depend on either ϵ or Φ. It defines the *virtual dimension* of the moduli space of J–holomorphic curves close to u. We say that u is *unobstructed* whenever the linearized Cauchy-Riemann operator at u is surjective: then the moduli space close to u is a smooth orbifold (or manifold if u is somewhere injective) of dimension $\mathrm{ind}(u)$. In the case where all orbits are nondegenerate, this follows from the Fredholm theory developed in [Dra04]; see [Wen05] or [Wend] for the Morse-Bott case.

The punctures $\Gamma \subset \Sigma$ can be divided into *even* punctures Γ_0 and *odd* punctures Γ_1 according to the parity of $\mu_{\mathrm{CZ}}^\Phi(\mathbf{A}_z - \epsilon)$, which is independent of Φ and $\epsilon > 0$ as noted above.[3] Now one can easily use the index formula to show that $\mathrm{ind}(u)$ and Γ_0 are either both even or both odd, so if Σ has genus g, there is an integer $c_N(u) \in \mathbb{Z}$ defined by the formula

$$\text{(A.2)} \qquad 2c_N(u) = \mathrm{ind}(u) - 2 + 2g + \#\Gamma_0.$$

We call this the *normal Chern number* of u, for reasons that are easy to see in the case where W is a closed manifold: then the combination of (A.1) and (A.2) yields the alternative definition $c_N(u) = c_1(u^*TW) - \chi(\Sigma)$, which is precisely the first Chern number of the normal bundle whenever u is immersed. As shown in [Wenb], this is also the appropriate interpretation of $c_N(u)$ in the punctured case. The following transversality criterion is a special case of a result proved in [Wenb]:

Proposition A.1. *If $u : \dot{\Sigma} \to W^\infty$ is an immersed finite energy J–holomorphic curve with $\mathrm{ind}(u) > c_N(u)$, then u is unobstructed.*

A stronger statement holds in the case where u is embedded with all asymptotic orbits distinct and simply covered, $\mathrm{ind}(u) = 2$ and $c_N(u) = 0$. Then a result in [Wen05, Wend] shows that the smooth 2–dimensional moduli space of curves near u foliates a neighborhood of $u(\dot{\Sigma})$ in W^∞. The reason is that tangent vectors to the moduli space can be identified with sections of the normal bundle $N_u \to \dot{\Sigma}$ that satisfy a linear Cauchy-Riemann type equation, and the condition $c_N(u) = 0$ constrains these sections to be nowhere zero. It follows that if we add one marked point and consider the resulting evaluation map from the moduli space into W^∞, this map is a local diffeomorphism.

A.2. **Asymptotic evaluation maps.** For the arguments in §5, it is convenient to have an asymptotic version of the above statement about the evaluation map. Consider a connected moduli space $\mathcal{M}$ of finite energy J–holomorphic curves $u : \dot{\Sigma} \to W^\infty$ that each have an odd puncture asymptotic to an orbit $x : S^1 \to M$ belonging to a 1–parameter family $\mathcal{P}$ of simply covered Morse-Bott orbits of period $T > 0$. To simplify the notation, we'll assume this is the only puncture, though the discussion can be generalized to multiple punctures in an obvious way. Let $\mathbf{A}_x$ denote the asymptotic operator for any $x \in \mathcal{P}$; since it is a 1–parameter family, $\dim\ker\mathbf{A}_x = 1$. We will use certain facts about the spectrum $\sigma(\mathbf{A}_x)$ of $\mathbf{A}_x$ that are proved in [HWZ95]: in particular, for any nontrivial eigenfunction $e_\lambda \in \Gamma(x^*\xi)$ with eigenvalue λ, the winding number $\mathrm{wind}^\Phi(\lambda) := \mathrm{wind}^\Phi(e_\lambda) \in \mathbb{Z}$ depends

[3] Note that we're assuming all punctures are positive here; if there were negative Morse-Bott punctures, both this definition of parity and the Fredholm index formula would need $\mathbf{A}_z + \epsilon$ instead of $\mathbf{A}_z - \epsilon$.

only on λ, so that the resulting function

$$\sigma(\mathbf{A}_x) \to \mathbb{Z} : \lambda \mapsto \operatorname{wind}^\Phi(\lambda)$$

is monotone and attains every integer value exactly twice (counting multiplicity of eigenvalues). If $0 \notin \sigma(\mathbf{A}_x)$, then one can also deduce the parity of $\mu_{\mathrm{CZ}}^\Phi(\mathbf{A}_x)$ from these winding numbers: it is even if and only if $\sigma(\mathbf{A}_x)$ contains a positive and negative eigenvalue for which the winding numbers match. It follows that if $\mu_{\mathrm{CZ}}^\Phi(\mathbf{A}_x - \epsilon)$ is odd and $\lambda_x < 0$ is the largest negative eigenvalue of $\mathbf{A}_x$, then the corresponding eigenspace $E_x \subset \Gamma(x^*\xi)$ is 1–dimensional and its eigenfunctions have zero winding relative to any nonzero element of $\ker \mathbf{A}_x$. The union of these eigenspaces for all $x \in \mathcal{P}$ defines a real line bundle

$$E \to \mathcal{P}.$$

The eigenfunctions of $\mathbf{A}_x$ appear naturally in the asymptotic formula proved in [HWZ96b] (see also [Sie08] for a fuller discussion) for a map $u \in \mathcal{M}$ asymptotic to $x_u \in \mathcal{P}$. Choose coordinates $(s,t) \in [0,\infty) \times S^1$ for a neighborhood of the puncture in $\dot{\Sigma}$, and assume without loss of generality that u maps this neighborhood into $[0,\infty) \times M$. Then using any $\mathbb{R}$–invariant connection to define the exponential map, one can choose the coordinates (s,t) so that for sufficiently large s, u satisfies

$$u(s,t) = \exp_{(Ts, x_u(t))}\left[e^{\lambda_x s}\left(f_u(t) + r_u(s,t)\right)\right], \tag{A.3}$$

where $f_u \in E_x$ and $r_u(s,t) \in \xi_{x_u(t)}$ is smooth and converges to 0 uniformly in t as $s \to \infty$. This formula defines an "asymptotic evaluation map"

$$\operatorname{ev} : \mathcal{M} \to E : u \mapsto (x_u, f_u).$$

Lemma A.2. *In the situation described above, if $u \in \mathcal{M}$ is immersed with $\operatorname{ind}(u) = 2$ and $c_N(u) = 0$, then $\operatorname{ev} : \mathcal{M} \to E$ is a local diffeomorphism near u.*

Proof. We will use the analytical setup in [Wenb] to show that under these conditions, $d\operatorname{ev}(u) : T_u\mathcal{M} \to T_{(x_u,f_u)}E$ is nonsingular. If $N_u \to \dot{\Sigma}$ denotes the normal bundle of u, $p > 2$ and $\epsilon > 0$ is small, we have $T_u\mathcal{M} = \ker \mathbf{D}_u^N$, where

$$\mathbf{D}_u^N : W^{1,p,-\epsilon}(N_u) \to L^{p,-\epsilon}(\overline{\operatorname{Hom}}_{\mathbb{C}}(T\dot{\Sigma}, N_u))$$

is the *normal Cauchy-Riemann operator*, defined on exponentially weighted Sobolev spaces

$$W^{k,p,-\epsilon} := \{ v \in W^{k,p}_{\mathrm{loc}} \mid e^{-\epsilon s} v(s,t) \in W^{k,p}([0,\infty) \times S^1) \}$$

for $k = \{0,1\}$. Note that by Prop. A.1, u is unobstructed and thus $\dim \ker \mathbf{D}_u^N = 2$. By an asymptotic version of local elliptic regularity (see [HWZ96a, Sie08]), any section $v \in \ker \mathbf{D}_u^N$ satisfies a linearized version of (A.3) in the form

$$v(s,t) = e^{\lambda s}(f_v(t) + r(s,t)), \tag{A.4}$$

where $f_v \in \Gamma(x_u^*\xi)$ is an eigenfunction of $\mathbf{A}_{x_u}$ with eigenvalue $\lambda < \epsilon$, and $r(s,t) \to 0$ as $s \to \infty$. In the present situation, the largest eigenvalue less than ϵ is 0, thus if v is nontrivial then $\operatorname{wind}^\Phi(f_v) \le \operatorname{wind}^\Phi(0)$. The zeroes of v are then isolated and positive, and can be counted by the normal Chern number: we have

$$Z(v) + Z_\infty(v) = c_N(u), \tag{A.5}$$

where $Z(v)$ is the algebraic count of zeros of v, and $Z_\infty(v)$ is a corresponding asymptotic contribution defined as $\operatorname{wind}^\Phi(0) - \operatorname{wind}^\Phi(f_v)$, and is thus also nonnegative. So the condition $c_N(u) = 0$ implies that f_v has winding number zero relative to any nontrivial section in $\ker \mathbf{A}_{x_u}$.

We can consider also the restriction of $\mathbf{D}_u^N$ to a smaller weighted domain,

$$\mathbf{D}' : W^{1,p,\epsilon}(N_u) \to L^{p,\epsilon}(\overline{\operatorname{Hom}}_{\mathbb{C}}(T\dot{\Sigma}, N_u)),$$

which amounts to linearizing the J–holomorphic curve problem with an added constraint fixing the asymptotic orbit at the puncture. This operator has index 1 and is also surjective, by the results in [Wenb]. It follows that there is a unique one-dimensional subspace $V_u \subset T_u\mathcal{M}$ consisting of sections $v \in \ker \mathbf{D}_u^N$ for which the eigenvalue λ in (A.4) is negative. For all $v \in \ker \mathbf{D}_u^N \setminus V_u$, this eigenvalue is zero, and we thus have $v(s,\cdot) \to f_v \in \ker \mathbf{A}_{x_u}$ as $s \to \infty$, implying that the derivative of the map $\mathcal{M} \to \mathcal{P} : u \mapsto x_u$ in this direction is nonzero.

Now fix an orbit $x \in \mathcal{P}$ and let $\mathcal{M}_x = \{ u \in \mathcal{M} \mid x_u = x \}$. By the remarks above, this is a 1–dimensional submanifold with $T_u\mathcal{M}_x = V_u$. The restriction of ev to $\mathcal{M}_x$ defines a map $\mathcal{M}_x \to E_x$, and we claim finally that for any nontrivial $v \in V_u$, the directional derivative of this map is nonzero. This follows from (A.4) and the fact that $Z_\infty(v) = 0$, as the nontrivial eigenfunction in (A.4) must have the same winding as a section in $\ker \mathbf{A}_{x_u}$, and therefore belongs to E_{x_u}. □

A.3. **Intersection numbers.** We discuss next the punctured generalization of the adjunction formula. These results are the topological consequences of the relative asymptotic analysis carried out by Siefring in [Sie08]; complete details are explained in [Sie] for curves with nondegenerate orbits and [SW] for the Morse-Bott case, and a summary with precise definitions may also be found in the last section of [Wenb]. We shall only need a few details, which we now state without proof. For any two finite energy curves u_1, u_2, there exists an *intersection number*

$$i(u_1; u_2) \in \mathbb{Z}$$

which algebraically counts actual intersections plus a certain "asymptotic contribution," which vanishes generically. The asymptotic contribution vanishes in particular whenever u_1 and u_2 have no asymptotic orbits in common, and it is otherwise analogous to the term $Z_\infty(v)$ in (A.5): it is a nonnegative measure of the winding numbers of certain asymptotic eigenfunctions that describe the relative approach of two distinct curves to the same orbit, and it vanishes if and only if these winding numbers attain the extremal values determined by the spectrum. Thus if u_1 and u_2 do not cover the same somewhere injective curve, both the actual intersection count and the asymptotic contribution are nonnegative, and moreover, their sum is invariant under deformations of both curves through the moduli space. The condition $i(u_1; u_2) = 0$ then suffices to ensure that u_1 and u_2 never have isolated intersections. For any somewhere injective curve u, there is also a *singularity number* $\operatorname{sing}(u) \in \mathbb{Z}$, which counts double points, critical points and "asymptotic singularities," each contributing nonnegatively. This sum is also invariant under deformations, and the condition $\operatorname{sing}(u) = 0$ suffices to ensure that a somewhere injective curve is embedded. The standard adjunction formula for closed holomorphic curves now generalizes to

$$i(u; u) = 2\operatorname{sing}(u) + c_N(u) + \sum_{z \in \Gamma} \operatorname{cov}_\infty(z), \tag{A.6}$$

where the terms $\operatorname{cov}_\infty(z)$ are nonnegative integers that vanish whenever certain asymptotic eigenfunctions are simply covered, so they depend only on the asymptotic orbit and sign of the respective puncture $z \in \Gamma$.

Finally, we observe one relevant situation where the left hand side of (A.6) is guaranteed to be zero. The proof below is only a sketch; we refer to [Sie] for details.

Lemma A.3. *Suppose that $u : \dot{\Sigma} \to W^\infty$ and $u' : \dot{\Sigma}' \to W^\infty$ are finite energy J–holomorphic curves that are both contained in $[0, \infty) \times M$ and have embedded projections to M that are either identical or disjoint. If also $c_N(u) = 0$, then $i(u; u') = 0$.*

Proof. The almost complex structure is $\mathbb{R}$–invariant in the region containing u and u', thus after translating u' upwards, we can assume without loss of generality that u and u' have no intersections. This $\mathbb{R}$-translation changes the asymptotic eigenfunctions at the ends of u' by multiplication with a positive number, thus we can also assume these eigenfunctions are not identical at any common asymptotic orbit of u and u'. Now the vanishing of $c_N(u)$ implies due to $\mathbb{R}$–invariance that u has no *asymptotic defect* (cf. [Wena]): this means its asymptotic eigenfunctions all attain the largest allowed winding number. The asymptotic analysis of [Sie08] then implies that the same is true for the eigenfunctions controlling the relative behavior of u and u' at infinity, so the asymptotic contribution to $i(u; u')$ is zero. □

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