A HIERARCHY OF LOCAL SYMPLECTIC FILLING OBSTRUCTIONS FOR CONTACT 3-MANIFOLDS

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Abstract. We generalize the familiar notions of overtwistedness and Giroux torsion in 3-dimensional contact manifolds, defining an infinite hierarchy of local filling obstructions called *planar torsion*, whose integer-valued *order* $k \geq 0$ can be interpreted as measuring a gradation in "degrees of tightness" of contact manifolds. We show in particular that any contact manifold with planar torsion admits no contact type embeddings into any closed symplectic 4-manifold, and has vanishing contact invariant in Embedded Contact Homology, and we give examples of contact manifolds that have planar k-torsion for any $k \geq 2$ but no Giroux torsion. We also show that the complement of the binding of a supporting open book never has planar torsion. The unifying idea in the background is a decomposition of contact manifolds in terms of contact fiber sums of open books along their binding. As the technical basis of these results, we establish existence, uniqueness and compactness theorems for certain classes of J-holomorphic curves in blown up summed open books; these also imply algebraic obstructions to planarity and embeddings of partially planar domains.

Contents

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1. Introduction

Contact structures for odd-dimensional manifolds arise naturally on boundaries of symplectic manifolds via the notion of *convexity*. A symplectic manifold (W, ω) is said to have **convex boundary** if, on a neighborhood of ∂W, there exists a vector field Y that points transversely outward at ∂W and whose flow is a symplectic dilation, i.e. $\mathcal{L}_Y \omega = \omega$. Writing $M = \partial W$, the co-oriented hyperplane field $\xi = \ker(\iota_Y \omega|_{TM}) \subset TM$ then satisfies a certain "maximal nonintegrability" condition which makes it a *contact structure*, and up to isotopy, it depends only on the symplectic structure of (W, ω) near M, not on the choice of vector field Y.

Given the above relationship, it is interesting to ask which isomorphism classes of contact manifolds (M, ξ) do *not* arise as boundaries of compact symplectic manifolds, i.e. which ones are not *symplectically fillable*. A variety of obstructions to symplectic filling are known, and the following two examples give some hint as to the diversity of such results:

- Lisca [Lis98, Lis99] used the Seiberg-Witten monopole invariants of Kronheimer and Mrowka [KM97] together with Donaldson's theorem on the intersection forms of smooth 4-manifolds [Don86] to find examples of oriented 3-manifolds that admit no symplectically fillable contact structures.
- The author [Wen10b] used punctured holomorphic curve techniques to show that a contact 3-manifold has no symplectic filling if it is supported by a planar open book whose monodromy is not a product of right-handed Dehn twists. (See [PV10, Pla12] for some applications of this result.)

One common feature of the above examples is that they depend fundamentally on the *global* properties of the manifolds involved. In contrast, one can also consider filling obstructions which are *local*, in the sense that they answer the following question:

> *What kinds of contact subdomains can never exist in the convex boundary of a compact symplectic manifold?*

The first known example of a symplectic filling obstruction was essentially local in this sense: Gromov [Gro85] and Eliashberg [Eli90] established that contact type boundaries of symplectic 4-manifolds can never contain an *overtwisted disk*, and significantly, the related distinction between so-called "overtwisted" and "tight" contact structures, discovered by Eliashberg [Eli89], has played a pivotal role in classification questions for contact structures in dimension three. This non-fillability result can be rephrased in terms of a certain 3-dimensional contact domain with boundary that we call a *Lutz tube*: this is a solid torus $S^1 \times \mathbb{D}$ with a radially symmetric contact structure that makes a half-twist along radii from the center to the boundary (see Figure 1 and Definition 2.17). One can show (e.g. using [Eli89]) that a closed contact 3-manifold contains an overtwisted disk if and only if it contains a Lutz tube, thus the latter may be regarded as the prototypical example of a local filling obstruction.

A more general local filling obstruction is furnished by the so-called *Giroux torsion domain*, a thickened torus $[0, 1] \times T^2$ with a T^2-invariant contact structure that makes one full twist from one end of the interval to the other (see Figure 2 and Definition 2.18). Contact manifolds containing such an object are said to have *Giroux torsion*, and the fact that they are not fillable in general is a comparatively recent result, due to Gay [Gay06]. Giroux torsion domains have also played an important role in the classification of contact structures, most notably through the work of Colin, Giroux and Honda [CGH03, CGH09].

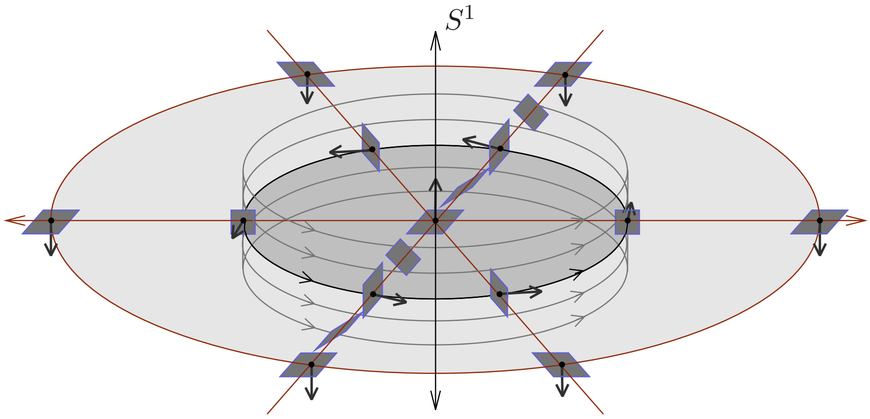

FIGURE 1. Contact planes twist around the radii emerging from the central axis of a Lutz tube. The picture also shows an embedded J-holomorphic plane asymptotic to a Reeb orbit of small period in a Morse-Bott family (arrows indicate the Reeb vector field); every Lutz tube contains such planes, which are the reason why the contact homology of an overtwisted contact manifold vanishes.

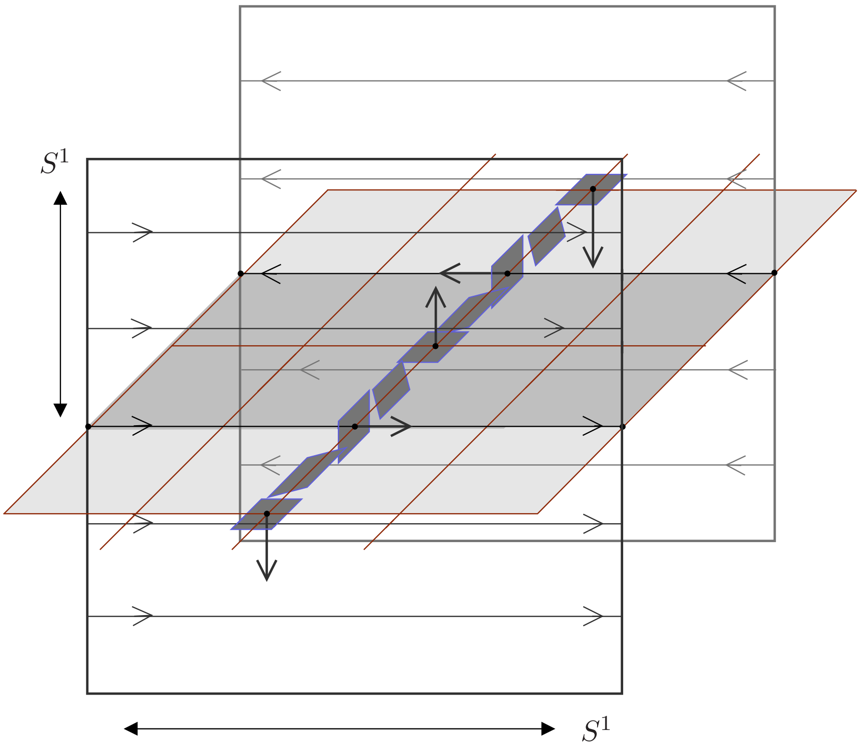

FIGURE 2. In a Giroux torsion domain $[0, 1] \times T^2$, contact planes twist around segments in the $[0, 1]$-direction. Such domains are foliated by J-holomorphic cylinders asymptotic to Morse-Bott Reeb orbits.

These two examples of local filling obstructions create the intuitive impression that contact manifolds tend to become non-fillable whenever they contain regions where the contact planes exhibit some threshold amount of *twisting*. In this paper we shall introduce a geometric formalism that makes this notion precise, and in so doing, greatly expands the known repertoire of local filling obstructions. We will demonstrate in particular that the examples above occupy the first two levels in an infinite *hierarchy*: for each integer $k \geq 0$, we shall define a special class of compact contact 3-manifolds, possibly with boundary, which we call *planar k-torsion domains*, such that the Lutz tube and Giroux torsion domain are special cases with $k = 0$ and 1 respectively. Our use of the word "hierarchy" is not incidental, as it turns out that a planar torsion domain yields quantifiably stricter or less strict filling obstructions depending on its *order*, i.e. the integer k. In particular, the overtwisted contact manifolds are precisely those which have planar 0-torsion, and these can be thought of as the "most non-fillable" among all contact 3-manifolds, while the fillable contact manifolds are the "tightest," and those which have only higher orders of planar torsion are non-fillable but are in some sense "tighter" than their lower order counterparts.

The definition of planar torsion, which will be given in a precise form in §2, combines the fundamental contact topological notion of a *supporting open book decomposition*, as introduced by Giroux [Gir], with a simple topological operation known as the *contact fiber sum* along codimension 2 contact submanifolds, originally due to Gromov [Gro86] and Geiges [Gei97]. Roughly speaking, a planar k-torsion domain is a compact contact 3-manifold (M, ξ), possibly with boundary, that contains a non-empty set of disjoint pre-Lagrangian tori dividing it into two pieces:

- A *planar piece* M^P, which is disjoint from ∂M and looks like a connected open book with some binding components blown up and/or attached to each other by contact fiber sums. The pages must have genus zero and $k + 1$ boundary components.
- The *padding* $M \setminus M^P$, which contains ∂M and consists of one or more arbitrary open books, again with some binding components blown up or fiber summed together.

Planar torsion domains are thus examples of what are called *partially planar domains*, a notion that was first hinted at in [ABW10]. The interior of such a domain M always contains a special set $\mathcal{I} \subset M$ of pre-Lagrangian tori which arise by blowing up binding components of open books: we refer to these tori all together as the *interface* of (M, ξ). Postponing the exact definitions until §2, let us for now merely point out that in a Lutz tube $M = S^1 \times \mathbb{D}$ (Figure 1), the planar piece is some smaller solid torus $M^P = S^1 \times \mathbb{D}_r$ for $0 < r < 1$, and the pages of the blown up open book in M^P are the disks $\{*\} \times \mathbb{D}_r$. Likewise, the planar piece in a Giroux torsion domain $M = [0, 1] \times T^2$ (Figure 2) is a smaller thickened torus $M^P = [r_1, r_2] \times T^2$ for $0 < r_1 < r_2 < 1$, foliated by cylindrical pages of the form $[r_1, r_2] \times S^1 \times \{*\}$, and for both examples $\mathcal{I} = \partial M^P$. We will see that in the more general definition, the topology of the planar piece and the whole domain may differ from each other considerably, and interface tori may also be found in the interior of the planar piece or the padding. Some simple examples of the form $S^1 \times \Sigma$ are shown in Figure 3. We should also mention that the idea of decomposing contact manifolds in this way via fiber sums of open books has further applications beyond filling obstructions, e.g. it is used in [Wen] to define a "blown up" version of Eliashberg's capping construction [Eli04], producing a wide range of existence results for non-exact symplectic cobordisms.

Let us now recall some basic definitions in preparation for stating the main results. A **contact structure** on an oriented 3-dimensional manifold is a hyperplane distribution ξ

that can be written locally as the kernel of a smooth 1-form α with $\alpha \wedge d\alpha \neq 0$. We call ξ **positive** if $\alpha \wedge d\alpha > 0$. Every contact structure in this paper will be assumed to be positive and to carry a *co-orientation*, which can be defined via a global choice of 1-form α; any α with $\ker \alpha = \xi$ that is compatible with the chosen co-orientation is called a **contact form** for (M, ξ). Note that a co-oriented contact structure also inherits a natural orientation. Given two contact 3-manifolds (M_0, ξ_0) and (M, ξ), a **contact embedding** of (M_0, ξ_0) into (M, ξ) is an orientation preserving embedding $\iota : M_0 \hookrightarrow M$ such that $\iota_* : TM_0 \hookrightarrow TM$ defines an orientation preserving map of ξ_0 to ξ.

Suppose (W, ω) is a compact 4-dimensional symplectic manifold (oriented by $\omega \wedge \omega$) and (M, ξ) is a closed contact 3-manifold. A **weak contact type embedding** of (M, ξ) into (W, ω) is an embedding $\iota : M \hookrightarrow W$ for which $\iota^*\omega|_\xi > 0$. It is called a (strong) **contact type embedding** if a neighborhood of $\iota(M) \subset W$ admits a 1-form λ such that $d\lambda = \omega$ and $\iota^*\lambda$ defines a contact form for (M, ξ); note that in this case, the vector field ω-dual to λ defines a symplectic dilation positively transverse to $\iota(M)$. The image of a (weak or strong) contact type embedding is called a (weak or strong) **contact type hypersurface** in (W, ω). If the image is ∂W and ι maps the orientation of M to the natural boundary orientation, then we say (W, ω) is a (weak or strong) **symplectic filling** of (M, ξ).

1.1. **Obstructions to symplectic fillings.** Given the notion of a planar k-torsion domain which was sketched above and will be explained fully in §2, it is natural to define the following.

Definition 1.1. A contact 3-manifold is said to have **planar torsion of order** k (or **planar** k**-torsion**) if it admits a contact embedding of a planar k-torsion domain (see Definition 2.13).

Theorem 1. *If (M, ξ) is a closed contact 3-manifold with planar torsion of any order, then it does not admit a contact type embedding into any closed symplectic 4-manifold. In particular, it is not strongly fillable.*

Though our proof of non-fillability will not depend on it, the implication that (M, ξ) is not strongly fillable follows from the above statement due to a result of Etnyre and Honda [EH02], that every contact 3-manifold is *concave* fillable: this means that strong fillings can always be capped off to produce closed symplectic 4-manifolds containing contact type hypersurfaces.

We will also prove an algebraic counterpart to the above result in terms of Embedded Contact Homology, or "ECH" for short (see e.g. [Hut10]). The definition of ECH will be reviewed in §4.2; for now it suffices to recall that given a closed contact 3-manifold (M, ξ) with nondegenerate contact form λ and generic compatible complex structure $J : \xi \to \xi$, one can define a chain complex generated by so-called *orbit sets*,

$$\boldsymbol{\gamma} = ((\gamma_1, m_1), \ldots, (\gamma_n, m_n)),$$

where $\gamma_1, \ldots, \gamma_n$ are distinct simply covered periodic Reeb orbits and $m_1, \ldots, m_n$ are positive integers, called *multiplicities*. A differential operator is then defined by counting a certain class of embedded rigid J-holomorphic curves in the symplectization of (M, ξ), which can be viewed as cobordisms between orbit sets. The homology of the resulting chain complex is the Embedded Contact Homology $\mathrm{ECH}_*(M, \lambda, J)$. Though the complex obviously depends on λ and J, Taubes has shown [Tau10a, Tau10b] that $\mathrm{ECH}_*(M, \lambda, J)$ is isomorphic to a version of Seiberg-Witten Floer homology, and thus actually only depends (up to natural isomorphisms) on the contact manifold (M, ξ), so we can write

$$\mathrm{ECH}_*(M, \xi) := \mathrm{ECH}(M, \lambda, J).$$

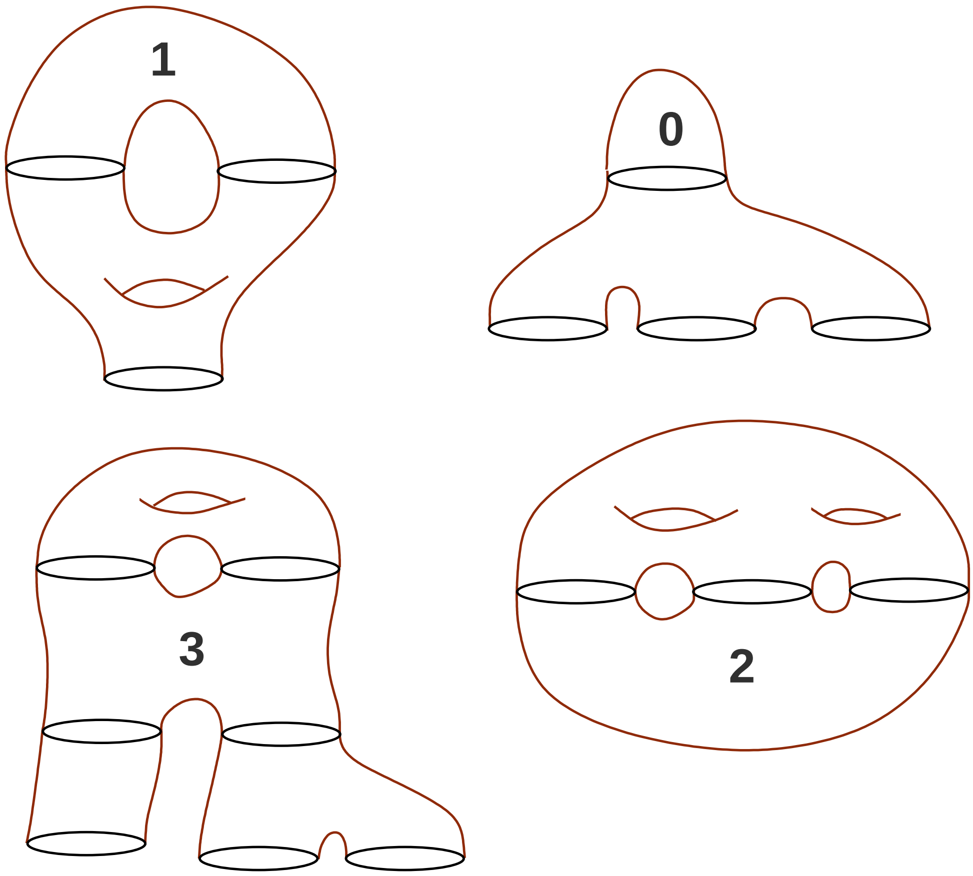

FIGURE 3. Various planar k-torsion domains, with the order $k \geq 0$ indicated within the planar piece. Each picture shows a surface Σ that defines a manifold $S^1 \times \Sigma$ with an S^1-invariant contact structure ξ. The multicurves that divide Σ are the sets of all points $z \in \Sigma$ at which $S^1 \times \{z\}$ is Legendrian. See also Example 2.15 and Figure 6.

The case $n = 0$ is also allowed among the generators, i.e. the "empty" orbit set $\emptyset := ()$, which is always a cycle in the homology, thus defining a distinguished class

$$c(\xi) := [\emptyset] \in \mathrm{ECH}(M, \xi),$$

which we call the **ECH contact invariant**. It corresponds under Taubes's isomorphism to a similar contact invariant in Seiberg-Witten theory, and conjecturally[1] also to the Ozsváth-Szabó contact invariant in Heegaard Floer homology.

Theorem 2. *If (M, ξ) is a closed contact 3-manifold with planar torsion of any order, then its ECH contact invariant $c(\xi)$ vanishes.*

This calculation is in some sense a generalization of the well-known fact that overtwisted contact manifolds have trivial contact homology (cf. Figure 1), and our proof of it has some

[1]Recent progress on this conjecture has been made in parallel projects by Colin-Ghiggini-Honda [CGHa] and Kutluhan-Lee-Taubes [KLTa, KLTb].

commonalities with the proof of the latter sketched by Eliashberg in the appendix of [Yau06]. The result implies another proof that planar torsion is a filling obstruction, albeit a very indirect one: under the isomorphism of Taubes [Tau10b], the ECH contact invariant corresponds to a similar invariant in Seiberg-Witten theory, whose vanishing gives a filling obstruction due to results of Kronheimer and Mrowka [KM97]. We will however give a proof of Theorem 1 that uses only holomorphic curve methods, requiring no assistance from Seiberg-Witten theory.

Remark 1.2. Aside from the direct holomorphic curve proof of Theorem 1 that we will give in §4.1, there are at least two alternative approaches/generalizations one can imagine:

(a) *Algebraic:* find a contact invariant whose vanishing contradicts symplectic filling, and which must always vanish in the presence of planar torsion.
(b) *Topological:* given (M, ξ) with planar torsion, find a symplectic cobordism with negative boundary (M, ξ) whose positive boundary is already known to be not fillable.

The first approach is pursued in the present article and in the related paper [LW11], however the second approach also works. Indeed, after the first version of this paper was completed, the author defined in [Wen] a generalized handle attaching construction which yields symplectic cobordisms from any contact manifold with planar torsion to another that is overtwisted. The decomposition of contact manifolds via blown up summed open books that we will explain in §2.1 is a crucial ingredient in this construction, which also yields alternative proofs of Theorem 5 and the weak filling obstructions of [NW11] mentioned below.

Under stronger geometric assumptions one also obtains stronger results in terms of ECH with *twisted coefficients*, which gives correspondingly stricter obstructions to symplectic fillings. As we will review in §4.2, a twisted version of the ECH chain complex can be defined as a module over the group ring $\mathbb{Z}[H_2(M;\mathbb{R})]$, so that the differential keeps track of the 2-dimensional relative homology classes of the holomorphic curves it counts. We shall denote this twisted version of ECH by $\widetilde{\mathrm{ECH}}_*(M, \xi)$. It also contains a preferred homology class $\tilde{c}(\xi) \in \widetilde{\mathrm{ECH}}_*(M, \xi)$ represented by the empty orbit set, called the **twisted ECH contact invariant**.

Definition 1.3. A contact 3-manifold is said to have **fully separating planar k-torsion** if it contains a planar k-torsion domain with a planar piece $M^P \subset M$ that has the following properties:

(1) There are no interface tori in the interior of M^P.
(2) Every connected component of ∂M^P separates M.

We will see that the fully separating condition is always satisfied if $k = 0$, and for the case of a Giroux torsion domain, it is satisfied if and only if the domain separates M.

Theorem 2′. *If (M, ξ) is a closed contact 3-manifold with fully separating planar torsion, then its twisted ECH contact invariant $\tilde{c}(\xi)$ vanishes.*

Appealing again to the isomorphism of [Tau10b] together with results from Seiberg-Witten theory [KM97] on weak symplectic fillings, we obtain the following consequence, which is also proved by a more direct holomorphic curve argument in joint work of the author with Klaus Niederkrüger [NW11].

Corollary 1. *If (M, ξ) is a closed contact 3-manifold with fully separating planar torsion, then it is not weakly fillable.*

As we will show shortly, Theorem 1 and Corollary 1 yield many previously unknown examples of non-fillable contact manifolds. Observe that the fully separating condition in Corollary 1 cannot be removed in general, as for instance, there are infinitely many tight 3-tori which have non-separating Giroux torsion (and hence planar 1-torsion by Theorem 3 below) but are weakly fillable by a construction of Giroux [Gir94]. Further examples of this phenomenon are constructed in [NW11] for planar k-torsion with any $k \geq 1$.

Remark 1.4. One can refine the above vanishing result with twisted coefficients as follows: for a given closed 2-form Ω on M, define (M, ξ) to have *Ω-separating planar torsion* if it contains a planar torsion domain such that every interface torus T lying in the planar piece satisfies $\int_T \Omega = 0$ (cf. Definition 2.12). Under this condition, our computation implies a similar vanishing result for the ECH contact invariant with twisted coefficients in $\mathbb{Z}[H_2(M; \mathbb{R}) / \ker \Omega]$, with the consequence that (M, ξ) admits no weak filling (W, ω) for which $\omega|_{TM}$ is cohomologous to Ω. A direct proof of the latter is given in [NW11].

We now consider examples of contact manifolds with planar torsion. We will show in §2.2 that the previously known local filling obstructions fit into the first two levels of the hierarchy, i.e. $k = 0$ and 1.

Theorem 3. *A closed contact* 3*-manifold has planar* 0*-torsion if and only if it is overtwisted, and every closed contact manifold with Giroux torsion also has planar* 1*-torsion.*

For this reason, Theorems 2 and 2′ imply ECH versions of the vanishing results of Ghiggini, Honda and Van Horn-Morris [GHV, GH] for the Ozsváth-Szabó contact invariant in the presence of Giroux torsion. We'll see below that it is also easy to construct examples of contact manifolds that have planar torsion of any order greater than 1 but no Giroux torsion. It is not clear whether there exist contact manifolds with planar 1-torsion but no Giroux torsion.

To find examples for $k \geq 2$, suppose Σ is a closed oriented surface containing a non-empty multicurve $\Gamma \subset \Sigma$ that divides it into two (possibly disconnected) pieces Σ_+ and Σ_-. We define the contact manifold (M_Γ, ξ_Γ), where

$$M_\Gamma := S^1 \times \Sigma$$

and ξ_Γ is the (up to isotopy) unique S^1-invariant contact structure that makes $\{\text{const}\} \times \Sigma$ into a convex surface with dividing set Γ. The existence and uniqueness of such a contact structure follows from a result of Lutz [Lut77]. We will see in Examples 2.10 and 2.15 that (M_Γ, ξ_Γ) is a partially planar domain whenever any connected component Σ_0 of $\Sigma \setminus \Gamma$ has genus zero: indeed, the surfaces $\{*\} \times \Sigma_0$ are then the pages of a blown up planar open book. Moreover, (M_Γ, ξ_Γ) is then a planar torsion domain unless $\Sigma \setminus \Gamma$ has exactly two connected components and they are diffeomorphic, and it is fully separating if every connected component of $\partial \overline{\Sigma}_0$ separates Σ.

Corollary 2. *Suppose $\Sigma \setminus \Gamma$ has a connected component Σ_0 of genus zero, and either $\Sigma \setminus \Gamma$ has more than two connected components or $\Sigma \setminus \overline{\Sigma}_0$ is not diffeomorphic to Σ_0. Then (M_Γ, ξ_Γ) has vanishing (untwisted) ECH contact invariant and is not strongly fillable. Moreover, if every connected component of $\partial \overline{\Sigma}_0$ separates Σ, then the invariant with twisted coefficients also vanishes and (M_Γ, ξ_Γ) is not weakly fillable.*

Note that (M_Γ, ξ_Γ) is always universally tight whenever Γ contains no contractible connected components. This follows from [Gir01, Prop. 4.1(b)], and can also be deduced (via [Hof93]) from the observation that (M_Γ, ξ_Γ) then admits contact forms with no contractible

Reeb orbits (e.g. any Giroux form in the sense of Definition 2.8 will have this property). Whenever this is true, an argument due to Giroux (see [Mas12, Theorem 3]) implies that (M_Γ, ξ_Γ) also has no Giroux torsion if no two connected components of Γ are isotopic. We thus obtain infinitely many examples of contact manifolds that have planar torsion of any order greater than 1 but no Giroux torsion:

Corollary 3. *For any integers $g \ge k \ge 1$, let (V_g, ξ_k) denote the S^1-invariant contact manifold (M_Γ, ξ_Γ) described above for the case where $\Gamma \subset \Sigma$ has k connected components and divides Σ into two connected components, one with genus zero and the other with genus $g - k + 1 > 0$. Then (V_g, ξ_k) has no Giroux torsion if $k \ge 3$, but for any $k \ge 1$ it has planar torsion of order $k-1$. In particular (V_g, ξ_k) always has vanishing ECH contact invariant and is not strongly fillable.*

Some more examples of planar torsion without Giroux torsion are shown in Figure 4.

Remark 1.5. In many cases, one can easily generalize the above results from products $S^1 \times \Sigma$ to general Seifert fibrations over Σ. In particular, whenever Σ has genus at least four, one can find dividing sets on Σ such that $(S^1 \times \Sigma, \xi_\Gamma)$ has no Giroux torsion but contains a proper subset that is a planar torsion domain (see Figure 4). Then modifications outside of the torsion domain can change the trivial fibration into arbitrary nontrivial Seifert fibrations with planar torsion but no Giroux torsion. This trick reproduces many (though not all) of the Seifert fibered 3-manifolds for which [Mas12] proves the vanishing of the Ozsváth-Szabó contact invariant.

Remark 1.6. There is a significant overlap between our ECH vanishing results and the Heegaard vanishing results proved by Massot in [Mas12] (see also [HKM, Mat11]), but neither set of results contains the other. In particular, the examples (V_g, ξ_k) in Corollary 3 with planar torsion of order greater than 1 seem thus far to be beyond the reach of Heegaard Floer homology.

By a recent result of Etnyre and Vela-Vick [EVV10], the complement of the binding of a supporting open book never contains a Giroux torsion domain. We will prove a natural generalization of this:

Theorem 4. *Suppose (M, ξ) is a contact 3-manifold supported by an open book $\pi : M \setminus B \to S^1$. Then any planar torsion domain in (M, ξ) must intersect the binding B.*

In order to explain our choice of terminology and the use of the word "hierarchy," we now mention some related joint results with Janko Latschev which are proved in [LW11]. These are most easily expressed by defining a contact invariant

$$\mathrm{PT}(M, \xi) := \sup \left\{ k \ge 0 \;\middle|\; (M, \xi) \text{ has no planar } \ell\text{-torsion for any } \ell < k \right\},$$

which takes values in $\mathbb{N} \cup \{0, \infty\}$ and is infinite if and only if (M, ξ) has no planar torsion. Then the results stated above show that $\mathrm{PT}(M, \xi) < \infty$ always implies (M, ξ) is not strongly fillable; moreover $\mathrm{PT}(M, \xi) \le 1$ whenever (M, ξ) has Giroux torsion, $\mathrm{PT}(M, \xi) = 0$ if and only if (M, ξ) is overtwisted, and there exist contact manifolds without Giroux torsion such that $\mathrm{PT}(M, \xi) < \infty$. We claim now that contact manifolds with larger values of $\mathrm{PT}(M, \xi)$ not only exist but are, in some quantifiable sense, "closer" to being fillable. This statement can be made precise by considering the existence or non-existence of symplectic cobordisms between contact manifolds with different values of $\mathrm{PT}(M, \xi)$, as in the following result.

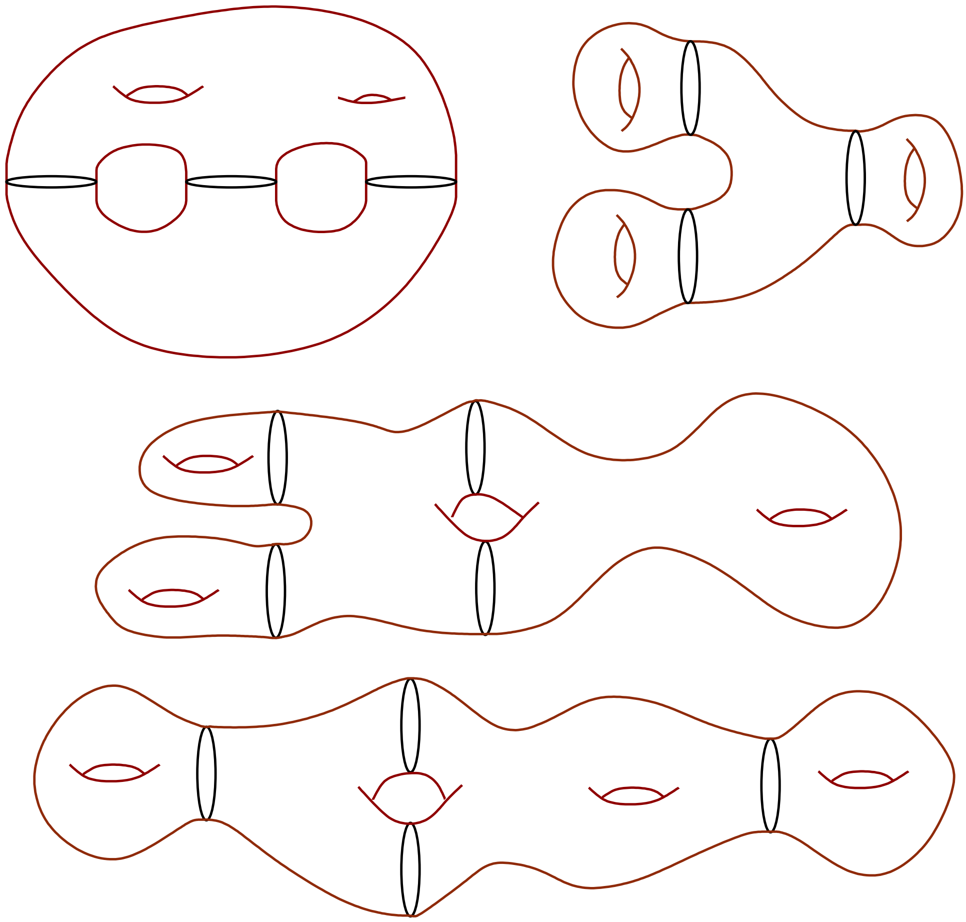

FIGURE 4. Some contact manifolds of the form $S^1 \times \Sigma$ that have no Giroux torsion but have planar torsion of orders 2, 2, 3 and 2 respectively. In each case the contact structure is S^1-invariant and induces the dividing set shown on Σ in the picture. For the example at the upper right, Theorem 2$'$ implies that the *twisted* ECH contact invariant also vanishes, so this one is not weakly fillable. In the bottom example, the planar torsion domain is a proper subset, thus one can make modifications outside of this subset to produce arbitrary nontrivial Seifert fibrations (see Remark 1.5).

Theorem ([LW11]). *For the contact manifold (V_g, ξ_k) in Corollary 3, $\mathrm{PT}(V_g, \xi_k) = k - 1$. Moreover, if (M, ξ) is any contact manifold that appears as the positive boundary of an exact symplectic cobordism whose negative boundary is (V_g, ξ_k), then $\mathrm{PT}(M, \xi) \geq k - 1$.*

Since a contact 3-manifold (M, ξ) is tight if and only if $\mathrm{PT}(M, \xi) \geq 1$, the above result can be regarded as demonstrating a "higher order" variant of the well-known conjecture that contact (-1)-surgery on a Legendrian in a closed tight contact manifold always produces something tight. Indeed, since contact surgery gives rise to a Stein cobordism, the above implies that contact surgery (or for that matter, contact connected sums) on (V_g, ξ_k) always produces examples with $\mathrm{PT}(M, \xi) \geq k - 1$.

Remark 1.7. It should be emphasized here that the scale defined by the invariant $\mathrm{PT}(M,\xi)$ measures something completely different from the standard quantitative measurement of Giroux torsion; the latter counts the maximum number of adjacent Giroux torsion domains that can be embedded in (M,ξ), and can take arbitrarily large values while $\mathrm{PT}(M,\xi) \le 1$. Likewise, (M,ξ) has Giroux torsion zero whenever $\mathrm{PT}(M,\xi) \ge 2$.

The theorem above follows from some results proved in [LW11] using notions from Symplectic Field Theory, which also lie in the background of our choice of terminology. Recall that SFT is a generalization of contact homology introduced by Eliashberg, Givental and Hofer [EGH00] (see also [CL09] for the reformulation discussed here), that defines contact invariants by counting J-holomorphic curves with arbitrary genus and positive and negative ends in symplectizations of arbitrary dimension. The chain complex of SFT is a graded algebra of the form $\mathcal{A}[[\hbar]]$, where $\hbar$ is an even variable and $\mathcal{A}$ is a graded unital algebra generated by symbols q_γ corresponding to closed Reeb orbits γ. There is then a differential operator $\mathbf{D}_{\mathrm{SFT}} : \mathcal{A}[[\hbar]] \to \mathcal{A}[[\hbar]]$ which counts holomorphic curves and vanishes by definition on the "constant" elements $\mathbb{R}[[\hbar]] \subset \mathcal{A}[[\hbar]]$, hence defining prefered homology classes in

$$H_*^{\mathrm{SFT}}(M,\xi) := H_*(\mathcal{A}[[\hbar]], \mathbf{D}_{\mathrm{SFT}}).$$

One then defines (M,ξ) to have **algebraic k-torsion** if the homology satisfies the relation

$$[\hbar^k] = 0 \in H_*^{\mathrm{SFT}}(M,\xi).$$

For $k = 0$, this means $[1] = 0$ and coincides with the notion of *algebraic overtwistedness* (cf. [BN10]). It follows easily from the formalism[2] of SFT that algebraic torsion of any order gives an obstruction to strong symplectic filling, but in fact it is stronger, as it also implies obstructions to the existence of exact symplectic cobordisms between certain contact manifolds. To state this succinctly, one can define an algebraic cousin of the invariant $\mathrm{PT}(M,\xi)$ by

$$\mathrm{AT}(M,\xi) := \sup\left\{k \ge 0 \;\middle|\; (M,\xi) \text{ has no algebraic } \ell\text{-torsion for any } \ell < k\right\}.$$

The above result is then a consequence of the following set of results, which serve as our main motivation for keeping track of the integer $k \ge 0$ in planar k-torsion.

Theorem ([LW11])**.** *The invariant* $\mathrm{AT}(M,\xi)$ *has the following properties.*

1. *Any contact manifold* (M,ξ) *with* $\mathrm{AT}(M,\xi) < \infty$ *is not strongly fillable.*
2. *If there is an exact symplectic cobordism with positive boundary* (M_+,ξ_+) *and negative boundary* (M_-,ξ_-)*, then* $\mathrm{AT}(M_-,\xi_-) \le \mathrm{AT}(M_+,\xi_+)$.
3. *Every contact* 3*-manifold* (M,ξ) *satisfies* $\mathrm{AT}(M,\xi) \le \mathrm{PT}(M,\xi)$.
4. *For the examples* (V_g,ξ_k) *in Corollary 3,* $\mathrm{AT}(V_g,\xi_k) = k-1$.

In particular, the computation $\mathrm{AT}(M,\xi) \le \mathrm{PT}(M,\xi)$ follows from a variation on our proof of Theorems 2 and 2′, and thus makes essential use of the holomorphic curve results in the present article.

[2]For this informal discussion we are taking it for granted that SFT is well defined, which was not proved in [EGH00] and is quite far from obvious. The rigorous definition of SFT, including the necessary abstract perturbations to achieve transversality, is a large project in progress by Hofer-Wysocki-Zehnder, see for example [Hof06]. The application stated above however does not depend on this, as it can also be proved using the ECH methods in Hutchings's appendix to [LW11].

1.2. **Obstructions to non-separating embeddings and planarity.** We now discuss a parallel stream of results that apply to a wider class of contact manifolds, some of which are fillable. Observe that in addition to ruling out symplectic fillings, Theorem 1 implies that contact manifolds with planar torsion can never appear as *non-separating* contact type hypersurfaces in any closed symplectic 4-manifold. This is actually a consequence of the following generalization of a result proved in [ABW10]:

Theorem 5. *Suppose* (M, ξ) *is a closed contact* 3*-manifold that contains a partially planar domain (see Definition 2.11) and admits a contact type embedding* $\iota : (M, \xi) \hookrightarrow (W, \omega)$ *into some closed symplectic* 4*-manifold* (W, ω)*. Then* ι *separates* W*.*

Corollary 4. *If* (M, ξ) *is a closed contact* 3*-manifold containing a partially planar domain, then it does not admit any strong symplectic semifilling with disconnected boundary.*

Recall that a **semifilling** of a contact manifold (M, ξ) is defined to be a filling of $(M, \xi) \sqcup (M', \xi')$ for any (perhaps empty) closed contact manifold (M', ξ'). The corollary follows from an observation due to Etnyre (cf. [ABW10, Example 1.3]), that given a filling of $(M, \xi) \sqcup (M', \xi')$ with M' non-empty, one can attach a symplectic 1-handle to connect M and M' and then cap off the resulting boundary in order to realize (M, ξ) as a non-separating contact type hypersurface. Corollary 4 also generalizes similar results proved by McDuff for the tight 3-sphere [McD91] and Etnyre for all planar contact manifolds [Etn04].

The algebraic counterpart to Corollary 4 involves the so-called *U-map* in Embedded Contact Homology. This is a natural endomorphism

$$U : \mathrm{ECH}_*(M, \xi) \to \mathrm{ECH}_{*-2}(M, \xi)$$

defined at the chain level by counting embedded index 2 holomorphic curves through a generic point in the symplectization. The same definition also gives a map on ECH with twisted coefficients,

$$\widetilde{U} : \widetilde{\mathrm{ECH}}_*(M, \xi) \to \widetilde{\mathrm{ECH}}_{*-2}(M, \xi).$$

Theorem 6. *If* (M, ξ) *is a closed contact* 3*-manifold containing a partially planar domain, then for all integers* $d \geq 1$*, the image of* $U^d : \mathrm{ECH}_*(M, \xi) \to \mathrm{ECH}_*(M, \xi)$ *contains* $c(\xi)$*.*

This implies Corollary 4 due to some recent results involving maps on ECH induced by cobordisms (cf. [HT]), though again, those results depend on Seiberg-Witten theory, and our proof of Theorem 5 will not.

Theorem 6 applies in particular to all planar contact manifolds and can thus be viewed as an obstruction to planarity. The corresponding obstruction in Heegaard Floer homology is a known result of Ozsváth, Stipsicz and Szabó [OSS05]. Our version of the obstruction can easily be strengthened by observing that a planar open book is also a *fully separating* partially planar domain, so analogously to Theorem 2′, it yields a result with twisted coefficients—the Heegaard Floer theoretic analogue of this result is apparently not known.

Theorem 6′. *If* (M, ξ) *is a planar contact manifold, then for all integers* $d \geq 1$*, the image of* $\widetilde{U}^d : \widetilde{\mathrm{ECH}}_*(M, \xi) \to \widetilde{\mathrm{ECH}}_*(M, \xi)$ *contains* $\tilde{c}(\xi)$*.*

Remark 1.8. Similarly to Remark 1.4, one can generalize the above by defining (cf. Definition 2.12) the notion of an Ω*-separating* embedding of a partially planar domain, where Ω is a closed 2-form on M. Then such an embedding produces a version of Theorem 6′ for ECH with coefficients in $\mathbb{Z}[H_2(M; \mathbb{R})/\ker \Omega]$, and implies corresponding generalizations of Corollary 4.

Remark 1.9. Note that by Theorem 6 above, there are also many non-planar examples for which $c(\xi)$ is in the image of U^d, but the corresponding statement with twisted coefficients is not true. The most obvious example is the standard T^3, which is a partially planar domain (see Example 2.5) but also admits weak semifillings with disconnected boundary (due to Giroux [Gir94]).

1.3. **Holomorphic curves and open book decompositions.** The technical work in the background of the above results is a set of theorems that we will prove in §3 relating holomorphic curves and a suitably generalized notion of open book decompositions. For illustration purposes, we now state some simplified versions of these results.

Recall that if M is a closed and oriented 3-manifold, an **open book decomposition** is a fibration

$$\pi : M \setminus B \to S^1,$$

where $B \subset M$ is an oriented link called the **binding**, and the closures of the fibers are called **pages**: these are compact, oriented and embedded surfaces with oriented boundary equal to B. An open book is called **planar** if the pages are connected and have genus zero, and it is said to **support** a contact structure ξ if the latter can be written as $\ker \alpha$ for some contact form α (called a **Giroux form**) whose induced Reeb vector field X_α is positively transverse to the interiors of the pages and positively tangent to the binding. The latter definition is due to Giroux [Gir], who established a groundbreaking one-to-one correspondence between isomorphism classes of contact manifolds and their supporting open books up to right-handed stabilization.

We refer to §3.1 for all the technical definitions needed to understand the following statement. A substantial generalization will appear in §3.2 as Theorem 7.

Proposition 1.10. *Suppose (M, ξ) is a closed connected contact 3-manifold with a supporting open book decomposition $\pi : M \setminus B \to S^1$ whose pages have genus $g \geq 0$. Then for any numbers $\tau_0 > 0$ and $m_0 \in \mathbb{N}$, (M, ξ) admits a nondegenerate Giroux form α and generic compatible almost complex structure J on its symplectization such that the following conditions hold:*

1. *The Reeb orbits in B have minimal period less than τ_0, and their covers up to multiplicity m_0 all have Conley-Zehnder index 1 with respect to the framing determined by the open book. All Reeb orbits in $M \setminus B$ have period at least 1.*
2. *If $g = 0$, then after a small isotopy of π fixing the binding, there is an $(\mathbb{R} \times S^1)$-parametrized family of embedded finite energy punctured J-holomorphic curves*
$$u_{(\sigma,\tau)} : \dot{\Sigma} \to \mathbb{R} \times M, \qquad (\sigma, \tau) \in \mathbb{R} \times S^1$$
which are Fredholm regular and have index 2 and have only positive ends, such that for each $(\sigma, \tau) \in \mathbb{R} \times S^1$, the projection of $u_{(\sigma,\tau)}$ to M is an embedding that parametrizes $\pi^{-1}(\tau)$.
3. *If $g = 0$, then every somewhere injective finite energy punctured J-holomorphic curve in $\mathbb{R} \times M$ whose positive ends all approach orbits in B of covering multiplicity up to m_0 is part of the $(\mathbb{R} \times S^1)$-family described above.*
4. *If $g > 0$, then there is no J-holomorphic curve in $\mathbb{R} \times M$ whose positive ends all approach distinct simply covered orbits in B.*

The $(\mathbb{R} \times S^1)$-parametrized family of J-holomorphic curves in this theorem is called a *holomorphic open book*; such objects have appeared previously in the work of Hofer-Wysocki-Zehnder [HWZ95b, HWZ98] and Abbas [Abb11]. Their existence for the case $g = 0$ was already established in [Wen10c] and generalized in [Abb11], and lies in the background of various contact topological results on planar contact manifolds, such as the proof of the Weinstein conjecture by Abbas-Cieliebak-Hofer [ACH05] and the author's proof that strong and Stein fillability are equivalent [Wen10b]. Given existence, the uniqueness statement for the $g = 0$ case follows from a straightforward but surprisingly powerful intersection theoretic argument, using the homotopy invariant intersection number for punctured holomorphic curves developed by Siefring [Sie11]. The non-existence result for $g > 0$ relies on this same argument but is much subtler, because for analytical reasons, the existence part of the above theorem fails in the case $g > 0$.[3] The situation is saved by the observation, explained in [Wen10c], that one can find a highly *non-generic* choice of data for which higher genus holomorphic open books exist, and this data is compatible with an exact stable Hamiltonian structure, which admits a well behaved perturbation to a suitable contact form.

In §3.2, we will state and prove a generalization of Proposition 1.10 in the context of blown up and summed open books, which gives us existence and uniqueness for certain holomorphic curves in partially planar domains that have only positive ends. Such results make it easy to find orbit sets in the ECH chain complex that satisfy $\partial\boldsymbol{\gamma} = \boldsymbol{\emptyset}$ or $U^d\boldsymbol{\gamma} = \boldsymbol{\emptyset}$, thus proving Theorems 2, 2′, 6 and 6′.

As already mentioned, our main results on fillability and embeddability (Theorems 1, 4 and 5) can also be proved without recourse to ECH and Seiberg-Witten theory, and we shall do this in §4.1. The main idea behind such arguments appeared already in [Wen10b]: given a strong filling whose boundary contains a planar torsion domain, we can attach a cylindrical end and use the above correspondence between open books and holomorphic curves to find a region near infinity that is foliated by a stable 2-dimensional family of holomorphic curves. This family can then be expanded into the filling and, due to the analytical properties of the holomorphic curves in question, must foliate it. But the latter produces a contradiction, as one can then follow the family back into a different region of the cylindrical end where our uniqueness statement in fact *excludes* the existence of such holomorphic curves.

To make this type of argument work, we need compactness and deformation results for families of curves in a symplectic filling that arise from the pages of a holomorphic open book. An example of such a result is the following. Suppose (M, ξ) is supported by a planar open book $\pi : M \setminus B \to S^1$, and α and J_+ are the contact form and almost complex structure respectively provided by Proposition 1.10. Assume also that (M, ξ) is the contact type boundary of a compact symplectic manifold (W, ω) such that near ∂W, $\omega = d\lambda$ for a 1-form λ that matches α at $M = \partial W$. We can then complete (W, ω) to a noncompact symplectic manifold by attaching a cylindrical end

$$(W^\infty, \omega) := (W, \omega) \cup_M \left([0, \infty) \times M, d(e^t\alpha)\right).$$

Let $u_+ : \dot{\Sigma} \to \mathbb{R}\times M$ denote one of the holomorphic planar pages provided by Proposition 1.10; applying a suitable $\mathbb{R}$-translation to u_+, we may assume without loss of generality that it lies in $[0, \infty) \times M \subset W^\infty$. Now choose an open neighborhood $\mathcal{N}(B) \subset M$ of the binding B and

[3]Holomorphic open books with pages of positive genus cannot be expected to exist in general because the necessary moduli spaces of holomorphic curves have negative virtual dimension. Hofer [Hof00] suggested that this problem might be solved by introducing a "cohomological perturbation" into the nonlinear Cauchy-Riemann equation in order to raise the Fredholm index. This program has recently been carried out by Casim Abbas [Abb11] (see also [vB]), though applications to problems such as the Weinstein conjecture are as yet elusive, as the compactness theory for the modified nonlinear Cauchy-Riemann equation is quite difficult.

an open subset $\mathcal{U} \subset M$ such that

$$u_+(\dot{\Sigma}) \subset [0, \infty) \times \mathcal{U}.$$

Finally, choose any set of data α', ω', J'_+ and J' with the following properties:

- α' is a nondegenerate contact form on M that matches α in $\mathcal{U} \cup \mathcal{N}(B)$ and has only Reeb orbits of period at least 1 outside of $\mathcal{N}(B)$
- ω' is a symplectic form on W^∞ that matches $d(e^t\alpha')$ on $[0, \infty) \times M$
- J'_+ is a generic almost complex structure on $\mathbb{R} \times M$ compatible with α' that matches J_+ on $\mathbb{R} \times (\mathcal{U} \cup \mathcal{N}(B))$
- J' is an ω'-compatible almost complex structure on W^∞ which is generic in W and matches J'_+ in $[0, \infty) \times M$

We then denote by $\mathcal{M}(J')$ the moduli space of all unparametrized finite energy J'-holomorphic curves in W^∞, and let $\mathcal{M}_0(J')$ denote the connected component of this space containing u_+. A standard application of the implicit function theorem (see e.g. [ABW10, Theorem 4.7]) shows that $\mathcal{M}_0(J')$ is a smooth 2-dimensional manifold whose elements are all embedded and do not intersect each other; in particular they foliate an open subset of W^∞. The key to the proofs in §4.1 as well as various other applications in [NW11, LVW] is to show that the curves in $\mathcal{M}_0(J')$ also fill a *closed* subset outside of some harmless subvariety of codimension two. That is the main point of the following result, which is a simplified version of Theorem 8 proved in §3.3.

Proposition 1.11. *$\mathcal{M}_0(J')$ is compact except for convergence in the sense of* [BEH+03] *to holomorphic buildings of the following types:*

(1) *Buildings with empty main level and a single non-empty upper level curve in $\mathbb{R} \times M$ whose projection to M is embedded,*
(2) *Finitely many nodal curves in W^∞ consisting of two embedded index 0 components that intersect each other transversely.*

It is instructive perhaps to compare this with the results of McDuff [McD90]: in particular, the role of McDuff's symplectic sphere with nonnegative self-intersection is played by our holomorphic curve u_+, which generates a smooth 2-dimensional family of curves that, due to the above compactness result and the aforementioned implicit function theorem, must fill the entirety of W^∞. In the form stated above, this result follows from [ABW10, Theorem 4.8]. The version we will prove in §3.3 for a general partially planar domain is more complicated because one cannot generally avoid holomorphic buildings with multiply covered components, nonetheless one can still show that only finitely many such buildings can appear.

1.4. **Open questions and recent progress.** Let us now discuss a few questions that arise from the above results, some of which have been partially answered since the first version of this paper appeared. In light of the equivalence between the ECH and Ozsváth-Szabó contact invariants, recently established in independent work of Colin-Ghiggini-Honda [CGHb] and Kutluhan-Lee-Taubes [KLTc], our vanishing results for the ECH contact invariants imply corresponding results in Heegaard Floer homology. Some of these were already known from the work of various authors [GHV, GH, HKM, Mas12, Mat11], but their results appear thus far to recognize planar torsion only up to order 1.

Question. *Can one prove within the context of Heegaard Floer homology (i.e. without using ECH) that the contact invariant vanishes in the presence of planar k-torsion for $k \geq 2$?*

As we sketched in the above discussion of related results in [LW11], the hierarchical structure encoded by the order $k \geq 0$ of planar k-torsion can be detected algebraically via Symplectic Field Theory, and it also can be detected by a refinement of the ECH contact invariant explained in Hutchings's appendix to [LW11]. The latter raises the question of what structure in Heegaard Floer homology might also be able to see this hierarchy, but apparently nothing is yet known about this.

Question. *Can Heegaard Floer homology distinguish between two contact manifolds with vanishing Ozsváth-Szabó invariant but differing minimal orders of planar torsion? Does this imply obstructions to the existence of exact or Stein cobordisms?*

It should be mentioned that in presenting this introduction to planar torsion, we neither claim nor believe it to be the most general source of vanishing results for the various invariants under discussion. For the Ozsváth-Szabó invariant, [Mas12] produces vanishing results on some Seifert fibered 3-manifolds that fall under the umbrella of our Corollary 2 and Remark 1.5, but also some that do not since there is no condition requiring the existence of a planar piece. This phenomenon appears to be related to a generalization of planar torsion that has recently emerged from joint work of the author with Lisi and Van Horn-Morris: the idea is to replace the contact fiber sum with a more general "plumbing" construction that produces a notion of "higher genus binding." Among its applications, this allows a substantial generalization of Corollary 2 that encompasses all of the examples in [Mas12] and many more; details of this will appear in the forthcoming paper [LVW].

And now the obvious question: what can be done in higher dimensions? There has been significant activity in this area in the last few years. Atsuhide Mori [Mor] showed that certain blown up open books in dimension 5 produce a filling obstruction that strongly resembles the Lutz tube and is related to Niederkrüger's speculative notion of higher-dimensional overtwistedness [Nie06]. After the preprint version of the present article first appeared, Mori's construction was generalized to all dimensions in a joint paper of the author with Massot and Niederkrüger [MNW] which also defined a higher-dimensional notion of Giroux torsion, giving the first examples of non-fillable contact manifolds in all dimensions that cannot be called "overtwisted" in any reasonable sense. The constructions in [MNW] also give some hints as to how one might define something analogous to higher-order planar torsion that could be detected algebraically via SFT in all dimensions. This subject is still in its infancy, but it now at least seems safe to state the following conjecture:

Conjecture. *For all $n \geq 1$ and $k \geq 0$, there exist $(2n+1)$-dimensional contact manifolds (M, ξ) with $\mathrm{AT}(M, \xi) = k$. In particular, there exists in every dimension greater than one a sequence of non-fillable contact manifolds $\{(M_k, \xi_k)\}_{k \geq 0}$ such that (M_k, ξ_k) admits exact symplectic cobordisms to (M_ℓ, ξ_ℓ) if and only if $k \leq \ell$.*

2. The definition of planar torsion

2.1. **Blown up summed open books.** We now explain the decomposition of a contact manifold into "binding sums" of supporting open books, which underlies the notion of a planar torsion domain.

Assume M is an oriented smooth manifold containing two disjoint oriented submanifolds $N_1, N_2 \subset M$ of real codimension 2, which admit an orientation preserving diffeomorphism $\varphi : N_1 \to N_2$ covered by an orientation reversing isomorphism $\Phi : \nu N_1 \to \nu N_2$ of their normal bundles. Then we can define a new smooth manifold M_Φ, the **normal sum** of M along Φ, by removing neighborhoods $\mathcal{N}(N_1)$ and $\mathcal{N}(N_2)$ of N_1 and N_2 respectively, then gluing together the resulting manifolds with boundary along an orientation reversing diffeomorphism

$$\partial\mathcal{N}(N_1) \to \partial\mathcal{N}(N_2)$$

determined by Φ. This operation determines M_Φ up to diffeomorphism, and is also well defined in the contact cateogory: if (M, ξ) is a contact manifold and N_1, N_2 are contact submanifolds with $\varphi : N_1 \to N_2$ a contactomorphism, then M_Φ admits a contact structure ξ_Φ, which agrees with ξ away from N_1 and N_2 (cf. [Gei08, §7.4]). Although the issue of uniqueness is not discussed in [Gei08, §7.4], one can show that the construction of ξ_Φ explained there is canonical up to isotopy; in the specific setting that we will be concerned with below, this is an obvious consequence of the uniqueness of "supported" contact structures (cf. Definition 2.8 and the ensuing discussion).

We will consider the special case of the contact fiber sum where N_1 and N_2 are disjoint components[4] of the binding of an open book decomposition

$$\pi : M \setminus B \to S^1$$

that supports ξ. Then N_1 and N_2 are automatically contact submanifolds, whose normal bundles come with distinguished trivializations determined by the open book. In the following, we shall always assume that M is oriented and the pages and binding are assigned the natural orientations determined by the open book, so in particular the binding is the oriented boundary of the pages.

Definition 2.1. Assume $\pi : M \setminus B \to S^1$ is an open book decomposition on M. By a **binding sum** of the open book, we mean any normal sum M_Φ along an orientation reversing bundle isomorphism $\Phi : \nu N_1 \to \nu N_2$ covering a diffeomorphism $\varphi : N_1 \to N_2$, where $N_1, N_2 \subset B$ are disjoint components of the binding and Φ is constant with respect to the distinguished trivialization determined by π. The resulting smooth manifold will be denoted by

$$M_{(\pi,\varphi)} := M_\Phi,$$

and we denote by $\mathcal{I}_{(\pi,\varphi)} \subset M_{(\pi,\varphi)}$ the closed hypersurface obtained by the identification of $\partial\mathcal{N}(N_1)$ with $\partial\mathcal{N}(N_2)$, which we'll also call the **interface**. We will then refer to the data (π, φ) as a **summed open book decomposition** of $M_{(\pi,\varphi)}$, whose **binding** is the (possibly empty) codimension 2 submanifold

$$B_\varphi := B \setminus (N_1 \cup N_2) \subset M_{(\pi,\varphi)}.$$

The **pages** of (π, φ) are the connected components of the fibers of the naturally induced fibration

$$\pi_\varphi : M_{(\pi,\varphi)} \setminus (B_\varphi \cup \mathcal{I}_{(\pi,\varphi)}) \to S^1;$$

if $\dim M = 3$, then these are naturally oriented open surfaces whose closures are generally immersed (distinct boundary components may sometimes coincide).

If ξ is a contact structure on M supported by π, we will denote the induced contact structure on $M_{(\pi,\varphi)}$ by

$$\xi_{(\pi,\varphi)} := \xi_\Phi$$

and say that $\xi_{(\pi,\varphi)}$ is **supported by** the summed open book (π, φ).

[4]We use the word *component* throughout to mean any open and closed subset, i.e. a disjoint union of connected components.

It follows from the corresponding fact for ordinary open books that every summed open book decomposition supports a contact structure, which is unique up to isotopy: in fact it depends only on the isotopy class of the open book $\pi : M \setminus B \to S^1$, the choice of binding components $N_1, N_2 \subset B$ and isotopy class of diffeomorphism $\varphi : N_1 \to N_2$.

Throughout this discussion, M, N_1, N_2 and the pages of π are all allowed to be disconnected (note that $\pi : M \setminus B \to S^1$ will have disconnected pages if M itself is disconnected). In this way, we can incorporate the notion of a binding sum of *multiple*, separate (perhaps summed) open books, e.g. given (M_i, ξ_i) supported by $\pi_i : M_i \setminus B_i \to S^1$ with components $N_i \subset B_i$ for $i = 1, 2$, and a diffeomorphism $\varphi : N_1 \to N_2$, a binding sum of (M_1, ξ_1) with (M_2, ξ_2) can be defined by applying the above construction to the disjoint union $M_1 \sqcup M_2$. We will generally use the shorthand notation

$$M_1 \boxplus M_2$$

to indicate manifolds constructed by binding sums of this type, where it is understood that M_1 and M_2 both come with contact structures and supporting summed open books, which combine to determine a summed open book and supported contact structure on $M_1 \boxplus M_2$.

Example 2.2. Consider the tight contact structure on $M := S^1 \times S^2$ with its supporting open book decomposition

$$\pi : M \setminus (\gamma_0 \cup \gamma_\infty) \to S^1 : (t, z) \mapsto z/|z|,$$

where $S^2 = \mathbb{C} \cup \{\infty\}$, $\gamma_0 := S^1 \times \{0\}$, $\gamma_\infty := S^1 \times \{\infty\}$ and S^1 is identified with the unit circle in $\mathbb{C}$. This open book has cylindrical pages and trivial monodromy. Now let M' denote a second copy of the same manifold and

$$\pi' : M' \setminus (\gamma_0' \cup \gamma_\infty') \to S^1$$

the same open book. Defining the binding sum $M \boxplus M'$ by pairing γ_0 with γ_0' and γ_∞ with γ_∞', we obtain the standard contact T^3. In fact, each of the tight contact tori (T^3, ξ_n), where

$$\xi_n = \ker \left[\cos(2\pi n\theta)\, dx + \sin(2\pi n\theta)\, dy\right]$$

in coordinates $(x, y, \theta) \in S^1 \times S^1 \times S^1$, can be obtained as a binding sum of $2n$ copies of the tight $S^1 \times S^2$; see Figure 5.

Example 2.3. Using the same open book decomposition on the tight $S^1 \times S^2$ as in Example 2.2, one can take only a single copy and perform a binding sum along the two binding components γ_0 and γ_∞. The contact manifold produced by this operation is the quotient of (T^3, ξ_1) by the contact involution $(x, y, \theta) \mapsto (-x, -y, \theta + 1/2)$, and is thus the torus bundle over S^1 with monodromy -1. The resulting summed open book on $T^3/\mathbb{Z}_2$ has connected cylindrical pages, empty binding and a single interface torus of the form $\mathcal{I}_{(\pi,\varphi)} = \{2\theta = 0\}$, inducing a fibration

$$\pi_\varphi : (T^3/\mathbb{Z}_2) \setminus \mathcal{I}_{(\pi,\varphi)} \to S^1 : [(x, y, \theta)] \mapsto \begin{cases} y & \text{if } \theta \in (0, 1/2), \\ -y & \text{if } \theta \in (1/2, 1). \end{cases}$$

The following two special cases of summed open books are of crucial importance.

Example 2.4. An ordinary open book can also be regarded as a summed open book: we simply take N_1 and N_2 to be empty.

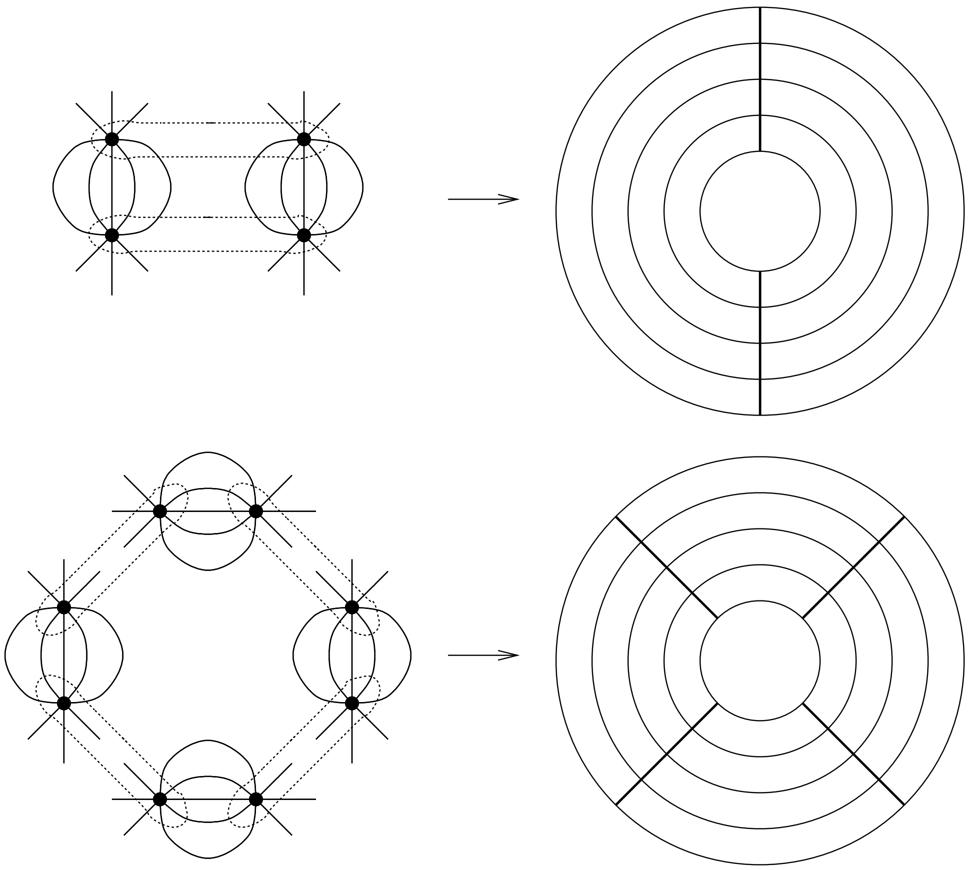

FIGURE 5. Two ways of producing tight contact tori from $2n$ copies of the tight $S^1 \times S^2$. At left, copies of $S^1 \times S^2$ are represented by open books with two binding components (depicted here through the page) and cylindrical pages. For each dotted oval surrounding two binding components, we construct the binding sum to produce the manifold at right, containing $2n$ special pre-Lagrangian tori (the black line segments) that separate regions foliated by cylinders. The results are (T^3, ξ_n) for $n = 1, 2$.

Example 2.5. Suppose (M_i, ξ_i) for $i = 1, 2$ are closed connected contact 3-manifolds with supporting open books π_i whose pages are diffeomorphic. Then we can set $N_1 = B_1$ and $N_2 = B_2$, choose a diffeomorphism $B_1 \to B_2$ and define $M = M_1 \boxplus M_2$ accordingly. The resulting summed open book is called **symmetric**; observe that it has empty binding, since every binding component of π_1 and π_2 has been summed. A simple example of this construction is (T^3, ξ_1) as explained in Example 2.2, and for an even simpler example, summing two open books with disk-like pages produces the tight $S^1 \times S^2$.

Remark 2.6. There is a close relationship between summed open books and the notion of open books with quasi-compatible contact structures, introduced by Etnyre and Van Horn-Morris [EV11]. A contact structure ξ is said to be *quasi-compatible* with an open book if it admits a contact vector field that is positively transverse to the pages and positively tangent to the binding; if the contact vector field is also positively transverse to ξ, then this is precisely the supporting condition, but quasi-compatibility is quite a bit more general, and can allow e.g. open books with empty binding. A summed open book on a 3-manifold gives rise to an open book with quasi-compatible contact structure whenever a certain orientation condition is satisfied: this is the result in particular whenever we construct binding sums of separate open books that are labeled with signs in such a way that every interface torus separates a positive piece from a negative piece. Thus the tight 3-tori in Figure 5 are examples, in this case producing an open book with empty binding (i.e. a fibration over S^1) that is quasi-compatible with all of the contact structures ξ_n. However, it is easy to construct binding sums for which this is not possible, e.g. Example 2.3.

We now generalize the discussion to include manifolds with boundary. Suppose $M_{(\pi,\varphi)}$ is a closed 3-manifold with summed open book (π, φ), which has binding B_φ and interface $\mathcal{I}_{(\pi,\varphi)}$, and $N \subset B_\varphi$ is a component of its binding. For each connected component $\gamma \subset N$, identify a tubular neighborhood $\mathcal{N}(\gamma)$ of γ with a solid torus $S^1 \times \mathbb{D}$, defining coordinates $(\theta, \rho, \phi) \in S^1 \times \mathbb{D}$, where (ρ, ϕ) denote polar coordinates[5] on the disk $\mathbb{D}$ and γ is the subset $S^1 \times \{0\} = \{\rho = 0\}$. Assume also that these coordinates are adapted to the summed open book, in the sense that the orientation of γ as a binding component agrees with the natural orientation of $S^1 \times \{0\}$, and the intersections of the pages with $\mathcal{N}(\gamma)$ are of the form $\{\phi = \text{const}\}$. This condition determines the coordinates up to isotopy. Then we define the *blown up* manifold $M_{(\pi,\varphi,\gamma)}$ from $M_{(\pi,\varphi)}$ by replacing $\mathcal{N}(\gamma) = S^1 \times \mathbb{D}$ with $S^1 \times [0,1] \times S^1$, using the same coordinates (θ, ρ, ϕ) on the latter, i.e. the binding circle γ is replaced by a 2-torus, which now forms the boundary of $M_{(\pi,\varphi,\gamma)}$. If $\xi_{(\pi,\varphi)}$ is a contact structure on $M_{(\pi,\varphi)}$ supported by (π, φ), then we can define an appropriate contact structure $\xi_{(\pi,\varphi,\gamma)}$ on $M_{(\pi,\varphi,\gamma)}$ as follows. Since γ is a positively transverse knot, the contact neighborhood theorem allows us to choose the coordinates (θ, ρ, ϕ) so that

$$\xi_{(\pi,\varphi)} = \ker\left(d\theta + \rho^2 d\phi\right)$$

in a neighborhood of γ. This formula also gives a well defined distribution on $M_{(\pi,\varphi,\gamma)}$, but the contact condition fails at the boundary $\{\rho = 0\}$. We fix this by making a C^0-small change in $\xi_{(\pi,\varphi)}$ to define a contact structure of the form

$$\xi_{(\pi,\varphi,\gamma)} = \ker\left[d\theta + g(\rho)\ d\phi\right],$$

where $g(\rho) = \rho^2$ for ρ outside a neighborhood of zero, $g'(\rho) > 0$ everywhere and $g(0) = 0$.

Performing the above operation at all connected components $\gamma \subset N \subset B_\varphi$ yields a compact manifold $M_{(\pi,\varphi,N)}$, generally with boundary, carrying a still more general decomposition determined by the data (π, φ, N), which we'll call a **blown up summed open book**. We define its **interface** to be the original interface $\mathcal{I}_{(\pi,\varphi)}$, and its **binding** is

$$B_{(\varphi,N)} = B_\varphi \setminus N.$$

There is a natural diffeomorphism

$$M_{(\pi,\varphi)} \setminus B_\varphi = M_{(\pi,\varphi,N)} \setminus \left(B_{(\varphi,N)} \cup \partial M_{(\pi,\varphi,N)}\right),$$

so the fibration $\pi_\varphi : M_{(\pi,\varphi)} \setminus \left(B_\varphi \cup \mathcal{I}_{(\pi,\varphi)}\right) \to S^1$ carries over to $M_{(\pi,\varphi,N)} \setminus (B_{(\varphi,N)} \cup \mathcal{I}_{(\pi,\varphi)} \cup \partial M_{(\pi,\varphi,N)})$, and can then be extended smoothly to the boundary to define a fibration

$$\pi_{(\varphi,N)} : M_{(\pi,\varphi,N)} \setminus \left(B_{(\varphi,N)} \cup \mathcal{I}_{(\pi,\varphi)}\right) \to S^1.$$

[5]Throughout this paper, we use polar coordinates (ρ, ϕ) on subdomains of $\mathbb{C}$ with the angular coordinate ϕ normalized to take values in $S^1 = \mathbb{R}/\mathbb{Z}$, i.e. the actual *angle* is $2\pi\phi$.

We will again refer to the connected components of the fibers of $\pi_{(\varphi,N)}$ as the **pages** of (π, φ, N), and orient them in accordance with the co-orientations induced by the fibration. Their closures are immersed surfaces which occasionally may have pairs of boundary components that coincide as oriented 1-manifolds, e.g. this can happen whenever two binding circles within the same connected open book are summed to each other.

Note that the fibration $\pi_{(\varphi,N)} : M_{(\pi,\varphi,N)} \setminus \big(B_{(\varphi,N)} \cup \mathcal{I}_{(\pi,\varphi)}\big) \to S^1$ is not enough information to fully determine the blown up open book (π, φ, N), as it does not uniquely determine the "blown down" manifold $M_{(\pi,\varphi)}$. Indeed, $M_{(\pi,\varphi)}$ determines on each boundary torus $T \subset \partial M_{(\pi,\varphi,N)}$ a distinguished basis

$$\{m_T, \ell_T\} \subset H_1(T),$$

where ℓ_T is a boundary component of a page and m_T is determined by the meridian on a small torus around the binding circle to be blown up. Two different manifolds $M_{(\pi,\varphi)}$ may sometimes produce diffeomorphic blown up manifolds $M_{(\pi,\varphi,N)}$, which will however have different meridians m_T on their boundaries. Similarly, each interface torus $T \subset \mathcal{I}_{(\pi,\varphi)}$ inherits a distinguished basis

$$\{\pm m_T, \ell_T\} \subset H_1(T)$$

from the binding sum operation, with the difference that the meridian m_T is only well defined up to a sign.

The binding sum of an open book $\pi : M \setminus B \to S^1$ along components $N_1 \cup N_2 \subset B$ can now also be understood as a two step operation, where the first step is to blow up N_1 and N_2, and the second is to attach the resulting boundary tori to each other via a diffeomorphism determined by $\Phi : \nu N_1 \to \nu N_2$. One can choose a supported contact structure on the blown up open book which fits together smoothly under this attachment to reproduce the construction of $\xi_{(\pi,\varphi,N)}$ described above.

Definition 2.7. A blown up summed open book (π, φ, N) is called **irreducible** if the fibers of the induced fibration $\pi_{(\varphi,N)}$ are connected.

In the irreducible case, the pages can be parametrized in a single S^1-family, e.g. an ordinary connected open book is irreducible, but a symmetric summed open book is not. Any blown up summed open book can however be decomposed uniquely into **irreducible subdomains**

$$M_{(\pi,\varphi,N)} = M^1_{(\pi,\varphi,N)} \cup \ldots \cup M^\ell_{(\pi,\varphi,N)},$$

where each piece $M^i_{(\pi,\varphi,N)}$ for $i = 1, \ldots, \ell$ is a compact manifold, possibly with boundary, defined as the closure in $M_{(\pi,\varphi,N)}$ of the region filled by some smooth S^1-family of pages. Thus $M^i_{(\pi,\varphi,N)}$ carries a natural blown up summed open book of its own, whose binding and interface are subsets of B_φ and $\mathcal{I}_{(\pi,\varphi)}$ respectively, and $\partial M^i_{(\pi,\varphi,N)} \subset \mathcal{I}_{(\pi,\varphi)} \cup \partial M_{(\pi,\varphi,N)}$. One can also write

$$M_{(\pi,\varphi,N)} = \check{M}^1_{(\pi,\varphi,N)} \boxplus \ldots \boxplus \check{M}^\ell_{(\pi,\varphi,N)},$$

where the manifolds $\check{M}^i_{(\pi,\varphi,N)}$ also naturally carry blown up summed open books and can be obtained from $M^i_{(\pi,\varphi,N)}$ by blowing down $\partial M^i_{(\pi,\varphi,N)} \cap \mathcal{I}_{(\pi,\varphi)}$.

Definition 2.8. Given a blown up summed open book (π, φ, N) on a manifold $M_{(\pi,\varphi,N)}$ with boundary, a **Giroux form** for (π, φ, N) is a contact form λ on $M_{(\pi,\varphi,N)}$ with Reeb vector field X_λ satisfying the following conditions:

(1) X_λ is positively transverse to the interiors of the pages,

(2) X_λ is positively tangent to the boundaries of the closures of the pages,
(3) $\ker\lambda$ on each interface or boundary torus $T \subset \mathcal{I}_{(\pi,\varphi)} \cup \partial M_{(\pi,\varphi,N)}$ induces a characteristic foliation with closed leaves homologous to the meridian m_T.

We will say that a contact structure on $M_{(\pi,\varphi,N)}$ is **supported** by (π,φ,N) whenever it is the kernel of a Giroux form. By the procedure described above, one can easily take a Giroux form for the underlying open book $\pi : M \setminus B \to S^1$ and modify it near B to produce a Giroux form for the blown up summed open book on $M_{(\pi,\varphi,N)}$. Moreover, the same argument that proves uniqueness of contact structures supported by open books (cf. [Etn06, Prop. 3.18]) shows that any two Giroux forms are homotopic to each other through a family of Giroux forms. We thus obtain the following uniqueness result for supported contact structures.

Proposition 2.9. *Suppose $M_{(\pi,\varphi,N)}$ is a compact 3-manifold with boundary, with a contact structure $\xi_{(\pi,\varphi,N)}$ supported by the blown up summed open book (π,φ,N), and $(M_{(\pi,\varphi,N)}, \xi_{(\pi,\varphi,N)})$ admits a contact embedding into some closed contact 3-manifold (M', ξ'). If λ is a contact form on M' such that*

(1) *λ defines a Giroux form on $M_{(\pi,\varphi,N)} \subset M'$, and*
(2) *$\ker\lambda = \xi'$ on $M' \setminus M_{(\pi,\varphi,N)}$,*

then $\ker\lambda$ is isotopic to ξ'.

Example 2.10. Suppose Σ is a compact, oriented and connected surface, possibly with boundary, containing a non-empty multicurve $\Gamma \subset \Sigma$ such that $\partial\Sigma \subset \Gamma$ and Γ divides Σ into two (possibly disconnected) pieces

$$\Sigma = \Sigma_+ \cup_\Gamma \Sigma_-.$$

By Lutz [Lut77], $S^1 \times \Sigma$ admits an S^1-invariant contact structure ξ_Γ which is determined uniquely up to isotopy by the condition that the loops $S^1 \times \{z\}$ be positively/negatively transverse to ξ_Γ for $z \in \operatorname{int}\Sigma_\pm$ and Legendrian for $z \in \Gamma$. Then $(S^1 \times \Sigma, \xi_\Gamma)$ is supported by a blown up summed open book with empty binding, interface $\mathcal{I} = S^1 \times (\Gamma \setminus \partial\Sigma)$ and fibration

$$\pi : (S^1 \times \Sigma) \setminus \mathcal{I} \to S^1 : (\phi, z) \mapsto \begin{cases} \phi & \text{for } z \in \Sigma_+, \\ -\phi & \text{for } z \in \Sigma_-. \end{cases}$$

Indeed, one can write ξ_Γ as the kernel of a contact form whose Reeb vector field is positively/negatively transverse to the interior of $\{*\} \times \Sigma_\pm$ and admits closed orbits of the form $\{*\} \times \gamma$ for each dividing curve $\gamma \subset \Gamma$. (An explicit construction of such a contact form may be found e.g. in [LW11].) The distinguished meridians at $\mathcal{I}$ and $\partial(S^1 \times \Sigma)$ are generated by the Legendrians $S^1 \times \{*\}$.

2.2. **Partially planar domains and planar torsion.** We are now ready to state the most important definitions in this paper.

Definition 2.11. A blown up summed open book on a compact manifold M is called **partially planar** if $M \setminus \partial M$ contains a planar page. A **partially planar domain** is then any contact 3-manifold (M, ξ) with a supporting blown up summed open book that is partially planar. An irreducible subdomain

$$M^P \subset M$$

that contains planar pages and doesn't touch ∂M is called a **planar piece**, and we will refer to the complementary subdomain $\overline{M \setminus M^P}$ as the **padding**.

By this definition, every planar contact manifold is a partially planar domain (with empty padding), as is the symmetric summed open book obtained by summing together two planar open books with the same number of binding components (here one can call either side the planar piece, and the other side the padding). As we'll soon see, one can also use partially planar domains to characterize the solid torus that appears in a Lutz twist, or the thickened torus in the definition of Giroux torsion, as well as many more general objects.

Definition 2.12. We say that a contact 3-manifold (M, ξ) with a closed 2-form Ω contains an **Ω-separating** partially planar domain if there exists a partially planar domain (M_0, ξ_0) with planar piece $M_0^P \subset M_0$ and a contact embedding $\iota : (M_0, \xi_0) \hookrightarrow (M, \xi)$ such that for every interface torus T of M_0 lying in M_0^P, $\int_T \iota^*\Omega = 0$. We say that the domain is **fully separating** if this is true for all choices of Ω.

Note that in general, a 2-torus T embedded in a closed oriented 3-manifold M satisfies $\int_T \Omega = 0$ for all closed 2-forms Ω on M if and only if T separates M. In a partially planar domain, any interface torus in the interior of the planar piece is necessarily non-separating, thus the fully separating condition implies that there are no such interface tori, and each component of the boundary of the planar piece also separates (cf. Definition 1.3).

We now come to the definition of a new symplectic filling obstruction.

Definition 2.13. For any integer $k \geq 0$, a contact manifold (M, ξ), possibly with boundary, is called a **planar torsion domain of order** k (or briefly a **planar k-torsion domain**) if it is supported by a partially planar blown up summed open book (π, φ, N) with a planar piece $M^P \subset M$ satisfying the following conditions:

(1) The pages in M^P have $k + 1$ boundary components.
(2) The padding $\overline{M \setminus M^P}$ is not empty.
(3) (π, φ, N) is not a symmetric summed open book (cf. Example 2.5).

We say that a contact 3-manifold (M, ξ) has (perhaps Ω-separating or fully separating) **planar k-torsion** if it admits a (perhaps Ω-separating or fully separating) contact embedding of a planar k-torsion domain.

Remark 2.14. The planar piece of a planar 0-torsion domain has no interior interface tori and only one boundary component, thus planar 0-torsion is *always* fully separating. It is easy to see from examples (cf. Example 2.15) that this is not true for $k \geq 1$. Observe also that whenever (M, ξ) is closed and connected and contains a fully separating partially planar domain $M_0 \subset M$, one of the following must be true:

(i) (M_0, ξ) is a planar torsion domain,
(ii) $M_0 = M$ and the interface is empty, i.e. (M, ξ) is supported by an ordinary planar open book,
(iii) $M_0 = M$ and it carries a symmetric summed open book with disk-like pages.

In the last case, (M, ξ) is contactomorphic to the tight $S^1 \times S^2$ (see Example 2.5), which is planar. We thus conclude that under these assumptions, (M, ξ) always either has planar torsion or is planar.

Example 2.15. The S^1-invariant contact manifold $(S^1 \times \Sigma, \xi_\Gamma)$ from Example 2.10 is a partially planar domain whenever $\Sigma \setminus \Gamma$ has a connected component Σ_0 of genus zero with $\overline{\Sigma}_0 \cap \partial\Sigma = \emptyset$. In this case $S^1 \times \overline{\Sigma}_0$ is the planar piece, and $S^1 \times \Sigma$ is also a planar torsion domain unless the blown up summed open book from Example 2.10 is symmetric, which would mean

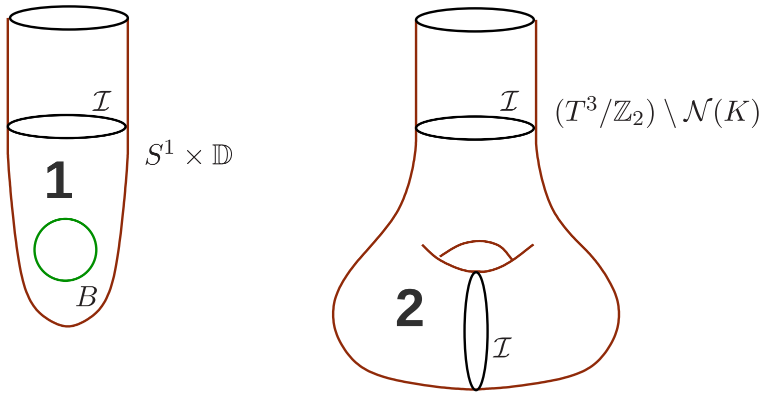

FIGURE 6. Schematic representations of two planar torsion domains as described in Example 2.16.

$\partial\Sigma = \emptyset$ and $\Sigma \setminus \Gamma$ has exactly two connected components, which are diffeomorphic to each other. Some special cases are shown in Figures 3 and 4.

Example 2.16. More generally than the S^1-invariant examples described above, blown up summed open books can always be represented by schematic pictures as in Figure 6, which shows two examples of planar torsion domains, each with the order labeled within the planar piece. Here each picture shows a surface Σ containing a multicurve Γ: each connected component $\Sigma_0 \subset \Sigma \setminus \Gamma$ then represents an irreducible subdomain with pages diffeomorphic to Σ_0, and the components of Γ represent interface tori (labeled in the picture by $\mathcal{I}$). Each irreducible subdomain may additionally have binding circles, shown in the picture as circles with the label B. The information in these pictures, together with a specified monodromy map for each component of $\Sigma \setminus \Gamma$, determine a blown up summed open book and supported contact structure uniquely up to contactomorphism. If we take these particular pictures with the assumption that all monodromy maps are trivial, then the first shows a solid torus $S^1 \times \mathbb{D}$ with an overtwisted contact structure that makes one full twist along a ray from the center (the binding B) to the boundary. The other picture shows the complement of a solid torus in the torus bundle $T^3/\mathbb{Z}_2$ from Example 2.3. More precisely, one can construct it by taking a loop $K \subset T^3/\mathbb{Z}_2$ transverse to the pages in that example, modifying the contact structure ξ near K by a full Lutz twist, and then removing a smaller neighborhood $\mathcal{N}(K)$ of K on which ξ makes a quarter twist. Note that the appearance of genus in this picture is a bit misleading; due to the interface torus in the interior of the bottom piece, it has planar pages with three boundary components.

We can now proceed toward the proof of Theorem 3.

Definition 2.17. A **Lutz tube** is the solid torus $S^1 \times \mathbb{D}$ with coordinates (θ, ρ, ϕ), where (ρ, ϕ) are polar coordinates on the closed unit disk $\mathbb{D} \subset \mathbb{C}$, together with the contact structure ξ defined as the hyperplane field

$$\xi = \ker\left[f(\rho)\, d\theta + g(\rho)\, d\phi\right] \tag{2.1}$$

for some pair of smooth functions f, g such that the path

$$[0,1] \to \mathbb{R}^2 \setminus \{0\} : \rho \mapsto (f(\rho), g(\rho))$$

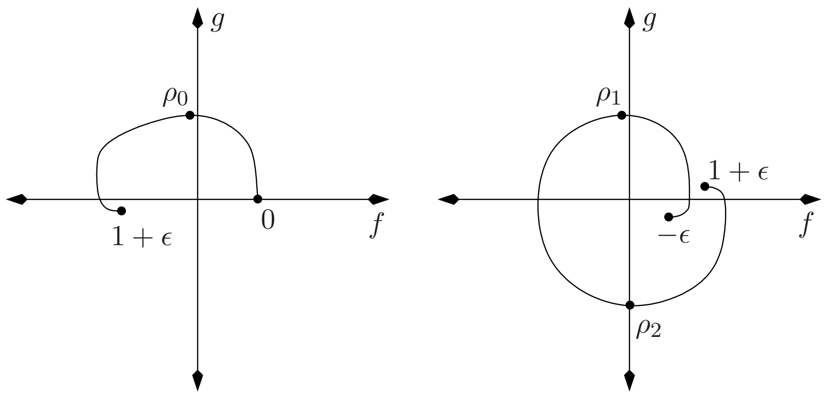

FIGURE 7. The path $\rho \mapsto (f(\rho), g(\rho))$ used to define the contact form on L_ϵ (for the Lutz tube at the left and Giroux torsion domain at the right) in the proof of Prop. 2.19.

makes exactly one half-turn (counterclockwise) about the origin, moving from the positive to the negative x-axis. (See Figure 1.)

Definition 2.18. A **Giroux torsion domain** is the thickened torus $[0, 1] \times T^2$ with coordinates $(\rho, \phi, \theta) \in [0, 1] \times S^1 \times S^1$, together with the contact structure ξ defined via these coordinates as in (2.1), where the path $\rho \mapsto (f(\rho), g(\rho))$ makes one full (counterclockwise) turn about the origin, beginning and ending on the positive x-axis. (See Figure 2.)

Proposition 2.19. *If $L \subset M$ is a Lutz tube in a closed contact 3-manifold (M, ξ), then any open neighborhood of L contains a planar 0-torsion domain. Similarly if L is a Giroux torsion domain, then any open neighborhood of L contains a planar 1-torsion domain.*

Proof. Suppose $L \subset M$ is a Lutz tube. Then for some $\epsilon > 0$, an open neighborhood of L contains a region identified with

$$L_\epsilon := S^1 \times \mathbb{D}_{1+\epsilon},$$

where $\mathbb{D}_r$ denotes the closed disk of radius r and $\xi = \ker \lambda_\epsilon$ for a contact form

$$\lambda_\epsilon = f(\rho)\ d\theta + g(\rho)\ d\phi$$

with the following properties (see Figure 7, left):

(1) $f(0) > 0$ and $g(0) = 0$,
(2) $f(1) < 0$ and $g(1) = 0$,
(3) $f(\rho)g'(\rho) - f'(\rho)g(\rho) > 0$ for all $\rho > 0$,
(4) $g'(1 + \epsilon) = 0$,
(5) $f(1 + \epsilon)/g(1 + \epsilon) \in \mathbb{Z}$.

Setting $D(\rho) := f(\rho)g'(\rho) - f'(\rho)g(\rho)$, the Reeb vector field defined by λ_ϵ in the region $\rho > 0$ is

$$X_\epsilon = \frac{1}{D(\rho)} \left[g'(\rho)\partial_\theta - f'(\rho)\partial_\phi\right],$$

and at $\rho = 0$, $X_\epsilon = \frac{1}{f(0)}\partial_\theta$. Thus X_ϵ in these coordinates depends only on ρ and its direction is always determined by the slope of the path $\rho \mapsto (f(\rho), g(\rho))$ in $\mathbb{R}^2$; in particular, X_ϵ points in the $-\partial_\phi$-direction at $\rho = 1 + \epsilon$, and in the $+\partial_\phi$-direction at some other radius $\rho_0 \in (0, 1)$. We can choose f and g without loss of generality so that these are the only radii at which X_ϵ is parallel to $\pm\partial_\phi$.

We claim now that L_ϵ is a planar 0-torsion domain with planar piece $L_\epsilon^P := S^1 \times \mathbb{D}_{\rho_0}$. Indeed, L_ϵ^P can be obtained from the open book on the tight 3-sphere with disk-like pages by blowing up the binding: the pages in the interior of L_ϵ^P are defined by $\{\theta = \text{const}\}$. Similarly, the θ-level sets in the closure of $L_\epsilon \setminus L_\epsilon^P$ form the pages of a blown up open book, obtained from an open book with cylindrical pages. The condition $f(1+\epsilon)/g(1+\epsilon) \in \mathbb{Z}$ implies that the characteristic foliation on $T := \partial L_\epsilon$ has closed leaves homologous to a primitive class $m_T \in H_1(T)$, which together with the homology class of the Reeb orbits on T forms a basis of $H_1(T)$. Thus our chosen contact form λ_ϵ is a Giroux form for some blown up summed open book. (Note that the monodromy of the blown up open book in $L_\epsilon \setminus L_\epsilon^P$ is not trivial since the distinguished meridians on ∂L_ϵ and ∂L_ϵ^P are not homologous.)

The argument for Giroux torsion is quite similar, so we'll only sketch it: given $L = [0, 1] \times T^2 \subset M$, we can expand L slightly on *both* sides to create a domain

$$L_\epsilon = [-\epsilon, 1 + \epsilon] \times T^2,$$

with a contact form λ_ϵ that induces a suitable characteristic foliation on ∂L_ϵ and whose Reeb vector field points in the $\pm\partial_\phi$-direction at $\rho = -\epsilon$, $\rho = 1 + \epsilon$ and exactly two other radii $0 < \rho_1 < \rho_2 < 1$ (see Figure 7, right). This splits L_ϵ into three pieces, of which $L_\epsilon^P := \{\rho \in [\rho_1, \rho_2]\}$ is the planar piece of a planar 1-torsion domain, as it can be obtained from an open book with cylindrical pages and trivial monodromy by blowing up both binding components. The padding now consists of two separate blown up open books with cylindrical pages and nontrivial monodromy. □

Proof of Theorem 3. The only claim in the theorem that doesn't follow immediately from Prop. 2.19 is that (M, ξ) must be overtwisted if it contains a planar 0-torsion domain M_0. One can see this as follows: note first that if we write

$$M_0 = M_0^P \cup M_0',$$

where M_0^P is the planar piece and $M_0' = \overline{M_0 \setminus M_0^P}$ is the padding, then M_0' carries a blown up summed open book with pages that are not disks (which means (M_0, ξ) is not the tight $S^1 \times S^2$). If the pages in M_0' are surfaces with positive genus and one boundary component, then one can glue one of these together with a page in M_0^P to form a convex surface $\Sigma \subset M_0$ whose dividing set is $\partial M_0^P \cap \Sigma$. The latter is the boundary of a disk in Σ, so Giroux's criterion (see [Gir01, Théorème 4.5(a)] or [Gei08, Prop. 4.8.13]) implies the existence of an overtwisted disk near Σ.

In all other cases the pages Σ in M_0' have multiple boundary components

$$\partial\Sigma = C^P \cup C',$$

where we denote by C^P the connected component situated near the interface ∂M_0^P, and $C' = \partial\Sigma \setminus C^P$. We can then find overtwisted disks by constructing a particular Giroux form using a small variation on the Thurston-Winkelnkemper construction as described e.g. in [Etn06, Theorem 3.13]. Namely, choose coordinates $(s, t) \in (1/2, 1] \times S^1$ on a collar neighborhood of each component of $\partial\Sigma$ and define a 1-form λ_1 on Σ with the following properties:

(1) $d\lambda_1 > 0$
(2) $\lambda_1 = (1 + s)\, dt$ near each component of C'

(3) $\lambda_1 = (-1+s)\, dt$ near C^P

Observe that all three conditions cannot be true unless C' is non-empty, due to Stokes's theorem. Now following the construction described in [Etn06], one can produce a Giroux form λ on M_0' which annihilates some boundary parallel curve ℓ near ∂M_0^P in a page, and fits together smoothly with some Giroux form in M_0^P, so that $\ker \lambda$ is a supported contact structure and is isotopic to ξ by Prop. 2.9. Then ℓ is the boundary of an overtwisted disk. □

3. Holomorphic summed open books

3.1. **Technical background.** We begin by collecting some definitions and background results on punctured holomorphic curves that will be important for understanding the remainder of the paper.

A **stable Hamiltonian structure** on an oriented 3-manifold M is a pair $\mathcal{H} = (\lambda, \omega)$ consisting of a 1-form λ and 2-form ω such that $d\omega = 0$, $\lambda \wedge \omega > 0$ and $\ker \omega \subset \ker d\lambda$. Given this data, we define the co-oriented 2-plane distribution $\xi = \ker \lambda$ and nowhere vanishing vector field X, called the **Reeb vector field**, which is determined by the conditions

$$\omega(X, \cdot) \equiv 0, \qquad \lambda(X) \equiv 1.$$

The conditions on λ and ω imply that $\omega|_\xi$ gives ξ the structure of a symplectic vector bundle over M, and this distribution with its symplectic structure is preserved by the flow of X. As an important special case, if λ is a contact form, then one can define a stable Hamiltonian structure in the form $\mathcal{H} = (\lambda, h\, d\lambda)$ for any smooth function $h : M \to (0, \infty)$ such that $dh \wedge d\lambda \equiv 0$. Then ξ is a positive and co-oriented contact structure, and X is the usual contact geometric notion of the Reeb vector field: we will often denote it in this case by X_λ, since it is uniquely determined by λ.

For the rest of this section, assume $\mathcal{H} = (\lambda, \omega)$ is a stable Hamiltonian structure with the usual attached data ξ and X. We say that an almost complex structure J on $\mathbb{R} \times M$ is **compatible with** $\mathcal{H}$ if it satisfies the following conditions:

(1) The natural $\mathbb{R}$-action on $\mathbb{R} \times M$ preserves J.
(2) $J\partial_t \equiv X$, where ∂_t denotes the unit vector in the $\mathbb{R}$-direction.
(3) $J(\xi) = \xi$ and $\omega(\cdot, J\cdot)$ defines a symmetric, positive definite bundle metric on ξ.

Denote by $\mathcal{J}(\mathcal{H})$ the (non-empty and contractible) space of almost complex structures compatible with $\mathcal{H}$. Note that if λ is contact then $\mathcal{J}(\mathcal{H})$ depends only on λ; we will in this case say that J is **compatible with** λ.

A periodic orbit γ of X is determined by the data (x, T), where $x : \mathbb{R} \to M$ satisfies $\dot{x} = X(x)$ and $x(T) = x(0)$ for some $T > 0$. We sometimes abuse notation and identify γ with the submanifold $x(\mathbb{R}) \subset M$, though technically the period is also part of the data defining γ. If $\tau > 0$ is the smallest positive number for which $x(\tau) = x(0)$, we call it the **minimal period** of this orbit, and say that $\gamma = (x, \tau)$ is a **simple**, or **simply covered** orbit. The **covering multiplicity** of an orbit (x, T) is the unique integer $k \geq 1$ such that $T = k\tau$ for a simple orbit (x, τ).

If $\gamma = (x, T)$ is a periodic orbit and φ_X^t denotes the flow of X for time $t \in \mathbb{R}$, then the restriction of the linearized flow to $\xi_{x(0)}$ defines a symplectic isomorphism

$$(\varphi_X^T)_* : (\xi_{x(0)}, \omega) \to (\xi_{x(0)}, \omega).$$

We call γ **nondegenerate** if 1 is not in the spectrum of this map. More generally, a **Morse-Bott submanifold** of T-periodic orbits is a closed submanifold $N \subset M$ fixed by φ_X^T such that for any $p \in N$,

$$\ker\left((\varphi_X^T)_* - \mathbb{1}\right) = T_p N.$$

We will call a single orbit $\gamma = (x, T)$ **Morse-Bott** if it lies on a Morse-Bott submanifold of T-periodic orbits. Nondegenerate orbits are clearly also Morse-Bott, with $N \cong S^1$. We say that the vector field X is **Morse-Bott** (or **nondegenerate**) if all of its periodic orbits are Morse-Bott (or nondegenerate respectively). Since X never vanishes, every Morse-Bott submanifold $N \subset M$ of dimension 2 is either a torus or a Klein bottle. One can show (cf. [Wen10a, Prop. 4.1]) that in the former case, if X is Morse-Bott, then every orbit contained in N has the same minimal period.

To every orbit $\gamma = (x, T)$, one can associate an **asymptotic operator**, which is morally the Hessian of a certain functional whose critical points are the periodic orbits. To write it down, choose $J \in \mathcal{J}(\mathcal{H})$, let $\mathbf{x} : S^1 \to M : t \mapsto x(Tt)$, choose a symmetric connection ∇ on M and define

$$\mathbf{A}_\gamma : \Gamma(\mathbf{x}^*\xi) \to \Gamma(\mathbf{x}^*\xi) : \eta \mapsto -J(\nabla_t \eta - T\nabla_\eta X).$$

One can show that this operator is well defined independently of the choice of connection, and it extends to an unbounded self-adjoint operator on the complexification of $L^2(\mathbf{x}^*\xi)$, with domain $H^1(\mathbf{x}^*\xi)$. Its spectrum $\sigma(\mathbf{A}_\gamma)$ consists of real eigenvalues with multiplicity at most 2, which accumulate only at $\pm\infty$. It is straightforward to show that solutions of the equation $\mathbf{A}_\gamma \eta = 0$ are given by $\eta(t) = (\varphi_X^{Tt})_* \eta(0)$, thus γ is nondegenerate if and only if $0 \notin \sigma(\mathbf{A}_\gamma)$, and in general if γ belongs to a Morse-Bott submanifold $N \subset M$, then

$$\dim \ker \mathbf{A}_\gamma = \dim N - 1.$$

Choosing a unitary trivialization Φ of (ξ, J, ω) along the parametrization $\mathbf{x} : S^1 \to M$ identifies $\mathbf{A}_\gamma$ with a first-order differential operator of the form

$$H^1(S^1, \mathbb{R}^2) \to L^2(S^1, \mathbb{R}^2) : \eta \mapsto -J_0 \dot{\eta} - S\eta, \tag{3.1}$$

where J_0 denotes the standard complex structure on $\mathbb{R}^2 = \mathbb{C}$ and $S : S^1 \to \mathrm{End}_{\mathbb{R}}(\mathbb{R}^2)$ is a smooth loop of symmetric real 2-by-2 matrices. Seen in this trivialization, $\mathbf{A}_\gamma \eta = 0$ defines a linear Hamiltonian equation $\dot{\eta} = J_0 S \eta$ corresponding to the linearized flow of X along γ, thus its flow defines a smooth family of symplectic matrices

$$\Psi : [0, 1] \to \mathrm{Sp}(2)$$

for which $1 \notin \sigma(\Psi(1))$ if and only if γ is nondegenerate. In this case, the homotopy class of the path Ψ is described by its **Conley-Zehnder index** $\mu_{\mathrm{CZ}}(\Psi) \in \mathbb{Z}$, which we use to define the Conley-Zehnder index of the orbit γ and of the asymptotic operator $\mathbf{A}_\gamma$ with respect to the trivialization Φ,

$$\mu_{\mathrm{CZ}}^\Phi(\gamma) := \mu_{\mathrm{CZ}}^\Phi(\mathbf{A}_\gamma) := \mu_{\mathrm{CZ}}(\Psi).$$

Note that in this way, $\mu_{\mathrm{CZ}}^\Phi(\mathbf{A})$ can be defined for *any* injective operator $\mathbf{A} : \Gamma(\mathbf{x}^*\xi) \to \Gamma(\mathbf{x}^*\xi)$ that takes the form (3.1) in a local trivialization. In particular then, even if γ is degenerate, we can pick any $\epsilon \in \mathbb{R} \setminus \sigma(\mathbf{A}_\gamma)$ and define the "perturbed" Conley-Zehnder index

$$\mu_{\mathrm{CZ}}^\Phi(\gamma - \epsilon) := \mu_{\mathrm{CZ}}^\Phi(\mathbf{A}_\gamma - \epsilon) := \mu_{\mathrm{CZ}}(\Psi_\epsilon),$$

where $\Psi_\epsilon : [0, 1] \to \mathrm{Sp}(2)$ is the path of symplectic matrices determined by the equation $(\mathbf{A}_\gamma - \epsilon)\eta = 0$ in the trivialization Φ. It is especially convenient to define Conley-Zehnder indices in this way for orbits that are degenerate but Morse-Bott: then the discreteness of the spectrum implies that for sufficiently small $\epsilon > 0$, the integer $\mu_{\mathrm{CZ}}^\Phi(\gamma \pm \epsilon)$ depends only on γ, Φ and the choice of sign.

The eigenfunctions of $\mathbf{A}_\gamma$ are nowhere vanishing sections $e \in \Gamma(\mathbf{x}^*\xi)$ and thus have well defined winding numbers $\mathrm{wind}^\Phi(e)$ with respect to any trivialization Φ. As shown in [HWZ95a], all sections in the same eigenspace have the same winding, thus defining a function

$$\sigma(\mathbf{A}_\gamma) \to \mathbb{Z} : \mu \mapsto \mathrm{wind}^\Phi(\mu),$$

where we set $\mathrm{wind}^\Phi(\mu) := \mathrm{wind}^\Phi(e)$ for any nontrivial $e \in \ker(\mathbf{A}_\gamma - \mu)$. In fact, [HWZ95a] shows that this function is nondecreasing and surjective: counting with multiplicity there are exactly two eigenvalues $\mu \in \sigma(\mathbf{A}_\gamma)$ such that $\mathrm{wind}^\Phi(\mu)$ equals any given integer. It is thus sensible to define the integers,

$$\begin{aligned}
\alpha_-^\Phi(\gamma - \epsilon) &= \max\{\mathrm{wind}^\Phi(\mu) \mid \mu \in \sigma(\mathbf{A}_\gamma - \epsilon),\, \mu < 0\},\\
\alpha_+^\Phi(\gamma - \epsilon) &= \min\{\mathrm{wind}^\Phi(\mu) \mid \mu \in \sigma(\mathbf{A}_\gamma - \epsilon),\, \mu > 0\},\\
p(\gamma - \epsilon) &= \alpha_+^\Phi(\gamma - \epsilon) - \alpha_-^\Phi(\gamma - \epsilon).
\end{aligned}$$

Note that the **parity** $p(\gamma - \epsilon)$ does not depend on Φ, and it always equals either 0 or 1 if $\epsilon \notin \sigma(\mathbf{A}_\gamma)$. In this case, the Conley-Zehnder index can be computed as

$$\mu_{\mathrm{CZ}}^\Phi(\gamma - \epsilon) = 2\alpha_-^\Phi(\gamma - \epsilon) + p(\gamma - \epsilon) = 2\alpha_+^\Phi(\gamma - \epsilon) - p(\gamma - \epsilon). \tag{3.2}$$

Given $\mathcal{H} = (\lambda, \omega)$ and $J \in \mathcal{J}(\mathcal{H})$, fix $c_0 > 0$ sufficiently small so that $(\omega + c\, d\lambda)|_\xi > 0$ for all $c \in [-c_0, c_0]$, and define

$$\mathcal{T} = \{\varphi \in C^\infty(\mathbb{R}, (-c, c)) \mid \varphi' > 0\}.$$

For $\varphi \in \mathcal{T}$, we can define a symplectic form on $\mathbb{R} \times M$ by

$$\omega_\varphi = \omega + d(\varphi\lambda), \tag{3.3}$$

where ω and λ are pulled back through the projection $\mathbb{R} \times M \to M$ to define differential forms on $\mathbb{R} \times M$, and $\varphi : \mathbb{R} \to (-c, c)$ is extended in the natural way to a function on $\mathbb{R} \times M$. Then any $J \in \mathcal{J}(\mathcal{H})$ is compatible with ω_φ in the sense that $\omega_\varphi(\cdot, J\cdot)$ defines a Riemannian metric on $\mathbb{R} \times M$. We therefore consider punctured pseudoholomorphic curves

$$u : (\dot\Sigma, j) \to (\mathbb{R} \times M, J)$$

where (Σ, j) is a closed Riemann surface with a finite subset of punctures $\Gamma \subset \Sigma$, $\dot\Sigma := \Sigma \setminus \Gamma$, and u is required to satisfy the *finite energy* condition

$$E(u) := \sup_{\varphi \in \mathcal{T}} \int_{\dot\Sigma} u^*\omega_\varphi < \infty. \tag{3.4}$$

An important example is the following: for any closed Reeb orbit $\gamma = (x, T)$, the map

$$u_\gamma : \mathbb{R} \times S^1 \to \mathbb{R} \times M : (s, t) \mapsto (Ts, x(Tt))$$

is a finite energy J-holomorphic cylinder (or equivalently punctured plane), which we call the **trivial cylinder** over γ. More generally, we are most interested in punctured J-holomorphic curves $u : \dot\Sigma \to \mathbb{R} \times M$ that are asymptotically cylindrical, in the following sense. Define the standard half cylinders

$$Z_+ = [0, \infty) \times S^1 \quad \text{and} \quad Z_- = (-\infty, 0] \times S^1.$$

We say that a smooth map $u : \dot\Sigma \to \mathbb{R} \times M$ is **asymptotically cylindrical** if the punctures can be partitioned into positive and negative subsets

$$\Gamma = \Gamma^+ \cup \Gamma^-$$

such that for each $z \in \Gamma^\pm$, there is a Reeb orbit $\gamma_z = (x, T)$, a closed neighborhood $\mathcal{U}_z \subset \Sigma$ of z and a diffeomorphism $\varphi_z : Z_\pm \to \mathcal{U}_z \setminus \{z\}$ such that for sufficiently large $|s|$,

$$u \circ \varphi_z(s,t) = \exp_{(Ts, x(Tt))} h_z(s,t), \tag{3.5}$$

where h_z is a section of ξ along u_{γ_z} with $h_z(s,t) \to 0$ for $s \to \pm\infty$, and the exponential map is defined with respect to any choice of $\mathbb{R}$-invariant connection on $\mathbb{R} \times M$. We often refer to the punctured neighborhoods $\mathcal{U}_z \setminus \{z\}$ or their images in $\mathbb{R} \times M$ as the positive and negative *ends* of u, and we call γ_z the **asymptotic orbit** of u at z.

Definition 3.1. Suppose $N \subset M$ is a submanifold which is the union of a family of Reeb orbits that all have the same minimal period. Consider an asymptotically cylindrical map $u : \dot{\Sigma} \to \mathbb{R} \times M$ with punctures $\Gamma^+ \cup \Gamma^- \subset \Sigma$ and corresponding asymptotic orbits γ_z with covering multiplicities $k_z \ge 1$ for each $z \in \Gamma^\pm$. Then if $k_N^\pm \ge 0$ denotes the sum of the multiplicities k_z for all punctures $z \in \Gamma^\pm$ at which γ_z lies in N, we shall say that u approaches N with **total multiplicity** $k_N^\pm$ at its positive or negative ends respectively.

Every asymptotically cylindrical map defines a **relative homology class** in the following sense. Suppose $\boldsymbol{\gamma} = \{(\gamma_1, m_1), \dots, (\gamma_N, m_N)\}$ is an **orbit set**, i.e. a finite collection of distinct simply covered Reeb orbits γ_i paired with positive integers m_i. This defines a 1-dimensional submanifold of M,

$$\bar{\boldsymbol{\gamma}} = \gamma_1 \cup \ldots \cup \gamma_N,$$

together with homology classes

$$[\boldsymbol{\gamma}] = m_1[\gamma_1] + \ldots + m_N[\gamma_N]$$

in both $H_1(M)$ and $H_1(\bar{\boldsymbol{\gamma}})$. Given two orbit sets $\boldsymbol{\gamma}^+$ and $\boldsymbol{\gamma}^-$ with $[\boldsymbol{\gamma}^+] = [\boldsymbol{\gamma}^-] \in H_1(M)$, denote by $H_2(M, \boldsymbol{\gamma}^+ - \boldsymbol{\gamma}^-)$ the affine space over $H_2(M)$ consisting of equivalence classes of 2-chains C in M with boundary ∂C in $\bar{\boldsymbol{\gamma}}^+ \cup \bar{\boldsymbol{\gamma}}^-$ representing the homology class $[\boldsymbol{\gamma}^+] - [\boldsymbol{\gamma}^-] \in H_1(\bar{\boldsymbol{\gamma}}^+ \cup \bar{\boldsymbol{\gamma}}^-)$, where $C \sim C'$ whenever $C - C'$ is the boundary of a 3-chain in M. Now, the projection of any asymptotically cylindrical map $u : \dot{\Sigma} \to \mathbb{R} \times M$ to M can be extended as a continuous map from a compact surface with boundary (the circle compactification of $\dot{\Sigma}$) to M, which then represents a relative homology class

$$[u] \in H_2(M, \boldsymbol{\gamma}^+ - \boldsymbol{\gamma}^-)$$

for some unique choice of orbit sets $\boldsymbol{\gamma}^+$ and $\boldsymbol{\gamma}^-$.

As is well known (cf. [Hof93, HWZ96a, HWZ96b]), every finite energy J-holomorphic curve with nonremovable punctures is asymptotically cylindrical if the contact form is Morse-Bott. Moreover in this case, the section h_z in (3.5), which controls the asymptotic approach of u to γ_z at $z \in \Gamma^\pm$, either is identically zero or satisfies a formula of the form[6]

$$h_z(s,t) = e^{\mu s}(e_\mu(t) + r(s,t)), \tag{3.6}$$

where $\mu \in \sigma(\mathbf{A}_\gamma)$ with $\pm\mu < 0$, e_μ is a nontrivial eigenfunction in the μ-eigenspace, and the remainder term $r(s,t) \in \xi_{x(Tt)}$ decays to zero as $s \to \pm\infty$. It follows that unless $h_z \equiv 0$, which is true only if u is a cover of a trivial cylinder, u has a well defined **asymptotic winding** about γ_z,

$$\operatorname{wind}_z^\Phi(u) := \operatorname{wind}^\Phi(e_\mu),$$

[6]The asymptotic formula (3.6) is a stronger version of a somewhat more complicated formula originally proved in [HWZ96a, HWZ96b]. The stronger version is proved in [Mor03], and another exposition is given in [Sie08].

which is necessarily either bounded from above by $\alpha_-^\Phi(\gamma_z)$ or from below by $\alpha_+^\Phi(\gamma_z)$, depending on the sign $z \in \Gamma^\pm$. We say that this winding is **extremal** whenever the bound is not strict.

Denote by $\mathcal{M}(J)$ the moduli space of unparametrized finite energy punctured J-holomorphic curves in $\mathbb{R} \times M$: this consists of equivalence classes of tuples (Σ, j, Γ, u), where $\dot{\Sigma} = \Sigma \setminus \Gamma$ is the domain of a pseudoholomorphic curve $u : (\dot{\Sigma}, j) \to (\mathbb{R} \times M, J)$, and we define $(\Sigma, j, \Gamma, u) \sim (\Sigma', j', \Gamma', u')$ if there is a biholomorphic map $\varphi : (\dot{\Sigma}, j) \to (\dot{\Sigma}', j')$ such that $u = u' \circ \varphi$. We assign to $\mathcal{M}(J)$ the natural topology defined by C^∞_{loc}-convergence on $\dot{\Sigma}$ and C^0-convergence up to the ends. It is often convenient to abuse notation by writing equivalence classes $[(\Sigma, j, \Gamma, u)] \in \mathcal{M}(J)$ simply as u when there is no danger of confusion.

If $u \in \mathcal{M}(J)$ has asymptotic orbits $\{\gamma_z\}_{z\in\Gamma}$ that are all Morse-Bott, then a neighborhood of u in $\mathcal{M}(J)$ can be described as the zero set of a Fredholm section of a Banach space bundle (see e.g. [Wen10a]). We say that u is **Fredholm regular** if this section has a surjective linearization at u, in which case a neighborhood of u in $\mathcal{M}(J)$ is a smooth finite dimensional orbifold. Its dimension is then equal to its **virtual dimension**, which is given by the **index** of u,

$$\operatorname{ind}(u) := -\chi(\dot{\Sigma}) + 2c_1^\Phi(u) + \sum_{z\in\Gamma^+} \mu_{\mathrm{CZ}}^\Phi(\gamma_z - \epsilon) - \sum_{z\in\Gamma^-} \mu_{\mathrm{CZ}}^\Phi(\gamma_z + \epsilon), \tag{3.7}$$

where $\epsilon > 0$ is any small positive number, Φ is an arbitrary choice of unitary trivialization of ξ along all the asymptotic orbits γ_z, and we abbreviate

$$c_1^\Phi(u) := c_1^\Phi(u^*T(\mathbb{R} \times M)),$$

where the latter denotes the relative first Chern number with respect to Φ of the complex vector bundle $u^*T(\mathbb{R} \times M) \to \dot{\Sigma}$. Since $T(\mathbb{R} \times M)$ splits into the direct sum of ξ with a trivial complex line bundle, this Chern number is the same as $c_1^\Phi(u^*\xi)$, which can be computed by counting the zeroes of a generic section of $u^*\xi$ that is nonzero and constant at infinity with respect to Φ.

We say that an almost complex structure $J \in \mathcal{J}(\mathcal{H})$ is **Fredholm regular** if all somewhere injective curves in $\mathcal{M}(J)$ are Fredholm regular. As shown in [Dra04] or the appendix of [Bou06], the set of Fredholm regular almost complex structures is of second category in $\mathcal{J}(\mathcal{H})$; one therefore often refers to them as *generic* almost complex structures.

It is sometimes convenient to have an alternative formula for $\operatorname{ind}(u)$ in the case where u is immersed. Indeed, the linearization of the Fredholm operator that describes $\mathcal{M}(J)$ near u acts on the space of sections of $u^*T(\mathbb{R}\times M)$, which then splits naturally as $T\dot{\Sigma} \oplus N_u$, where $N_u \to \dot{\Sigma}$ is the normal bundle, defined so that it matches ξ at the asymptotic ends of u. As explained e.g. in [Wen10a], the restriction of the linearization to N_u defines a linear Cauchy-Riemann type operator

$$\mathbf{D}_u^N : \Gamma(N_u) \to \Gamma(\overline{\mathrm{Hom}}_{\mathbb{C}}(T\dot{\Sigma}, N_u)),$$

called the **normal Cauchy-Riemann operator** at u, and the Fredholm index of this operator is precisely $\operatorname{ind}(u)$. Thus whenever u is immersed, we can compute $\operatorname{ind}(u)$ directly from the punctured version of the Riemann-Roch formula proved in [Sch95]:

$$\operatorname{ind}(\mathbf{D}_u^N) = \chi(\dot{\Sigma}) + 2c_1^\Phi(N_u) + \sum_{z\in\Gamma^+} \mu_{\mathrm{CZ}}^\Phi(\gamma_z - \epsilon) - \sum_{z\in\Gamma^-} \mu_{\mathrm{CZ}}^\Phi(\gamma_z + \epsilon). \tag{3.8}$$

Finally, let us briefly summarize the intersection theory of punctured J-holomorphic curves introduced by R. Siefring [Sie11]. Given any asymptotically cylindrical smooth maps $u : \dot{\Sigma} \to$

$\mathbb{R} \times M$ and $v : \dot{\Sigma}' \to \mathbb{R} \times M$, there is a symmetric pairing

$$u * v \in \mathbb{Z}$$

with the following properties:

(1) $u * v$ depends only on the asymptotic orbits of u and v and the relative homology classes $[u]$ and $[v]$.
(2) If u and v represent curves in $\mathcal{M}(J)$ with non-identical images, then their algebraic count of intersections $u \bullet v$ satisfies $0 \le u \bullet v \le u * v$. In particular, $u * v = 0$ implies that u and v never intersect.

The first property amounts to homotopy invariance: it implies that $u_0 * v = u_1 * v$ whenever u_0 and u_1 are connected to each other by a continuous family of curves $u_\tau \in \mathcal{M}(J)$ with fixed asymptotic orbits. The second property gives a *sufficient* condition for two curves to have disjoint images, but this condition is not in general *necessary*: sometimes one may have $0 = u \bullet v < u * v$ if u and v have an asymptotic orbit in common, and one must then expect intersections to emerge from infinity under generic perturbations. The number $u * v$ can also be defined when u and v are holomorphic *buildings* in the sense of [BEH$^+$03], so that it satisfies a similar continuity property under convergence of curves to buildings. The computation of $u * v$ is then a sum of the intersection numbers between corresponding levels, plus some additional nonnegative terms that count "hidden" intersections at the breaking orbits.

Remark 3.2. The version of homotopy invariance described above assumes that u and v vary as asymptotically cylindrical maps with *fixed* asymptotic orbits, but if any of the orbits belong to Morse-Bott families, one can define an alternative version of $u * v$ that permits the orbits to move continuously. This more general theory is sketched in the last section of [Wen10a]. In general, the intersection number defined in this way is greater than or equal to $u * v$, because it counts additional nonnegative contributions for intersections that may emerge from infinity as the asymptotic orbits move. It's useful to observe however that in the situation we will consider, both versions agree: in particular, if u and v are disjoint curves with $u * v = 0$ and a common positive asymptotic orbit that is (for both curves) simply covered and belongs to a Morse-Bott torus that doesn't intersect the images of u and v, then no new intersections can appear under a perturbation that moves the orbit (independently for both curves). This follows from an easy computation of asymptotic winding numbers using the definitions given in [Wen10a].

Similarly, if $u \in \mathcal{M}(J)$ is somewhere injective, one can define the integer $\delta(u) \ge 0$, which algebraically counts the self-intersections of u after perturbing away its critical points, but in the punctured case this need not be homotopy invariant. One fixes this by introducing the **asymptotic contribution** $\delta_\infty(u) \in \mathbb{Z}$, which is also nonnegative and counts "hidden" self-intersections that may emerge from infinity under generic perturbations. We then have

$$0 \le \delta(u) \le \delta(u) + \delta_\infty(u),$$

and the punctured version of the adjunction formula takes the form

$$u * u = 2\left[\delta(u) + \delta_\infty(u)\right] + c_N(u) + \left[\bar{\sigma}(u) - \#\Gamma\right], \tag{3.9}$$

where $\bar{\sigma}(u)$ is an integer that depends only on the asymptotic orbits and satisfies $\bar{\sigma}(u) \geq \#\Gamma$, and $c_N(u)$ is the **constrained normal Chern number**, which can be defined as[7]

$$c_N(u) = c_1^\Phi(u) - \chi(\dot{\Sigma}) + \sum_{z\in\Gamma^+} \alpha_-^\Phi(\gamma_z + \epsilon) - \sum_{z\in\Gamma^-} \alpha_+^\Phi(\gamma_z - \epsilon). \tag{3.10}$$

Observe that $c_N(u)$ also depends only on the asymptotic orbits $\{\gamma_z\}_{z\in\Gamma}$ and the relative homology class $[u]$.

3.2. **An existence and uniqueness theorem.** We now prove a theorem on holomorphic open books which lies in the background of all the results that were stated in §1. The setup is as follows. Assume (M', ξ) is a closed 3-manifold with a positive, co-oriented contact structure, and it contains a compact 3-dimensional submanifold $M \subset M'$, possibly with boundary, on which ξ is supported by a partially planar blown up summed open book

$$\check{\boldsymbol{\pi}} = (\check{\pi}, \check{\varphi}, \check{N}).$$

We will denote its binding and interface by B and $\mathcal{I}$ respectively, and denote the induced fibration by

$$\pi : M \setminus (B \cup \mathcal{I}) \to S^1.$$

Denote the irreducible subdomains by M_i for $i = 0, \dots, N$, so

$$M = M_0 \cup M_1 \cup \dots \cup M_N$$

for some $N \geq 0$. If B_i and $\mathcal{I}_i$ denote the intersections of B and $\mathcal{I}$ respectively with the interior of M_i, then the restriction of π to the interior of $M_i \setminus (B_i \cup \mathcal{I}_i)$ extends smoothly to its boundary as a fibration

$$\pi_i : M_i \setminus (B_i \cup \mathcal{I}_i) \to S^1.$$

Denote by $g_i \geq 0$ the genus of the fibers of π_i, and assume without loss of generality that M_0 is a planar piece, thus $g_0 = 0$ and $M_0 \cap \partial M = \emptyset$; in particular $\partial M_0 \subset \mathcal{I}$.

Definition 3.3. Given the above setup, an integer $m \in \mathbb{N}$ and an almost complex structure J compatible with some contact form on (M', ξ), we shall say that a finite energy J-holomorphic curve $u : \dot{\Sigma} \to \mathbb{R} \times M'$ is **subordinate to π_0 up to multiplicity** m if the following conditions hold:

- u is not a cover of a trivial cylinder,
- All positive ends of u approach Reeb orbits in $B_0 \cup \mathcal{I}_0 \cup \partial M_0$,
- Each positive asymptotic orbit of u in B_0 has covering multiplicity at most m.

Moreover, u is **strongly subordinate to π_0** if the following also holds:

- At its positive ends, u approaches each connected component of $B_0 \cup \partial M_0$ with total multiplicity at most 1, and each connected component of $\mathcal{I}_0$ with total multiplicity at most 2.

See Definition 3.1 for an explanation of the term *total multiplicity.* Note that the above condition allows the total multiplicity at any given component of $B_0 \cup \mathcal{I}_0 \cup \partial M_0$ to be 0, which would mean that the curve has no asymptotic orbits in that component.

[7] The version of $c_N(u)$ defined in (3.10) is adapted to the condition that homotopies in $\mathcal{M}(J)$ are required to fix asymptotic orbits. A more general definition is given in [Wen10a] (see also Remark 3.2).

Theorem 7. *For any numbers $\tau_0 > 0$ and $m_0 \in \mathbb{N}$, the contact manifold (M', ξ) with subdomain $M \subset M'$ carrying the blown up summed open book $\check{\boldsymbol{\pi}}$ described above admits a Morse-Bott contact form λ and compatible Fredholm regular almost complex structure J with the following properties.*

(1) *The contact structure* $\ker \lambda$ *is isotopic to ξ.*
(2) *On M, λ is a Giroux form for $\check{\boldsymbol{\pi}}$.*
(3) *The components of $\mathcal{I} \cup \partial M$ are all Morse-Bott submanifolds, while the Reeb orbits in B are nondegenerate and elliptic, and their covers for all multiplicities up to m_0 have Conley-Zehnder index* 1 *with respect to the natural trivialization determined by the pages.*
(4) *All Reeb orbits in $B_0 \cup \mathcal{I}_0 \cup \partial M_0$ have minimal period at most τ_0, while every other closed orbit of X_λ in M' has minimal period at least* 1*.*
(5) *For each component M_i with $g_i = 0$, the fibration $\pi_i : M_i \setminus (B_i \cup \mathcal{I}_i) \to S^1$ admits a C^∞-small perturbation $\hat{\pi}_i : M_i \setminus (B_i \cup \mathcal{I}_i) \to S^1$ such that the interior of each fiber $\hat{\pi}_i^{-1}(\tau)$ for $\tau \in S^1$ lifts uniquely to an $\mathbb{R}$-invariant family of properly embedded surfaces*
$$S^{(i)}_{\sigma,\tau} \subset \mathbb{R} \times M_i, \qquad (\sigma, \tau) \in \mathbb{R} \times S^1,$$
which are the images of embedded finite energy J-holomorphic curves
$$u^{(i)}_{\sigma,\tau} = (a^{(i)}_\tau + \sigma, F^{(i)}_\tau) : \dot{\Sigma}_i \to \mathbb{R} \times M_i,$$
all of them Fredholm regular with index 2*, and with only positive ends.*
(6) *A finite energy J-holomorphic curve u in $\mathbb{R} \times M'$ parametrizes one of the planar surfaces $S^{(i)}_{\sigma,\tau}$ described above whenever either of the following holds:*
   - *u is strongly subordinate to π_0,*
   - *u is somewhere injective, subordinate to π_0 up to multiplicity m_0 and intersects the interior of M_0.*

In addition to the applications treated in §4, Theorem 7 implies a wide range of existence results for finite energy foliations, e.g. it could be used to reduce the construction in [Wen08] to a few lines, after observing that every overtwisted contact structure is supported by a variety of summed open books with only planar pages. The proof of the theorem will occupy the remainder of §3.2.

3.2.1. *A family of stable Hamiltonian structures.* The first step in the proof is to construct a specific almost complex structure on $\mathbb{R} \times M$ for which all pages of $\check{\boldsymbol{\pi}}$ admit holomorphic lifts. We will follow the approach in [Wen10c] and refer to the latter for details in a few places where no new arguments are required. The idea is to present each subdomain M_i as an abstract open book that supports a stable Hamiltonian structure which is contact near $B \cup \mathcal{I} \cup \partial M$ and integrable elsewhere.

We must choose suitable coordinate systems near each component of the binding, interface and boundary. Choose $r > 0$ and let $\mathbb{D}_r \subset \mathbb{R}^2$ denote the closed disk of radius r. For each binding circle $\gamma \subset B$, choose a small tubular neighborhood $\mathcal{N}(\gamma)$ and identify it with the solid torus $S^1 \times \mathbb{D}_r$ with coordinates (θ, ρ, ϕ), where (ρ, ϕ) denote polar coordinates on $\mathbb{D}_r$. If r is sufficiently small then we can arrange these coordinates so that the following conditions are satisfied:

- $\gamma = S^1 \times \{0\}$, with the natural orientation of S^1 matching the co-orientation of ξ along γ

- $\pi(\theta, \rho, \phi) = \phi$ on $\mathcal{N}(\gamma) \setminus \gamma$
- $\xi = \ker(d\theta + \rho^2\, d\phi)$

Similarly, for each connected component $T \subset \partial M$, let $\widehat{\mathcal{N}(T)} \subset M'$ denote a neighborhood that is split into two connected components by T, and denote $\mathcal{N}(T) = \widehat{\mathcal{N}(T)} \cap M$. Identify $\widehat{\mathcal{N}(T)}$ with $S^1 \times [-r, r] \times S^1$ with coordinates (θ, ρ, ϕ) such that:

- $\mathcal{N}(T) = S^1 \times [0, r] \times S^1$
- For each $\phi_0 \in S^1$ the oriented loop $S^1 \times \{0, \phi_0\}$ in T is positively transverse to ξ
- $\pi(\theta, \rho, \phi) = \phi$ on $\mathcal{N}(T)$
- $\xi = \ker(d\theta + \rho\, d\phi)$

Finally, we choose two coordinate systems for neighborhoods $\mathcal{N}(T)$ of each interface torus $T \subset \mathcal{I}$, assuming that T divides $\mathcal{N}(T)$ into two connected components

$$\mathcal{N}(T) \setminus T = \mathcal{N}_+(T) \cup \mathcal{N}_-(T).$$

Choose an identification of $\mathcal{N}(T)$ with $S^1 \times [-r, r] \times S^1$ and denote the resulting coordinates by $(\theta_+, \rho_+, \phi_+)$, which we arrange to have the following properties:

- $T = S^1 \times \{0\} \times S^1$, $\mathcal{N}_+(T) = S^1 \times (0, r] \times S^1$ and $\mathcal{N}_-(T) = S^1 \times [-r, 0) \times S^1$
- For each $\phi_0 \in S^1$ the oriented loop $S^1 \times \{0, \phi_0\}$ in T is positively transverse to ξ
- $\pi(\theta_+, \rho_+, \phi_+) = \phi_+$ on $\mathcal{N}_+(T)$ and $\pi(\theta_+, \rho_+, \phi_+) = -\phi_+ + c$ on $\mathcal{N}_-(T)$ for some constant $c \in S^1$
- $\xi = \ker(d\theta_+ + \rho_+\, d\phi_+)$

Given these coordinates, it is natural to define a second coordinate system $(\theta_-, \rho_-, \phi_-)$ by

$$(\theta_-, \rho_-, \phi_-) = (\theta_+, -\rho_+, -\phi_+ + c). \tag{3.11}$$

Then the coordinates $(\theta_-, \rho_- \phi_-)$ satisfy minor variations on the properties listed above: in particular $\xi = \ker(d\theta_- + \rho_-\, d\phi_-)$ and $\pi(\theta_-, \rho_-, \phi_-) = \phi_-$ on $\mathcal{N}_-(T)$. In the following, we will use separate coordinates on the two components of $\mathcal{N}(T) \setminus T$, denoting both by (θ, ρ, ϕ):

$$(\theta, \rho, \phi) := \begin{cases} (\theta_+, \rho_+, \phi_+) & \text{on } \mathcal{N}_+(T), \\ (\theta_-, \rho_-, \phi_-) & \text{on } \mathcal{N}_-(T). \end{cases}$$

Then $\pi(\theta, \rho, \phi) = \phi$ and $\xi = \ker(d\theta + \rho\, d\phi)$ everywhere on $\mathcal{N}(T) \setminus T$. Observe that these coordinates on $\mathcal{N}_+(T)$ or $\mathcal{N}_-(T)$ separately can be extended smoothly to the closures $\overline{\mathcal{N}_+(T)}$ and $\overline{\mathcal{N}_-(T)}$, though in particular the two ϕ-coordinates are different where they overlap at T.

Notation. For any open and closed subset $N \subset B \cup \mathcal{I} \cup \partial M$, we shall in the following denote by $\mathcal{N}(N)$ the union of all the neighborhoods $\mathcal{N}(\gamma)$ and $\mathcal{N}(T)$ constructed above for the connected components $\gamma, T \subset N$. Thus for example,

$$\mathcal{N}(B \cup \mathcal{I} \cup \partial M)$$

denotes the union of all of them.

The complement $M \setminus \mathcal{N}(B \cup \mathcal{I} \cup \partial M)$ is diffeomorphic to a mapping torus. Indeed, let P denote the closure of $\pi^{-1}(0) \cap (M \setminus \mathcal{N}(B \cup \mathcal{I} \cup \partial M))$, a compact surface whose boundary components are in one to one correspondence with the connected components of $\mathcal{N}(B \cup \mathcal{I} \cup \partial P) \setminus \mathcal{I}$. The monodromy map of the fibration π defines a diffeomorphism $\psi : P \to P$, which preserves connected components and without loss of generality has support away from ∂P, so we define the mapping torus

$$P_\psi = (\mathbb{R} \times P)/\sim,$$

where $(t+1,p) \sim (t,\psi(p))$. This comes with a natural fibration $\phi : P_\psi \to S^1$ which is trivial near the boundary, so for a sufficiently small collar neighborhood $\mathcal{U} \subset P$ of ∂P, a neighborhood of ∂P_ψ can be identified with $S^1 \times \mathcal{U}$. Choose positively oriented coordinates on each connected component of $\mathcal{U}$

$$(\theta,\rho) : \mathcal{U} \to [r-\delta, r+\delta) \times S^1$$

for some small $\delta > 0$. This defines coordinates (ϕ,θ,ρ) on a collar neighborhood of $\partial P_\psi = S^1 \times \partial P$, so identifying these for $\rho \in (r-\delta, r]$ with the (θ,ρ,ϕ) coordinates chosen above on the corresponding components of $\mathcal{N}(B \cup \mathcal{I} \cup \partial M) \setminus \mathcal{I}$ defines an attaching map, such that the union

$$P_\psi \cup \mathcal{N}(B \cup \mathcal{I} \cup \partial M)$$

is diffeomorphic to M, and the ϕ-coordinate, which is globally defined outside of $B \cup \mathcal{I}$, corresponds to the fibration $\pi : M \setminus (B \cup \mathcal{I}) \to S^1$.

Choose a number $\delta' > \delta$ with $r - \delta' > 0$, and for each of the coordinate neighborhoods in $\mathcal{N}(B \cup \mathcal{I} \cup \partial M) \setminus \mathcal{I}$, define a 1-form of the form

$$\lambda_0 = f(\rho)\, d\theta + g(\rho)\, d\phi,$$

with smooth functions $f, g : [0,r] \to \mathbb{R}$ chosen so that

(1) $\ker \lambda_0 = \xi$ on a smaller neighborhood of $B \cup \mathcal{I} \cup \partial M$.
(2) For $\mathcal{N}(\mathcal{I}) \setminus \mathcal{I}$, $f(\rho)$ and $g(\rho)$ extend smoothly to $[-r,r]$ as even and odd functions respectively.
(3) The path $[0,r] \to \mathbb{R}^2 : \rho \mapsto (f(\rho), g(\rho))$ moves through the first quadrant from the positive real axis to $(0,1)$ and is constant for $\rho \in [r-\delta, r]$.
(4) The function
$$D(\rho) := f(\rho)g'(\rho) - f'(\rho)g(\rho)$$
is positive and $f'(\rho)$ is negative for all $\rho \in (0, r-\delta)$.
(5) $g(\rho) = 1$ for all $\rho \in [r-\delta', r]$.

Some possible pictures of the path $\rho \mapsto (f(\rho), g(\rho)) \in \mathbb{R}^2$ (with extra conditions that will be useful in the proof of Lemma 3.7) are shown in Figure 8. Note that the functions f and g must generally be chosen individually for each connected component of $\mathcal{N}(B \cup \mathcal{I} \cup \partial M)$. Extend λ_0 over $M' \setminus M$ so that $\ker \lambda_0 = \xi$ on this region, and extend it over P_ψ as $\lambda_0 = d\phi$. The kernel $\xi_0 := \ker \lambda_0$ is then a confoliation on M': it is contact outside of M and near $B \cup \mathcal{I} \cup \partial M$, while integrable and tangent to the fibers on P_ψ. In particular λ_0 is contact in the region $\{\rho < r - \delta\}$ near $B \cup \mathcal{I} \cup \partial M$, and its Reeb vector field here is

$$X_0 = \frac{g'(\rho)}{D(\rho)} \partial_\theta - \frac{f'(\rho)}{D(\rho)} \partial_\phi, \tag{3.12}$$

which is positively transverse to the pages $\{\phi = \text{const}\}$ and reduces to ∂_ϕ for $\rho \in [r-\delta', r]$, which contains the region where P_ψ and $\mathcal{N}(B \cup \mathcal{I} \cup \partial M)$ overlap.

Proceeding as in [Wen10c], choose next a 1-form α on P_ψ such that $d\alpha$ is positive on the fibers and, in the chosen coordinates (ϕ,θ,ρ) near ∂P_ψ, α takes the form

$$\alpha = (1-\rho)\, d\theta,$$

where we assume $r > 0$ is small enough so that $1 - \rho > 0$ when $r \in [r-\delta, r+\delta)$. Then if $\epsilon > 0$ is sufficiently small, the 1-form

$$\lambda_\epsilon := d\phi + \epsilon\alpha$$

is contact on P_ψ. We extend it to the rest of M' by setting $\lambda_\epsilon = \lambda_0$ on $M' \setminus M$, and on $\mathcal{N}(B \cup \mathcal{I} \cup \partial M)$,

$$\lambda_\epsilon = f_\epsilon(\rho)\ d\theta + g_\epsilon(\rho)\ d\phi,$$

where the functions $f_\epsilon, g_\epsilon : [0, r] \to \mathbb{R}$ satisfy

(1) $(f_\epsilon(\rho)), g_\epsilon(\rho)) = (f(\rho), g(\rho))$ for $\rho \le r - \delta'$,
(2) $g_\epsilon(\rho) = 1$ and $f'_\epsilon(\rho) < 0$ for $\rho \in [r - \delta', r - \delta]$,
(3) $(f_\epsilon(\rho), g_\epsilon(\rho)) = (\epsilon(1 - \rho), 1)$ for $\rho \in [r - \delta, r]$,
(4) $f_\epsilon \to f$ and $g_\epsilon \to g$ in C^∞ as $\epsilon \to 0$.

Now λ_ϵ is a contact form everywhere on M', and $\lambda_\epsilon \to \lambda_0$ in C^∞ as $\epsilon \to 0$. Denote the corresponding contact structure by

$$\xi_\epsilon = \ker \lambda_\epsilon.$$

The Reeb vector field X_ϵ of λ_ϵ is defined by the obvious analogue of (3.12) near $B \cup \mathcal{I} \cup \partial M$, is independent of ϵ on $M' \setminus M$, and on P_ψ is determined uniquely by the conditions

$$d\alpha(X_\epsilon, \cdot) \equiv 0, \qquad d\phi(X_\epsilon) + \epsilon\alpha(X_\epsilon) \equiv 1.$$

It follows that as $\epsilon \to 0$, X_ϵ converges to a smooth vector field X_0 that matches (3.12) near $B \cup \mathcal{I} \cup \partial M$ and on P_ψ is determined by

$$d\alpha(X_0, \cdot) \equiv 0 \quad \text{and} \quad d\phi(X_0) \equiv 1. \tag{3.13}$$

Observing that X_ϵ is always positively transverse to the pages $\{\phi = \text{const}\}$, and applying Proposition 2.9, we have:

Lemma 3.4. *For $\epsilon > 0$ sufficiently small, ξ_ϵ is a contact structure on M' isotopic to ξ, and λ_ϵ is a Giroux form for $\check{\boldsymbol{\pi}}$.*

In order to turn λ_ϵ into a stable Hamiltonian structure, we define an exact taming form as follows. For each coordinate neighborhood in $\mathcal{N}(B \cup \mathcal{I} \cup \partial M) \setminus \mathcal{I}$, fix a smooth function $h : [r - \delta', r - \delta] \to \mathbb{R}$ such that $h' < 0$, $h(\rho) = f(\rho) + c$ for ρ near $r - \delta'$ and some constant $c \ge 0$, and $h(\rho) = 1 - \rho$ for ρ near $r - \delta$. For each interface torus $T \subset \mathcal{I}$ the function $f(\rho)$ is the same on $\mathcal{N}_+(T)$ as on $\mathcal{N}_-(T)$, thus we may assume the same is true of $h(\rho)$ and c. Then

$$F(\rho) := \begin{cases} 1 - \rho & \text{for } \rho \in [r - \delta, r], \\ h(\rho) & \text{for } \rho \in [r - \delta', r - \delta], \\ f(\rho) + c & \text{for } \rho \in [0, r - \delta'] \end{cases}$$

defines a smooth function on $[0, r)$ which, for components of $\mathcal{N}(\mathcal{I})$, has a smooth even extension to $[-r, r]$. By choosing $f(\rho)$ appropriately on the components of $\mathcal{N}(\partial M)$, one can also arrange $c = 0$; it will be convenient (e.g. for Lemma 3.7 below) to assume this for $\mathcal{N}(\partial M)$ but leave the choice of $c \ge 0$ and thus $f(\rho)$ arbitrary everywhere else. Now there is a smooth 1-form $\hat{\alpha}$ on M' such that

$$\hat{\alpha} = \begin{cases} \alpha + d\phi & \text{on } P_\psi, \\ F(\rho)\ d\theta + g(\rho)\ d\phi & \text{on } \mathcal{N}(B \cup \mathcal{I} \cup \partial M), \\ \lambda_0 & \text{on } M' \setminus M, \end{cases}$$

and we use this to define an exact 2-form

$$\omega = d\hat{\alpha}.$$

We claim that (λ_0, ω) defines a stable Hamiltonian structure on M'. Indeed, outside M and in a sufficiently small neighborhood of $B \cup \mathcal{I} \cup \partial M$ this is clear since λ_0 is contact and

$\omega = d\lambda_0$. On the subsets described in coordinates by $r - \delta' \le \rho < r - \delta$, λ_0 is still contact and $\omega = -h'(\rho)\, d\theta \wedge d\rho = \frac{h'(\rho)}{f'(\rho)} d\lambda_0$, thus ω has maximal rank and its kernel is spanned by X_0. On P_ψ, $d\lambda_0 = 0$ and $\omega = d\alpha$ annihilates X_0 by (3.13), so the claim is proved. In fact, for $\epsilon > 0$ sufficiently small, we still have $\omega|_{\xi_\epsilon} > 0$ and the kernel of ω is still spanned by X_ϵ, thus we've proved:

Proposition 3.5. *For sufficiently small $\epsilon \ge 0$,*

$$\mathcal{H}_\epsilon := (\lambda_\epsilon, \omega)$$

defines a stable Hamiltonian structure on M'.

Definition 3.6. Any smooth family $\mathcal{H}_\epsilon = (\lambda_\epsilon, \omega)$ of stable Hamiltonian structures on M' defined for small $\epsilon \ge 0$ by the procedure above will be said to be **adapted** to $\check{\boldsymbol{\pi}}$.

Lemma 3.7. *There exists a number $\tau_1 > 0$ so that for any $\tau_0 > 0$ and $m_0 \in \mathbb{N}$, a family of stable Hamiltonian structures $\mathcal{H}_\epsilon = (\lambda_\epsilon, \omega)$ on M' adapted to $\check{\boldsymbol{\pi}}$ can be constructed so as to satisfy the following additional conditions on the Reeb vector fields X_ϵ:*

(1) *The interface and boundary tori are Morse-Bott submanifolds, and all closed orbits in a neighborhood of $\mathcal{I} \cup \partial M$ are also Morse-Bott.*
(2) *Each connected component $\gamma \subset B$ and all its multiple covers are nondegenerate elliptic orbits, and their covers up to multiplicity m_0 all have Conley-Zehnder index* 1 *with respect to the natural trivialization of ξ along γ determined by the coordinates.*
(3) *All orbits in $B_0 \cup \mathcal{I}_0 \cup \partial M_0$ have minimal period at most τ_0, while all other orbits have period at least τ_1.*

Moreover for each $\epsilon > 0$ sufficiently small, the contact form λ_ϵ admits a C^∞-small perturbation to a globally Morse-Bott contact form whose Reeb vector field still satisfies the above conditions.

Proof. We first prove that the stated conditions can be established for X_0.

If $\gamma \subset B$ is a binding circle, then γ and all its multiple covers can be made nondegenerate and elliptic by choosing the functions f and g so that

$$f'(\rho)/g'(\rho) \in \mathbb{R} \setminus \mathbb{Q} \quad \text{for all } \rho > 0 \text{ sufficiently small.}$$

This implies that the slope of the curve $\rho \mapsto (f(\rho), g(\rho)) \in \mathbb{R}^2$ is constant for ρ near 0, and this slope determines the Conley-Zehnder index of γ; in particular, the stated condition is satisfied whenever $f''(0)/g''(0)$ is a negative number sufficiently close to 0. Assume this from now on.

Similarly, we make every orbit in a neighborhood of $\mathcal{I} \cup \partial M$ Morse-Bott by assuming that in such a neighborhood, $\lambda_0 = f(\rho)\, d\theta + g(\rho)\, d\phi$ where f and g satisfy

$$f'(\rho)g''(\rho) - f''(\rho)g'(\rho) > 0.$$

This means that the path $\rho \mapsto (f(\rho), g(\rho)) \in \mathbb{R}^2$ has nonzero inward angular acceleration as it winds (counterclockwise) about the origin; clearly for $\mathcal{N}(\mathcal{I})$ we can also still safely assume that f and g are restrictions of even and odd functions respectively on $[-r, r]$.

We now show that the periods of the orbits in $B_0 \cup \mathcal{I}_0 \cup \partial M_0$ can be made arbitrarily small compared to all other periods. Observe that by (3.12), the Reeb flow as we've constructed it preserves the concentric tori $\{\rho = \text{const}\}$ in the neighborhood $\mathcal{N}(B_0 \cup \mathcal{I}_0 \cup \partial M_0)$, thus it also preserves $M' \setminus \mathcal{N}(B_0 \cup \mathcal{I}_0 \cup \partial M_0)$. Since the latter has compact closure, there is a positive lower bound for the periods of all closed orbits in $M' \setminus \mathcal{N}(B_0 \cup \mathcal{I}_0 \cup \partial M_0)$, so it will suffice

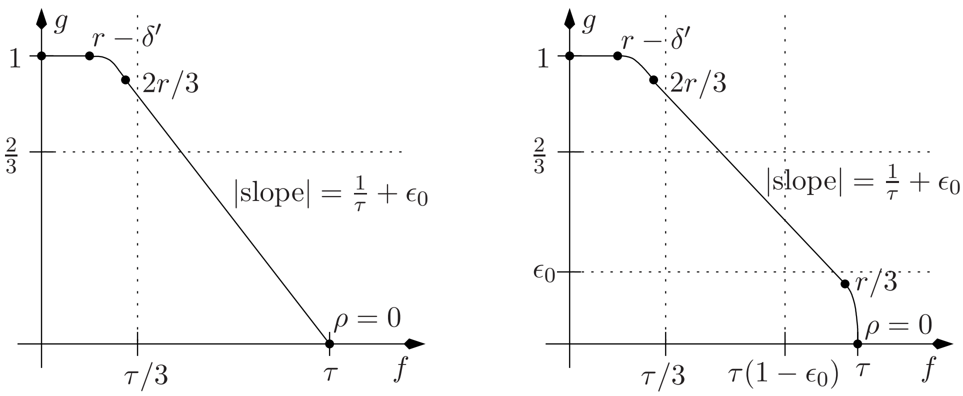

FIGURE 8. The path $\rho \mapsto (f(\rho), g(\rho)) \in \mathbb{R}^2$ with the extra conditions imposed in the proof of Lemma 3.7 for the nondegenerate case (left) and Morse-Bott case (right).

to leave λ_0 fixed in this region and reduce the periods in $B_0 \cup \mathcal{I}_0 \cup \partial M_0$ while preserving a lower bound for all other orbits in $\mathcal{N}(B_0 \cup \mathcal{I}_0 \cup \partial M_0)$.

Consider a binding orbit $\gamma \subset B_0$: writing λ_0 as $f(\rho)\, d\theta + g(\rho)\, d\phi$ near γ, the period of γ is $f(0) > 0$. Choosing sufficiently small constants $\tau > 0$ and $\epsilon_0 > 0$, we impose the following additional conditions on f and g (see Figure 8, left):

- $(f(0), g(0)) = (\tau, 0)$,
- For all $\rho \in (0, r]$,
$$\frac{g'(\rho)}{-f'(\rho)} \le \frac{1}{\tau} + \epsilon_0 \in \mathbb{R} \setminus \mathbb{Q},$$
with equality for $\rho \le 2r/3$.
- For $\rho \in [2r/3, r]$, $g(\rho) \ge 2/3$ and $f(\rho) \le \tau/3$.

Since $f'(\rho)/g'(\rho)$ is irrational for $\rho \le 2r/3$, all closed orbits in $\mathcal{N}(\gamma) \setminus \gamma$ are outside this region. For any $\rho_0 \in [2r/3, r]$, (3.12) implies that a Reeb orbit in $\{\rho = \rho_0\}$ has its ϕ-coordinate increasing at the constant rate of $-f'(\rho_0)/D(\rho_0)$. Its period is thus at least

$$\begin{aligned} \left|\frac{D(\rho_0)}{f'(\rho_0)}\right| &= \left|\frac{f(\rho_0)g'(\rho_0) - f'(\rho_0)g(\rho_0)}{f'(\rho_0)}\right| \ge |g(\rho_0)| - \left|f(\rho_0)\frac{g'(\rho_0)}{f'(\rho_0)}\right| \\ &\ge \frac{2}{3} - \left|\frac{\tau}{3}\left(\frac{1}{\tau} + \epsilon_0\right)\right| = \frac{2}{3} - \frac{1}{3}(1 + \tau\epsilon_0) > 0. \end{aligned} \tag{3.14}$$

We can therefore keep these periods bounded away from zero while shrinking $f(0) = \tau$ to make both the period at γ and the ratio $-f'(\rho)/g'(\rho)$ near γ arbitrarily small.

The above requires only a small modification for the neighborhood of a torus $T \subset \mathcal{I}_0 \cup \partial M_0$: here we need f and g to extend over $\rho \in [-r, r]$ as even and odd functions respectively, so it is no longer possible to fix the slope $f'(\rho)/g'(\rho)$ throughout $\rho \in [0, 2r/3]$. In fact $f'(0)$ must vanish, so we amend the above conditions by allowing them to hold for $\rho \in [r/3, r]$, but requiring the following for $\rho \in [0, r/3]$,

- $-g'(\rho)/f'(\rho) \ge 1/\tau + \epsilon_0$,
- $f(\rho) \ge \tau(1 - \epsilon_0)$,

- $g(\rho) \le \epsilon_0$.

This modification is shown at the right of Figure 8. Now for $\rho \le r/3$, the lower bound calculated in (3.14) becomes

$$\begin{aligned}\left|\frac{D(\rho_0)}{f'(\rho_0)}\right| &\ge \left|f(\rho_0)\frac{g'(\rho_0)}{f'(\rho_0)}\right| - |g(\rho_0)| \ge \tau(1-\epsilon_0)\left(\frac{1}{\tau}+\epsilon_0\right) - \epsilon_0 \\ &= 1 + \epsilon_0\left(\tau - 2 - \tau\epsilon_0^2\right) > 0.\end{aligned}$$

Thus we can freely shrink $f(0) = \tau$, the minimal period of the Morse-Bott family at T, while bounding all other periods away from zero.

Since X_ϵ is a small perturbation of X_0 outside a neighborhood of $B \cup \mathcal{I} \cup \partial M$, the same results immediately hold for X_ϵ: in particular, for any sequence $\epsilon_k \to 0$, $M' \setminus \mathcal{N}(B_0 \cup \mathcal{I}_0 \cup \partial M_0)$ cannot contain a sequence of orbits of X_{ϵ_k} with periods below a certain threshold, as a subsequence of these would converge (by Arzelà-Ascoli) to an orbit of X_0. Similarly, this constraint on the periods will be satisfied by any sufficiently small perturbation of X_ϵ. We can now choose such a perturbation to a globally Morse-Bott contact form as follows: let $\mathcal{U} \subset M'$ denote a union of coordinate neighborhoods of the form $\{|\rho| < r_0\}$ near each component of $B \cup \mathcal{I} \cup \partial M$, where $r_0 > 0$ is chosen such that all periodic orbits inside $\mathcal{U}$ are Morse-Bott and none exist near $\partial\overline{\mathcal{U}}$ (because f'/g' is irrational). After a generic perturbation of λ_ϵ in $M' \setminus \overline{\mathcal{U}}$, every Reeb orbit not fully contained in $\overline{\mathcal{U}}$ becomes nondegenerate (cf. the appendix of [ABW10]), which means all orbits outside $\overline{\mathcal{U}}$ are nondegenerate, while all the others (which are inside $\mathcal{U}$) are Morse-Bott by construction. □

Remark 3.8. To satisfy the conditions stated in Theorem 7, we need a version of Lemma 3.7 with $\tau_1 = 1$. This can always be achieved by rescaling λ_ϵ by a constant, and thus replacing $\mathcal{H}_\epsilon = (\lambda_\epsilon, \omega)$ by $(c\lambda_\epsilon, \omega)$ for some $c > 0$.

3.2.2. *A symplectic cobordism.* As a quick detour away from the proof of Theorem 7, we now explain a construction that will be useful for proving Theorem 4. Namely, we will need to know that the stable Hamiltonian structures $\mathcal{H}_0$ and $\mathcal{H}_\epsilon$ for some $\epsilon > 0$ can be related to each other by a cylindrical symplectic cobordism that looks standard near the binding.

To simplify the statement of the following result, let us restrict to the special case where $M = M'$ and $\pi : M \setminus B \to S^1$ is an ordinary (not summed or blown up) open book; this will suffice for the application we have in mind.

Proposition 3.9. *There exists a family of stable Hamiltonian structures $\mathcal{H}_\epsilon = (\lambda_\epsilon, \omega)$ on M adapted to the open book $\pi : M \setminus B \to S^1$ such that $[0,1] \times M$ admits a symplectic form Ω with the following properties:*

- *$\Omega = \omega + d(t\lambda_0)$ near $\{0\} \times M$.*
- *$\Omega = d(e^t\lambda)$ near $\{1\} \times M$ for some contact form λ with $\ker\lambda = \xi_\epsilon$ and some $\epsilon > 0$.*
- *$\Omega = d(\varphi(t)\lambda_0)$ on $[0,1] \times \mathcal{U}$ for some neighborhood $\mathcal{U} \subset M$ of B on which $\lambda_\epsilon = \lambda_0$, and some smooth function $\varphi : [0,1] \to (0,\infty)$ with $\varphi' > 0$.*

Remark 3.10. We are not claiming that $\mathcal{H}_\epsilon$ in this result can be chosen to make the periods of binding orbits small as in Lemma 3.7 and Theorem 7. For our application we will not need this.

Proof of Prop. 3.9. In (θ, ρ, ϕ)-coordinates on $\mathcal{N}(B)$, we can write $\lambda_0 = f(\rho)\, d\theta + g(\rho)\, d\phi$ with f and g chosen such that $f(\rho) = 1 - \rho$ for ρ near $r - \delta'$. Then setting

$$F(\rho) = \begin{cases} 1 - \rho & \text{for } \rho \in [r - \delta', r], \\ f(\rho) & \text{for } \rho \in [0, r - \delta'] \end{cases}$$

and defining $\hat{\alpha}$ and ω as before, we have $\omega \equiv d\hat{\alpha}$ where $\hat{\alpha} = \lambda_0$ on a neighborhood $\mathcal{U} := \{\rho < r - \delta'\}$ of B.

With this stipulation in place, construct the family λ_ϵ as before. Next choose small numbers $\epsilon, \epsilon_1 > 0$ and a smooth function $\beta : [0, \infty) \to [0, \epsilon]$ such that

- $\beta(t) = 0$ for t near 0,
- $\beta(t) = \epsilon$ for $t \geq \epsilon_1$.

Define a 1-form $\hat{\lambda}$ on $[0, \infty) \times M$ by

$$\hat{\lambda}|_{(t,p)} = \lambda_{\beta(t)}|_p$$

for all $(t, p) \in [0, \infty) \times M$, and then define

$$\Omega = \omega + d(t\hat{\lambda})$$

on $[0, \infty) \times M$. Note that $\omega + d(t\lambda_0)$ is symplectic on $[0, \epsilon_1] \times M$ if $\epsilon_1 > 0$ is sufficiently small, and Ω is C^∞-close to this if $\epsilon > 0$ is also small, implying that Ω is also symplectic on $[0, \epsilon_1] \times M$. It is also obviously symplectic on $[\epsilon_1, \infty) \times M$ since it then equals

$$\omega + d(t\lambda_\epsilon)$$

for some $\epsilon > 0$, where λ_ϵ is contact and ω is $d\lambda_\epsilon$ multiplied by a smooth positive function. This construction thus gives a symplectic form on $[0, \infty) \times M$ which has the desired form already near $\{0\} \times M$ and on $[0, \infty) \times \mathcal{U}$. To define a suitable top boundary for the cobordism, observe that $\Omega = d(\hat{\alpha} + t\hat{\lambda})$, thus the Ω-dual vector field to $\hat{\alpha} + t\hat{\lambda}$ is a Liouville vector field Y:

$$\iota_Y \Omega := \hat{\alpha} + t\hat{\lambda}.$$

We claim that on the hypersurface $\{T\} \times M$ for $T > 0$ sufficiently large, $dt(Y) > 0$. Indeed, this is equivalent to the statement that $\hat{\alpha} + t\hat{\lambda}$ defines a positive contact form on $\{T\} \times M$, which is true if T is large enough since its kernel is then a small perturbation of $\ker \lambda_\epsilon$. Thus fixing T sufficiently large, $\{T\} \times M$ is a convex boundary component of $[0, T] \times M$. Moreover since the primitive of Ω is just $(1 + t)\lambda_0$ in $[\epsilon_1, \infty) \times \mathcal{U}$, the vector field Y takes the simple form $(1 + t)\partial_t$ in this region. Using the flow of Y near $\{T\} \times M$, we can now identify a neighborhood of this hypersurface in $[0, T] \times M$ symplectically with a domain of the form

$$((1 - \epsilon_1, 1] \times M, d(e^t\lambda)),$$

where λ is a constant multiple of the contact 1-form $\hat{\alpha} + T\lambda_\epsilon$, which defines a contact structure isotopic to ξ_ϵ due to Gray's theorem. There is thus a diffeomorphism of $[0, T] \times M$ to $[0, 1] \times M$ that transforms Ω into the desired form. □

3.2.3. *Non-generic holomorphic curves and perturbation.* Returning to the proof of Theorem 7, assume $\mathcal{H}_\epsilon = (\lambda_\epsilon, \omega)$ is a family of stable Hamiltonian structures adapted to the blown up summed open book $\check{\boldsymbol{\pi}}$ on $M \subset M'$ and satisfying Lemma 3.7. Choose any compatible almost complex structure $J_0 \in \mathcal{J}(\mathcal{H}_0)$ which has the following properties in the coordinate neighborhoods $\mathcal{N}(B \cup \mathcal{I} \cup \partial M)$:

- J_0 is invariant under the T^2-action defined by translating the coordinates (θ, ϕ).

- $d\rho(J_0\partial_\rho) \equiv 0$.

Observe that $\partial_\rho \in \xi_0$ always, so the second condition says that J_0 maps ∂_ρ into the characteristic foliation defined by ξ_0 on the torus $\{\rho = \text{const}\}$. Note also that since ξ_0 is tangent to the fibers of P_ψ, these fibers naturally embed into $\mathbb{R} \times M'$ as J_0-holomorphic curves. The construction in [Wen10c, §3] now carries over directly to the present setting and gives the following result.

Proposition 3.11. *For each $i = 0, \dots, N$, the interior of $\mathbb{R} \times (M_i \setminus (B_i \cup \mathcal{I}_i))$ is foliated by an $\mathbb{R}$-invariant family of properly embedded surfaces*

$$\{S^{(i)}_{\sigma,\tau}\}_{(\sigma,\tau)\in\mathbb{R}\times S^1}$$

with J_0-invariant tangent spaces, where

$$S^{(i)}_{\sigma,\tau} \cap (\mathbb{R} \times P_\psi) = \{\sigma\} \times \left(\pi_i^{-1}(\tau) \cap P_\psi\right),$$

and its intersection with each connected component of $\mathbb{R}\times\mathcal{N}(B\cup\mathcal{I}\cup\partial M)$ can be parametrized in (θ, ρ, ϕ)-coordinates by a map of the form

$$[0,\infty) \times S^1 \to \mathbb{R} \times S^1 \times (0, r] \times S^1 : (s,t) \mapsto (a_i(s) + \sigma, t, \rho_i(s), \tau).$$

Here $a_i : [0,\infty) \to [0,\infty)$ is a fixed map with $a_i(0) = 0$ and $\lim_{s\to\infty} a_i(s) = +\infty$, and $\rho_i : [0,\infty) \to (0, r]$ is a fixed orientation reversing diffeomorphism.

Denote by $\mathcal{F}^{(i)}_0$ for $i = 0, \dots, N$ the resulting foliation on the interior of $\mathbb{R}\times(M_i\setminus(B_i\cup\mathcal{I}_i))$, whose leaves can each be parametrized by an embedded finite energy J_0-holomorphic curve

$$u^{(i)}_{\sigma,\tau} : \dot{\Sigma}_i \to \mathbb{R} \times M'.$$

The collection of all these curves together with the trivial cylinders over their asymptotic orbits (which include all orbits in $B\cup\mathcal{I}\cup\partial M$) defines a J_0-holomorphic finite energy foliation $\mathcal{F}_0$ of M, as defined in [HWZ03, Wen08]. It's important however to be aware that this foliation is not generally *stable*, due to the following index calculation. From now on we assume that $\mathcal{H}_\epsilon$ has the properties specified in Lemma 3.7.

Proposition 3.12. $\operatorname{ind}\left(u^{(i)}_{\sigma,\tau}\right) = 2 - 2g_i$.

Proof. Let Φ denote the natural trivialization of ξ_0 determined by the (θ, ρ, ϕ)-coordinates along each of the asymptotic orbits of $u^{(i)}_{\sigma,\tau}$. These orbits are in general a mix of nondegenerate binding circles $\gamma \subset B_i$ with $\mu^\Phi_{\mathrm{CZ}}(\gamma) = 1$ and Morse-Bott orbits that belong to S^1-families foliating $\mathcal{I} \cup \partial M$. If γ is one of the latter, then we observe that since $u^{(i)}_{\sigma,\tau}$ doesn't intersect $\mathbb{R}\times(\mathcal{I}\cup\partial M)$, the asymptotic winding of $u^{(i)}_{\sigma,\tau}$ as it approaches γ matches the winding of any nontrivial section in $\ker \mathbf{A}_\gamma$, which is zero in the chosen coordinates. Thus for sufficiently small $\epsilon > 0$, the two largest negative eigenvalues of $\mathbf{A}_\gamma - \epsilon$ both have zero winding, implying $\alpha^\Phi_-(\gamma - \epsilon) = 0$ and $p(\gamma - \epsilon) = 1$, hence by (3.2),

$$\mu^\Phi_{\mathrm{CZ}}(\gamma - \epsilon) = 2\alpha^\Phi_-(\gamma - \epsilon) + p(\gamma - \epsilon) = 1. \tag{3.15}$$

Since $u^{(i)}_{\sigma,\tau}$ projects to an embedding in M', it is everywhere transverse to the complex subspace in $T(\mathbb{R} \times M')$ spanned by ∂_t and X_0, though asymptotically $u^{(i)}_{\sigma,\tau}$ becomes tangent to this space. We can thus define a sensible normal bundle $N \to \dot{\Sigma}_i$ for $u^{(i)}_{\sigma,\tau}$ as follows: let X denote the smooth vector field on $M'\setminus(B\cup\mathcal{I}\cup\partial M)$ that equals ∂_ϕ in every (θ, ρ, ϕ)-coordinate neighborhood (except at $\{\rho = 0\}$, where this is not well defined), and X_0 everywhere outside of this. Then the J_0-complex span of this vector field defines a bundle that extends smoothly over $B \cup \mathcal{I} \cup \partial M$, and we define the normal bundle $N \to \dot{\Sigma}_i$ to be the restriction of this bundle to the image of $u^{(i)}_{\sigma,\tau}$. From this construction it is clear that $c_1^\Phi(N) = 0$. Now since $u^{(i)}_{\sigma,\tau}$ is embedded, its index is the index of the normal Cauchy-Riemann operator on the bundle $N \to \dot{\Sigma}_i$, so by (3.8),

$$\operatorname{ind}\left(u^{(i)}_{\sigma,\tau}\right) = \chi(\dot{\Sigma}_i) + 2c_1^\Phi(N) + \sum_\gamma \mu^\Phi_{\mathrm{CZ}}(\gamma - \epsilon) = \chi(\Sigma_i) = 2 - 2g_i,$$

where the summation is over all the asymptotic orbits of $u^{(i)}_{\sigma,\tau}$, whose Conley-Zehnder indices thus cancel out the terms in $\chi(\dot{\Sigma}_i)$ resulting from the punctures. □

From this calculation it follows that the higher genus curves in $\mathcal{F}_0$ will vanish under generic perturbations of the data. In contrast, the genus zero curves have exactly the right properties to apply the following useful perturbation result (cf. [Wen05, Theorem 4.5.44]):

Implicit Function Theorem. *Assume M is any closed 3-manifold with stable Hamiltonian structure $\mathcal{H} = (\lambda, \omega)$, $J \in \mathcal{J}(\mathcal{H})$, and*

$$u = \left(u^{\mathbb{R}}, u^M\right) : \dot{\Sigma} \setminus \Gamma \to \mathbb{R} \times M$$

is a finite energy J-holomorphic curve with positive/negative punctures $\Gamma^\pm \subset \Sigma$ and the following properties:

1. *u is embedded and asymptotic to simply covered periodic orbits at each puncture, and satisfies $\delta_\infty(u) = 0$.*
2. *$\dot{\Sigma}$ has genus zero.*
3. *All asymptotic orbits γ_z of u for $z \in \Gamma^\pm$ are either nondegenerate or belong to S^1-parametrized Morse-Bott families foliating tori, and*
$$p(\gamma_z \mp \epsilon) = 1$$
for all $z \in \Gamma^\pm$ and sufficiently small $\epsilon > 0$.
4. $\operatorname{ind}(u) = 2$.

Then u is Fredholm regular and belongs to a smooth 2-parameter family of embedded curves

$$u_{(\sigma,\tau)} = \left(u^{\mathbb{R}}_\tau + \sigma, u^M_\tau\right) : \dot{\Sigma} \to \mathbb{R} \times M, \qquad (\sigma, \tau) \in \mathbb{R} \times (-1, 1)$$

with $u_{(0,0)} = u$, whose images foliate an open neighborhood of $u(\dot{\Sigma})$ in $\mathbb{R} \times M$. Moreover, the maps $u^M_\tau : \dot{\Sigma} \to M$ are all embedded and foliate an open neighborhood of $u^M(\dot{\Sigma})$ in M, and if γ^τ_z denotes a degenerate Morse-Bott asymptotic orbit of $u_{(\sigma,\tau)}$ for some fixed puncture $z \in \Gamma$, then the map $\tau \mapsto \gamma^\tau_z$ parametrizes a neighborhood of γ^0_z in its S^1-family of orbits.

Using this and a simple topological argument in [Wen10c], it follows that whenever $g_i = 0$, the family $u^{(i)}_{\sigma,\tau}$ perturbs smoothly along with any sufficiently small perturbation of J_0. In particular, picking $\epsilon > 0$ small and $J_\epsilon \in \mathcal{J}(\mathcal{H}_\epsilon)$ close to J_0, there is a corresponding family of J_ϵ-holomorphic curves in $\mathbb{R} \times M_i$ that project to a blown up summed open book on M_i that is C^∞-close to the original one. Perturbing λ_ϵ a little bit further outside a suitable neighborhood of $B \cup \mathcal{I} \cup \partial M$, we can then also turn λ_ϵ into a globally Morse-Bott contact form, and a corresponding perturbation of J_ϵ makes the latter Fredholm regular. This proves the existence part of Theorem 7. We will continue to denote the J_ϵ-holomorphic pages constructed in this way by

$$u^{(i)}_{\sigma,\tau} : \dot{\Sigma}_i \to \mathbb{R} \times M_i,$$

for all $i = 0, \dots, N$ with $g_i = 0$.

3.2.4. *Uniqueness.* Despite their obvious instability, the higher genus curves in the foliation $\mathcal{F}_0$ are useful due to the following uniqueness result based on intersection theory. Here $m_0 \in \mathbb{N}$ denotes the multiplicity bound from Lemma 3.7, which we can assume to be arbitrarily large.

Proposition 3.13. *Suppose $v : \dot{\Sigma} \to \mathbb{R} \times M'$ is a somewhere injective finite energy J_0-holomorphic curve that intersects the interior of $\mathbb{R}\times M_i$ and has all its positive ends asymptotic to orbits in $B \cup \mathcal{I} \cup \partial M$, where the orbits in B_i each have covering multiplicity at most m_0. Then v parametrizes one of the surfaces $S^{(i)}_{\sigma,\tau}$.*

Proof. We use the homotopy invariant intersection number $u*v \in \mathbb{Z}$ defined by Siefring [Sie11] for asymptotically cylindrical maps u and v. If v does not parametrize any leaf of $\mathcal{F}^{(i)}_0$, then its intersection with $\mathbb{R} \times M_i$ implies that it has at least one isolated positive intersection with some leaf $S^{(i)}_{\sigma,\tau}$ with J_0-holomorphic parametrization $u^{(i)}_{\sigma,\tau}$, hence

$$u^{(i)}_{\sigma,\tau} * v > 0.$$

By changing τ slightly, we may assume without loss of generality that any ends of $u^{(i)}_{\sigma,\tau}$ approaching Morse-Bott orbits in $\mathcal{I} \cup \partial M$ are disjoint from the positive asymptotic orbits of v. By homotopy invariance, we can also take advantage of the lack of negative ends for $u^{(i)}_{\sigma,\tau}$ and $\mathbb{R}$-translate it until its image lies entirely in $[0,\infty) \times M'$. We can likewise change v by a homotopy through asymptotically cylindrical maps so that its intersection with $[0,\infty)\times M'$ lies entirely in the trivial cylinders over its positive asymptotic orbits, i.e. in $[0,\infty)\times(B\cup\mathcal{I}\cup\partial M)$. An example of this kind of homotopy is shown in Figure 9. The intersection number above is then a sum of the form

$$u^{(i)}_{\sigma,\tau} * v = \sum_{\gamma} u^{(i)}_{\sigma,\tau} * (\mathbb{R} \times \gamma),$$

where the summation is over some collection of orbits γ in $B \cup \mathcal{I} \cup \partial M$, and we use $\mathbb{R} \times \gamma$ as shorthand for a J_0-holomorphic curve that parametrizes the trivial cylinder over γ. Note that $u^{(i)}_{\sigma,\tau}$ never has an actual intersection with $\mathbb{R}\times\gamma$, so the intersections counted by $u^{(i)}_{\sigma,\tau} * (\mathbb{R}\times\gamma)$ are *asymptotic*, i.e. they are hidden intersections that could potentially emerge from infinity under small perturbations of the data. Since we've arranged for $u^{(i)}_{\sigma,\tau}$ and v to have no Morse-Bott orbits in common, the asymptotic intersections vanish except possibly for orbits $\gamma \subset B_i$ of covering multiplicity $m \le m_0$. As explained in [Sie11, §3.2], each such asymptotic intersection can be expressed in terms of the difference in the asymptotic winding of the m-fold cover of the end of $u^{(i)}_{\sigma,\tau}$ about γ from its maximum possible value, which (by standard results in [HWZ96a, HWZ95a]) is the winding number of the asymptotic eigenfunction with largest negative eigenvalue. In the natural trivialization Φ determined by the (θ, ρ, ϕ)-coordinates, each of the relevant orbits γ has $\mu^\Phi_{\mathrm{CZ}}(\gamma) = 1 = 2\alpha^\Phi_-(\gamma) + 1$, hence $\alpha^\Phi_-(\gamma) = 0$ using (3.2). By construction, the asymptotic winding of $u^{(i)}_{\sigma,\tau}$ as it approaches γ is also zero, hence this winding is extremal, and this implies

$$u^{(i)}_{\sigma,\tau} * (\mathbb{R} \times \gamma) = 0.$$

This is a contradiction. □

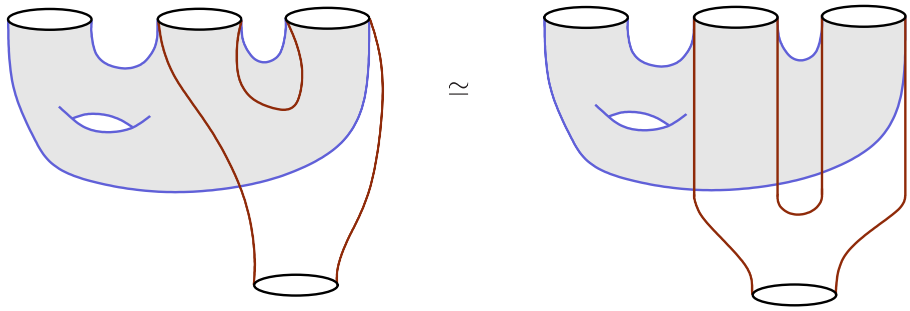

FIGURE 9. A homotopy of two asymptotically cylindrical maps, reducing the computation of the intersection number to the intersection of one holomorphic curve with the asymptotic trivial cylinders of the other.

The above proof also works for a J_ϵ-holomorphic curve if it passes through a region that is foliated by J_ϵ-holomorphic pages. In particular, since we've already shown this to be true in the planar piece M_0 for sufficiently small $\epsilon > 0$, we deduce the following parallel result:

Proposition 3.14. *For all sufficiently small $\epsilon > 0$, the following holds: if $v : \dot{\Sigma} \to \mathbb{R} \times M'$ is a somewhere injective finite energy J_ϵ-holomorphic curve that intersects the interior of $\mathbb{R} \times M_0$ and has all its positive ends asymptotic to orbits in $B \cup \mathcal{I} \cup \partial M$, where the orbits in B_0 have covering multiplicity at most m_0, then v is a reparametrization of one of the J_ϵ-holomorphic pages $u^{(0)}_{\sigma,\tau}$.*

We now prove the remainder of the uniqueness statement in Theorem 7. Choose a sequence $\epsilon_k > 0$ converging to zero, denote $\lambda_k := \lambda_{\epsilon_k}$ and $\xi_k := \ker \lambda_k$, and choose generic almost complex structures $J_k \in \mathcal{J}(\mathcal{H}_{\epsilon_k})$ with $J_k \to J_0$ in C^∞. By small perturbations we can assume the forms λ_k are all Morse-Bott and have the properties listed in Lemma 3.7: in particular the minimal periods of the orbits in $B_0 \cup \mathcal{I}_0 \cup \partial M_0$ are bounded by an arbitrarily small number $\tau > 0$, while all others are at least 1, and the orbits in B_0 have Conley-Zehnder index 1. We can also assume that for sufficiently large k, planar J_k-holomorphic pages $u^{(i)}_{\sigma,\tau}$ in $\mathbb{R} \times M_i$ exist whenever $g_i = 0$, and hence Prop. 3.14 holds. Now arguing by contradiction, suppose that for every k, there exists a finite energy J_k-holomorphic curve

$$v_k : (\dot{\Sigma}_k, j_k) \to (\mathbb{R} \times M', J_k)$$

which is strongly subordinate to π_0 and is (for large k) not equivalent to any of the planar curves $u^{(i)}_{\sigma,\tau}$. If v_k has any positive end asymptotic to an orbit in B_0 or $\mathcal{I}_0$, then it must intersect the interior of $\mathbb{R} \times M_0$ and Proposition 3.14 already gives a contradiction. We can therefore assume that the positive ends of v_k approach simply covered orbits in distinct connected components of ∂M_0. This implies that they are all somewhere injective.

Lemma 3.15. *A subsequence of v_k converges to one of the J_0-holomorphic leaves of the foliation $\mathcal{F}_0$.*

Proof. We proceed in three steps.

Step 1: Energy bounds. We use the stable Hamiltonian structure $\mathcal{H}_{\epsilon_k} = (\lambda_k, \omega)$ to define the energy of v_k. To be precise, choose $c_0 > 0$ small enough so that $\omega + d(t\lambda_0)$ is symplectic on $[-c_0, c_0] \times M'$; the same is then true for all $\omega + d(t\lambda_k)$ with k sufficiently large, so following

(3.3) and (3.4), define

$$E_k(v_k) = \int_{\dot{\Sigma}_k} v_k^*\omega + \sup_{\varphi \in \mathcal{T}} \int_{\dot{\Sigma}_k} v_k^* d(\varphi\lambda_k),$$

where $\mathcal{T} = \{\varphi \in C^\infty(\mathbb{R}, (-c_0, c_0)) \mid \varphi' > 0\}$. Since ω is exact, $E_k(v_k)$ depends only on the asymptotic behavior of v_k. Now since the positive ends all approach simple orbits in distinct connected components of ∂M_0, the number of ends and sum of their periods are uniformly bounded, implying a uniform bound on $E_k(v_k)$.

Step 2: Genus bounds. After taking a subsequence we may assume that all the curves v_k have the same number of positive and negative punctures. It is still possible however that the surfaces $\dot{\Sigma}_k$ could have unbounded topology, i.e. their genus could blow up as $k \to \infty$. To preclude this, we apply the currents version of Gromov compactness, see [Tau98, Prop. 3.3] or [Hut02, Lemma 9.9]. The key fact is that since $E_k(v_k)$ is uniformly bounded, $\mathcal{H}_k \to \mathcal{H}_0$ and $J_k \to J_0$, v_k as a sequence of currents has a convergent subsequence, and this implies in particular that the relative homology classes $[v_k]$ for this subsequence converge. We now plug this into the adjunction formula (3.9) for punctured holomorphic curves, which implies

$$v_k * v_k \geq 2\,[\delta(v_k) + \delta_\infty(v_k)] + c_N(v_k) \geq c_N(v_k).$$

Both the right and left hand sides of this expression depend only on $[v_k]$ and on certain integer valued winding numbers of eigenfunctions at the asymptotic orbits of v_k. As orbits vary in a Morse-Bott family that all have the same minimal period, these winding numbers remain constant, thus by the convergence of $[v_k]$, the sequence $v_k * v_k$ converges to a fixed integer, implying an upper bound on $c_N(v_k)$ for large k. The latter can be written as $c_1^\Phi(v_k) - \chi(\dot{\Sigma}_k)$ plus more winding numbers of eigenfunctions, thus every term other than $\chi(\dot{\Sigma}_k)$ converges, and we obtain a uniform upper bound on $-\chi(\dot{\Sigma}_k)$, or equivalently, an upper bound on the genus of $\dot{\Sigma}_k$.

Step 3: SFT compactness. We can now assume the domains $\dot{\Sigma}_k$ are a fixed surface $\dot{\Sigma}$, so the sequence v_k with uniform energy bound $E_k(v_k) < C$ satisfies the compactness theorem of Symplectic Field Theory [BEH+03]. There is one subtle point to be careful of here: since X_0 is not a Morse-Bott vector field, it is not clear at first whether the SFT compactness theory can be applied as $\mathcal{H}_{\epsilon_k} \to \mathcal{H}_0$. What saves us is the fact that v_k is asymptotic at $+\infty$ to orbits with arbitrarily small period: then for energy reasons, we may assume the only orbits that can appear under breaking or bubbling are other orbits in $B_0 \cup \mathcal{I}_0 \cup \partial M_0$, all of which are Morse-Bott. With this observation, the proof of SFT compactness in [BEH+03] goes through unchanged. We can thus assume that v_k converges to a J_0-holomorphic building v_∞. The positive asymptotic orbits of v_∞ are all simply covered and lie in distinct connected components of ∂M_0, thus the top level of v_∞ contains at least one somewhere injective curve v_+ that is strongly subordinate to π_0. Then Prop. 3.13 implies that v_+ parametrizes a leaf of the foliation $\mathcal{F}_0$, so it has no negative ends. The same is true for every other top level component of v_∞ unless it is a trivial cylinder, and nontrivial curves must all be distinct since they approach distinct orbits at their positive ends. It follows that they do not intersect each other, so there is no possibility of nodes connecting them, and the building must be disconnected unless it consists of only a single component, namely v_+. □

We are now just about done with the proof of Theorem 7: the implicit function theorem implies that if the limit $v_\infty = \lim v_k$ has genus zero, then v_k is always one of the J_k-holomorphic pages $u_{\sigma,\tau}^{(i)}$ for sufficiently large k. If on the other hand v_∞ has genus $g > 0$, then $\mathrm{ind}(v_k) = \mathrm{ind}(v_\infty) = 2 - 2g \leq 0$ by Prop. 3.12, yet v_k must be Fredholm regular since J_k was chosen generically, and this gives a contradiction.

3.3. **Deformation and compactness.** We now prove a compactness result for families of holomorphic curves in symplectic manifolds that emerge from the holomorphic pages provided by Theorem 7.

We recall first that every strong symplectic filling can be *completed* by attaching a cylindrical end. To be precise, assume (M', ξ) is a closed, connected contact 3-manifold with positive contact form α, and for any two smooth functions $f, g : M' \to [-\infty, \infty]$ with $g > f$, define a subdomain of the symplectization $(\mathbb{R} \times M', d(e^t\alpha))$ by

$$\mathcal{S}_f^g = \{(t, m) \in \mathbb{R} \times M' \mid f(m) \leq t \leq g(m) \}. \tag{3.16}$$

Here we include the cases $f \equiv -\infty$ and $g \equiv +\infty$ so that $\mathcal{S}_f^g$ may be unbounded. Now suppose $M' = \partial W$, where (W, ω) is a (not necessarily compact) symplectic manifold with contact type boundary, and λ is a primitive of ω defined near ∂W such that $\lambda|_{TM'} = e^f\alpha$ for some smooth function $f : M' \to \mathbb{R}$. Then using the flow of the Liouville vector field Y defined by $\iota_Y\omega = \lambda$, one can identify a neighborhood of M' in (W, ω) symplectically with a neighborhood of $\partial\mathcal{S}_{-\infty}^f$ in $(\mathcal{S}_{-\infty}^f, d(e^t\alpha))$. As a consequence, one can symplectically glue the cylindrical end $(\mathcal{S}_f^\infty, d(e^t\alpha))$ to (W, ω) along M', giving a noncompact symplectic manifold

$$(W^\infty, \omega) := (W, \omega) \cup_{M'} (\mathcal{S}_f^\infty, d(e^t\alpha)),$$

which necessarily contains the half-symplectization $([T, \infty) \times M', d(e^t\alpha))$ whenever $T \in \mathbb{R}$ is sufficiently large.

Adopting the notation from the setup for Theorem 7, assume now that in addition to the above, (M', ξ) contains a partially planar domain $M \subset M'$ with irreducible subdomains $M = M_0 \cup \ldots \cup M_N$ for $N \geq 0$, of which M_0 is a planar piece lying in the interior of M. By Theorem 7, we can then find a Morse-Bott contact form α on M' and generic compatible almost complex structure J_+ such that the planar pages in M_0 lift to an $\mathbb{R}$-invariant foliation by properly embedded J_+-holomorphic curves in $\mathbb{R} \times M'$, whose asymptotic orbits are simply covered and have minimal period less than an arbitrarily small number $\tau_0 > 0$, while all other closed orbits of X_α in M' have period at least 1. Assume that α is the contact form chosen for defining the symplectic cylindrical end in (W^∞, ω).

Choose an almost complex structure J on W^∞ which is compatible with ω, generic on $W \subset W^\infty$ and matches J_+ on $\mathcal{S}_f^\infty \subset W^\infty$. Then every leaf of the J_+-holomorphic foliation in $\mathbb{R} \times M_0$ has an $\mathbb{R}$-translation that can be regarded as a properly embedded surface in $\mathcal{S}_f^\infty \subset W^\infty$ parametrized by a finite energy J-holomorphic curve. The main idea used for the proofs in §4.1 is to show that these curves generate a moduli space of J-holomorphic curves that must fill the entirety of W^∞, and leads to a contradiction in any of the situations considered by Theorems 1, 4 and 5. To prove this, we need a deformation result and a corresponding compactness result to show that the region filled by these curves is open and closed respectively. We shall prove somewhat more general versions of these results than are immediately needed, as they are also useful for other applications (e.g. in [NW11, LVW]).

We now generalize the above setup as follows: let $u_+ : \dot{\Sigma} \to W^\infty$ denote one of the J-holomorphic planar pages living in the cylindrical end of (W^∞, ω), and pick any open neighborhood $\mathcal{U} \subset M'$ and $T > 0$ such that

$$u_+(\dot{\Sigma}) \subset [T, \infty) \times \mathcal{U}.$$

Choose any data (α', ω', J') with the following properties:

- α' is a Morse-Bott contact form on M' that matches α on $\mathcal{U} \cup \mathcal{N}(B_0 \cup \mathcal{I}_0 \cup \partial M_0)$ and has only Reeb orbits of period at least 1 outside of $\mathcal{N}(B_0 \cup \mathcal{I}_0 \cup \partial M_0)$
- ω' is a sympectic form on W^∞ that matches $d(e^t\alpha')$ on $\mathcal{S}_f^\infty$
- J' is an ω'-compatible almost complex structure on W^∞ that has an $\mathbb{R}$-invariant restriction
$$J'_+ := J'|_{\mathcal{S}_f^\infty}$$
that is generic and compatible with α' and matches J_+ on $\mathbb{R} \times (\mathcal{U} \cup \mathcal{N}(B_0 \cup \mathcal{I}_0 \cup \partial M_0))$, and J' is generic on W.

The advantage of this generalization is that fairly arbitrary changes to the data can be accommodated outside a neighborhood of a single page, which is useful for instance in the adaptation of these arguments for weak fillings (cf. [NW11]). Let $\mathcal{M}^*(J')$ denote the moduli space of all unparametrized somewhere injective finite energy J'-holomorphic curves in W^∞, which is non-empty by construction since it contains u_+, and define
$$\mathcal{M}_0^*(J') \subset \mathcal{M}^*(J')$$
to be the connected component of this space containing u_+. The curves $u \in \mathcal{M}_0^*(J')$ share all homotopy invariant properties of the planar J_+-holomorphic pages in $\mathbb{R} \times M'$, in particular:

(1) $\mathrm{ind}(u) = 2$,
(2) $u * u = \delta(u) + \delta_\infty(u) = 0$.

It follows that all curves in $\mathcal{M}_0^*(J')$ are embedded. This situation is a slight variation on the setup that was considered in [ABW10, §4], only with the added complication that curves in $\mathcal{M}_0^*(J')$ may have two ends approaching the same Morse-Bott Reeb orbit, which presents the danger of degeneration to holomorphic buildings with multiply covered components. The required deformation result is however exactly the same: it depends on the fact that a neighborhood of each embedded curve $u \in \mathcal{M}_0^*(J')$ can be described by sections of its normal bundle which are nowhere vanishing, because they satisfy a Cauchy-Riemann type equation and have vanishing first Chern number with respect to certain special trivializations at the ends.

Proposition 3.16 ([ABW10, Theorem 4.7])**.** *The moduli space $\mathcal{M}_0^*(J')$ is a smooth 2-dimensional manifold containing only proper embeddings that never intersect each other: in particular they foliate an open subset of W^∞.*

The compactness result we need is a variation on [ABW10, Theorem 4.8], but somewhat more complicated due to the appearance of multiple covers. For the statement of the result, recall that the compactification in [BEH+03] for the space of finite energy holomorphic curves in an almost complex manifold with cylindrical ends consists of so-called *stable holomorphic buildings*, which have one *main level* and potentially multiple *upper* and *lower levels*, each of which is a (perhaps disconnected) nodal holomorphic curve. We will be considering sequences of curves in W^∞ that stay within a bounded distance of the positive end, so there will be no lower levels in the limit. We shall use the term "smooth holomorphic curve" to mean a holomorphic building with only one level and no nodes. The following variation on Definition 3.3 will be convenient.

Definition 3.17. A J'-holomorphic curve $u : \dot{\Sigma} \to W^\infty$ will be called **subordinate to** π_0 if it has only positive ends, all of which approach Reeb orbits in $B_0 \cup \mathcal{I}_0 \cup \partial M_0$, with total multiplicity at most 1 for each connected component of $B_0 \cup \partial M_0$ and at most 2 for each connected component of $\mathcal{I}_0$.

Observe that all the curves in $\mathcal{M}_0^*(J')$ are subordinate to π_0. The intersection argument in the proof of Prop. 3.13 now implies:

Lemma 3.18. *If $u \in \mathcal{M}^*(J)$ is subordinate to π_0, then $u * u_+ = 0$.*

Theorem 8. *Choose an open subset $W_0 \subset W$ that contains ∂W and has compact closure, and let $W_0^\infty = W_0 \cup_{M'} \mathcal{S}_f^\infty$. Then there is a finite set of index 0 curves $\Theta(W_0) \subset \mathcal{M}^*(J')$ subordinate to π_0 and with images in $\overline{W}_0^\infty$ such that the following holds. Any sequence of curves $u_k \in \mathcal{M}_0^*(J')$ with images in $\overline{W}_0^\infty$ has a subsequence convergent (in the sense of* [BEH+03]*) to one of the following:*

1. *A curve in $\mathcal{M}_0^*(J')$*
2. *A holomorphic building with empty main level and one nontrivial upper level consisting of a single connected curve that can be identified (up to $\mathbb{R}$-translation) with a curve in $\mathcal{M}_0^*(J')$ with image in $\mathcal{S}_f^\infty$*
3. *A J'-holomorphic building whose upper levels contain only covers of trivial cylinders, and main level consists of a connected double cover of a curve in $\Theta(W_0)$*
4. *A J'-holomorphic building whose upper levels contain only covers of trivial cylinders, and main level contains at most two connected components, which are curves in $\Theta(W_0)$.*

Proof. Assume u_k is a sequence of either index 2 curves in $\mathcal{M}_0^*(J')$ or index 0 curves subordinate to π_0 with images in $\overline{W}_0^\infty$ and only simply covered asymptotic orbits. By [BEH+03], u_k has a subsequence converging to a stable J'-holomorphic building u_∞. The main idea is to add up the indices of all the connected components of u_∞ and use genericity to derive restrictions on the configuration of u_∞. To facilitate this, we introduce a variation on the usual Fredholm index formula (3.7): for any finite energy holomorphic curve $v : \dot{\Sigma} \to \mathbb{R} \times M'$ with positive and negative asymptotic orbits $\{\gamma_z\}_{z \in \Gamma^\pm}$, choose a small number $\epsilon > 0$ and trivializations Φ of the contact bundle along each γ_z and define the *constrained* index

$$\widehat{\mathrm{ind}}(v) = -\chi(\dot{\Sigma}) + 2c_1^\Phi(v) + \sum_{z \in \Gamma^+} \mu_{\mathrm{CZ}}^\Phi(\gamma_z - \epsilon) - \sum_{z \in \Gamma^-} \mu_{\mathrm{CZ}}^\Phi(\gamma_z - \epsilon).$$

The only difference here from (3.7) is that at the negative punctures we take $\mu_{\mathrm{CZ}}^\Phi(\gamma_z - \epsilon)$ instead of $\mu_{\mathrm{CZ}}^\Phi(\gamma_z + \epsilon)$, which geometrically means we compute the virtual dimension of a space of curves whose negative ends have all their Morse-Bott orbits fixed in place. So for curves without negative ends $\widehat{\mathrm{ind}}(v) = \mathrm{ind}(v)$, and the constrained index otherwise has the advantage of being *additive* across levels, i.e. if the building u_∞ has no nodes, then we obtain $\mathrm{ind}(u_k) = \mathrm{ind}(u_\infty)$ if the latter is defined as the sum of the constrained indices for all its connected components. Observe that trivial cylinders over Reeb orbits always have constrained index 0. If u_∞ does have nodes, the formula remains true after adding 2 for each node in the building, so we then take this as a definition of the index for a nodal curve or nodal holomorphic building. We now proceed in several steps.

Step 1: Curves in upper levels. We claim that every connected component of u_∞ either has no negative ends or is a cover of a trivial cylinder (in an upper level). Indeed, curves in the main level obviously have no negative ends, and if v is an upper level component with negative ends, the smallness of the periods in $B_0 \cup \mathcal{I}_0 \cup \partial M_0$ constrains these to approach other orbits in $B_0 \cup \mathcal{I}_0 \cup \partial M_0$, as otherwise v would have negative energy. Then if v does not cover a trivial cylinder, an intersection argument carried out in [ABW10, Proof of Theorem 4.8] implies that v must intersect u_+, contradicting Lemma 3.18 above. The key idea here is to consider the asymptotic winding numbers that control holomorphic curves approaching orbits at $B_0 \cup \mathcal{I}_0 \cup \partial M_0$, which differ for positive and negative ends at each of these orbits, and thus force v to intersect u_+ in the projection to M'. We refer to [ABW10] for the details; note that a similar argument has also appeared in [Mom08].

Step 2: Indices of connectors. Borrowing some terminology from Embedded Contact Homology, we refer to branched multiple covers of trivial cylinders as **connectors**. These can appear in the upper levels of u_∞, but can never have any curves above them except for further covers of trivial cylinders, due to Step 1. Since the positive ends of u_∞ approach any given orbit in $B_0 \cup \mathcal{I}_0 \cup \partial M_0$ with total multiplicity at most 2, only the following types of connectors can appear, both with genus zero:

- *Pair-of-pants* connectors: these have one positive end at a doubly covered orbit and two negative ends at the same simply covered orbit.
- *Inverted pair-of-pants* connectors: with two positive ends at the same simply covered orbit and one negative end at its double cover.

The second variety will be especially important, and we'll refer to it for short as an **inverted connector**. As we computed in (3.15), all of the simply covered Morse-Bott orbits under consideration have $\mu_{\mathrm{CZ}}^{\Phi}(\gamma - \epsilon) = 1$ in the natural trivialization, and in fact exactly the same argument produces the same result for their multiple covers. We thus find that the constrained Fredholm index is 0 for a pair-of-pants connector and 2 for the inverted variant.

Step 3: Indices of multiple covers. Suppose v is a connected component of u_∞ which is not a cover of a trivial cylinder: then it has no negative ends, and all its positive ends must approach orbits in $B_0 \cup \mathcal{I}_0 \cup \partial M_0$ with total multiplicity at most 2. Thus if v is a k-fold cover of a somewhere injective curve v', we have $k \in \{1, 2\}$, and all the asymptotic orbits of both v and v' have $\mu_{\mathrm{CZ}}^{\Phi}(\gamma - \epsilon) = 1$ in the natural trivialization. Assume $k = 2$, and label the positive punctures of v as $\Gamma = \Gamma_1 \cup \Gamma_2$, where a puncture is defined to belong to Γ_2 if its asymptotic orbit is doubly covered, and Γ_1 otherwise. For $i = 1, 2$, let Γ_i' denote the punctures of v' that are covered by Γ_i, so the set of all punctures Γ' of v' is $\Gamma_1' \cup \Gamma_2'$. Note that in this situation all the asymptotic orbits of v must have total multiplicity exactly 2, which implies that all asymptotic orbits of v' are distinct and simply covered, and we have $\#\Gamma_2 = \#\Gamma_2'$ and $\#\Gamma_1 = 2\#\Gamma_1'$. Both domains must also have genus zero, so we have

$$\begin{aligned}\operatorname{ind}(v) &= -(2 - \#\Gamma) + 2c_1^{\Phi}(v) + \#\Gamma = -2 + 2(\#\Gamma_2' + 2\#\Gamma_1') + 2kc_1^{\Phi}(v'),\\ \operatorname{ind}(v') &= -(2 - \#\Gamma') + 2c_1^{\Phi}(v') + \#\Gamma' = -2 + 2(\#\Gamma_2' + \#\Gamma_1') + 2c_1^{\Phi}(v'),\end{aligned}$$

hence

$$\operatorname{ind}(v) = k \operatorname{ind}(v') + 2(k-1)(1 - \#\Gamma_2). \tag{3.17}$$

This formula also trivially holds if $k = 1$. This gives a lower bound on $\operatorname{ind}(v)$ since $\operatorname{ind}(v')$ is bounded from below by either 1 (in $\mathbb{R} \times M'$) or 0 (in W^∞) due to genericity. Now observe that whenever Γ_2 is non-empty, the doubly covered orbit must connect v to an inverted connector, whose constrained index is 2, so for $k = 2$ we have

$$\operatorname{ind}(v) + \sum_C \widehat{\operatorname{ind}}(C) = k \operatorname{ind}(v') + 2(k-1) \ge 2, \tag{3.18}$$

where the sum is over all inverted connectors that connect to v along doubly covered breaking orbits.

Step 4: Indices of bubbles. There may also be closed components in the main level of u_∞: these are J'-holomorphic spheres v which are either constant (*ghost bubbles*) or are k-fold covers of somewhere injective spheres v' for some $k \in \mathbb{N}$. In the latter case, (3.17) also holds with $\#\Gamma_2 = 0$, implying $\operatorname{ind}(v) \geq 0$, and the inequality is strict whenever $k > 1$.

If v is a ghost bubble, then $\operatorname{ind}(v) = -2$, but then the stability condition implies the existence of at least three nodes connecting v to other components; let us refer to nodes of this type as *ghost nodes.* There is then a graph with vertices representing the ghost bubbles in u_∞ and edges representing the ghost nodes that connect two ghost bubbles together, and since u_∞ has arithmetic genus zero, every connected component of this graph is a tree. Let G denote such a connected component, with V vertices and E_i edges, which therefore satisfy $V - E_i = 1$, and suppose there are also E_e nodes connecting the ghost bubbles represented by G to nonconstant components; we can think of these as represented by "external" edges in G. By the stability condition, we have

$$2E_i + E_e \geq 3V,$$

which after replacing E_i by $V - 1$, becomes $E_e - 2 \geq V$. Then the total contribution to $\operatorname{ind}(u_\infty)$ from all the ghost bubbles and ghost nodes represented by G is

$$\begin{aligned} -2V + 2(E_i + E_e) &= [-2V + (2E_i + E_e)] + E_e \geq V + (2 + V) \\ &= 2V + 2 \geq 4, \end{aligned} \tag{3.19}$$

unless u_∞ has no ghost bubbles at all.

Step 5: The total index of u_∞. We can now break down $\operatorname{ind}(u_\infty) \in \{0, 2\}$ into a sum of nonnegative terms and use this to rule out most possibilities. Ghost bubbles are excluded immediately due to (3.19). Similarly, there cannot be any multiply covered bubbles, because these imply the existence of at least one node and thus contribute at least 4 to $\operatorname{ind}(u_\infty)$. The only remaining possibility for multiple covers (aside from connectors) is a component with only positive ends, whose index together with contributions from attached inverted connectors is given by (3.18) and is thus already at least 2. In fact, if this component exists in an upper level, then the underlying simple curve must have index at least 1, implying an even larger lower bound in (3.18) and hence a contradiction. The remaining possibility, which occurs in the case $\operatorname{ind}(u_\infty) = 2$, is therefore that the main level consists only of a connected double cover, and there are no nodes at all, nor anything other than trivial cylinders and connectors in the upper levels (Figure 10). The underlying simple curve in the main level has index 0 and has only simply covered asymptotic orbits, all in separate connected components of $B_0 \cup \mathcal{I}_0 \cup \partial M_0$, thus it is subordinate to π_0.

Assume now that u_∞ contains no multiply covered components except possibly for connectors. If there is an upper level component v that is not a cover of a trivial cylinder, then genericty implies $\operatorname{ind}(v) \geq 1$, and in fact the index must also be even since all the asymptotic orbits satisfy $\mu_{\mathrm{CZ}}^\Phi(\gamma - \epsilon) = 1$. Then $\operatorname{ind}(u_\infty) = \operatorname{ind}(v) = 2$ and there are no nodes or inverted connectors; the latter implies that all positive asymptotic orbits of v must be simply covered. Then there also cannot be any doubly covered breaking orbits, leaving only the possibility that v is the only nontrivial component in u_∞.

Next assume there are only covers of trivial cylinders in the upper levels, in which case the main level is necessarily non-empty. Each component in the main level has a nonnegative even index, so there can be at most one node or one inverted connector in u_∞, and only

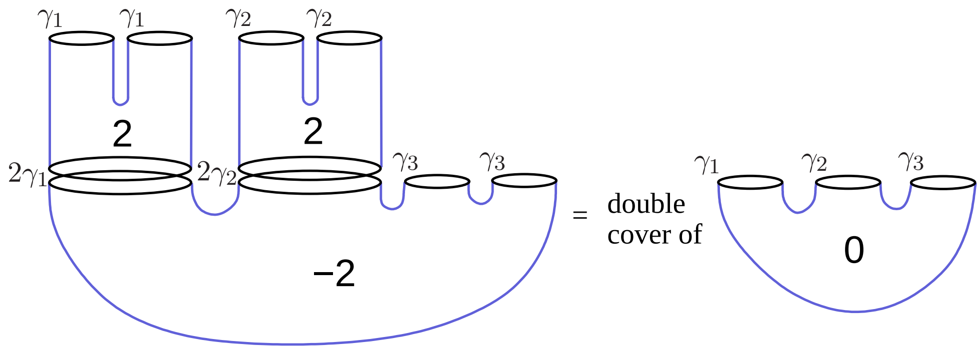

FIGURE 10. The limit building u_∞ in a case where all asymptotic orbits have total multiplicity two, so the main level may be a double cover of an index 0 curve, while the upper level includes connectors and trivial cylinders (the latter not shown in the picture). The numbers inside each component indicate the constrained index.

if $\operatorname{ind}(u_\infty) = 2$. If the main level contains a component v of index 2, then there are no nodes or inverted connectors. The latter precludes doubly covered breaking orbits, thus there are no connectors at all, and since v cannot have negative ends, we conclude that $u_\infty = v$ (Figure 11). Otherwise all main level components in u_∞ have index 0 and are subordinate to π_0. Examples of the possible configurations are shown in Figures 12–15.

Step 6: Compactness for index 0 *curves.* If $\operatorname{ind}(u_\infty) = 2$, then the somewhere injective index 0 curves that can appear in the building u_∞ are all subordinate to π_0 and come in two types:

- Type 1: Curves with only simply covered asymptotic orbits.
- Type 2: Curves with exactly one doubly covered asymptotic orbit and all others simply covered, and satisfying $v * v = 0$.

Indeed, the second type can occur as the unique main level curve in u_∞ if there is a single inverted connector in an upper level, attached along the doubly covered orbit (Figure 14). To see that $v * v = 0$ for such a curve, we use the continuity of the intersection number under convergence to buildings, and the fact that $u_k * u_k = 0$ since $u_k \in \mathcal{M}_0^*(J')$; a computation shows that the contribution to $u_\infty * u_\infty$ from trivial cylinders and connectors in the upper level plus breaking orbits adds up to 0. The index counting argument of the previous steps shows already that the curves of Type 1 form a compact and hence finite set. To finish the proof, we must show that the same is true for the Type 2 curves.

Suppose v_k is a sequence of Type 2 curves converging to a holomorphic building v_∞. Applying the index counting argument from the previous steps, v_∞ cannot contain any nodes or inverted connectors; the worst case scenario is that the upper levels contain only trivial cylinders and a single pair-of-pants connector, whose two negative ends connect to two main level components v_-^1 and v_-^2 that are both Type 1 curves (Figure 16). Since there are finitely many Type 1 curves, we may assume by genericity of J' that no two of them approach a common orbit in the Morse-Bott families $\mathcal{I}_0$, but this must be the case for v_-^1 and v_-^2 as they are both attached to a connector over an orbit in $\mathcal{I}_0$, so we conclude that both are the same curve, which we'll call v_-. We can rule out this scenario by computing the self-intersection

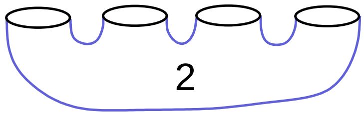

FIGURE 11. The limit building u_∞ in the simplest case, a smooth index 2 curve in the main level.

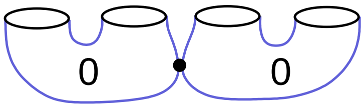

FIGURE 12. A nodal curve with two index 0 components.

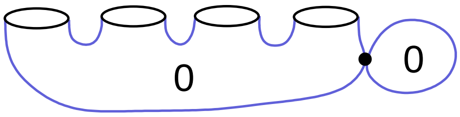

FIGURE 13. Another nodal curve, including a bubble.

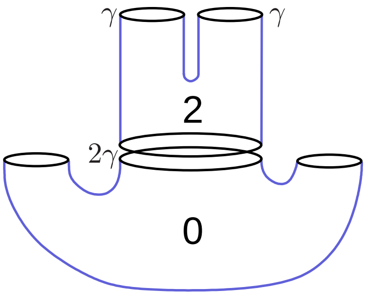

FIGURE 14. An inverted connector can appear in an upper level.

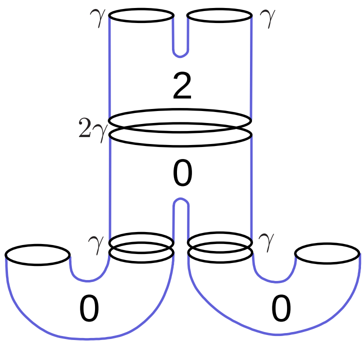

FIGURE 15. Both types of connectors can appear.

number $v_\infty * v_\infty$, which must a priori equal $v_k * v_k = 0$. Once more the connectors, trivial cylinders and breaking orbits contribute zero in total, so since the main level includes two copies of v_-, we deduce

$$0 = v_\infty * v_\infty = 4(v_- * v_-).$$

But we can also compute $v_- * v_-$ directly from the adjunction formula (3.9); indeed,

$$v_- * v_- = 2\left[\delta(v_-) + \delta_\infty(v_-)\right] + c_N(v_-),$$

where we've dropped the last term in (3.9) since all the asymptotic orbits are simple. The constrained normal Chern number $c_N(v_-)$ is defined in (3.10) and can be deduced from the fact that $\operatorname{ind}(v_-) = 0$: since all of the relevant orbits satisfy $\mu_{\mathrm{CZ}}^\Phi(\gamma - \epsilon) = 1$ and $\alpha_-^\Phi(\gamma + \epsilon) = 0$, we find $2c_1^\Phi(v_-) = \operatorname{ind}(v_-) + \chi(\dot{\Sigma}) - \sum_{z\in\Gamma} \mu_{\mathrm{CZ}}^\Phi(\gamma_z - \epsilon) = 2 - 2\#\Gamma$, hence

$$c_N(v_-) = c_1^\Phi(v_-) - \chi(\dot{\Sigma}) + \sum_{z\in\Gamma^+} \alpha_-^\Phi(\gamma_z + \epsilon) = 1 - \#\Gamma - (2 - \#\Gamma) = -1.$$

This implies that $v_- * v_-$ is odd, and is thus a contradiction. □

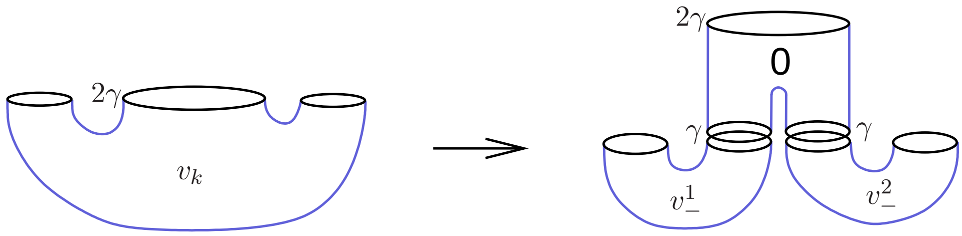

FIGURE 16. A possible limit of the sequence v_k.

4. Proofs of the main results

4.1. **Non-fillability.** We are now in a position to prove the main results on symplectic fillings.

Proof of Theorem 5 and Corollary 4. Given Proposition 3.16 (implicit function theorem) and Theorem 8 (compactness) above, the result follows from the same argument as in [ABW10]. For completeness let us briefly recall the main idea: if (M, ξ) is a closed contact 3-manifold which embeds as a non-separating contact type hypersurface into some closed symplectic 4-manifold (W, ω), then by cutting W open along M and gluing together an infinite chain of copies of the resulting symplectic cobordism between (M, ξ) and itself, we obtain a non-compact but geometrically bounded symplectic manifold $(\mathcal{W}, \omega)$ with contact type boundary (M, ξ). Attaching a cylindrical end and considering the moduli space $\mathcal{M}_0(J)$ that arises from a partially planar domain, one can use the monotonicity lemma to prevent the curves in $\mathcal{M}_0(J)$ from escaping beyond a compact subset of $\mathcal{W}$, thus the compactness result Theorem 8 applies. In combination with Prop. 3.16, this implies that outside a subset of codimension 2 (the images of finitely many curves from Theorem 8), the set of all points in $\mathcal{W}$ filled by curves in $\mathcal{M}_0(J)$ must be open and closed, and is therefore everything; since those curves are confined to a compact subset, this implies $\mathcal{W}$ is compact and is thus a contradiction.

By a similar argument one can prove Corollary 4 independently of Theorem 5, for if (W, ω) is a strong filling with at least two boundary components (M, ξ) and (M', ξ'), then the curves in $\mathcal{M}_0(J)$ emerging from the cylindrical end at M will foliate W^∞ except at a subset of codimension 2; yet they cannot enter the cylindrical end at M' due to convexity, and this is again a contradiction. $\square$

Proof of Theorem 1. Assume (W, ω) is a strong filling of (M, ξ) and the partially planar domain $M_0 \subset M$ is a planar torsion domain. It therefore has a planar piece $M_0^P \subset M_0$, which is a proper subset of its interior. Combining Prop. 3.16 (implicit function theorem) and Theorem 8 (compactness) as in the proof of Theorem 5 above, the curves in $\mathcal{M}_0(J)$ that emerge from M_0^P in the cylindrical end of W^∞ form a foliation of W^∞ outside a subset of codimension 2. We can therefore pick a point $p \in M \setminus M_0^P$ and find a sequence of curves $u_k \in \mathcal{M}_0(J)$ for $k \to \infty$ whose images contain $(k, p) \in [T, \infty) \times M \subset W^\infty$. Applying Theorem 8 again, these have a subsequence which converges to a J_+-holomorphic curve u' in $\mathbb{R} \times M$, whose asymptotic orbits are in the same Morse-Bott families as the curves in $\mathcal{M}_0(J)$. The uniqueness statement in the holomorphic open book result (Theorem 7) then implies that u' is a lift of a page in the blown up summed open book on M_0, which proves that $M_0 = M$, and $M_0 \setminus M_0^P$ consists of a single family of pages diffeomorphic to the planar pages in M_0^P and approaching

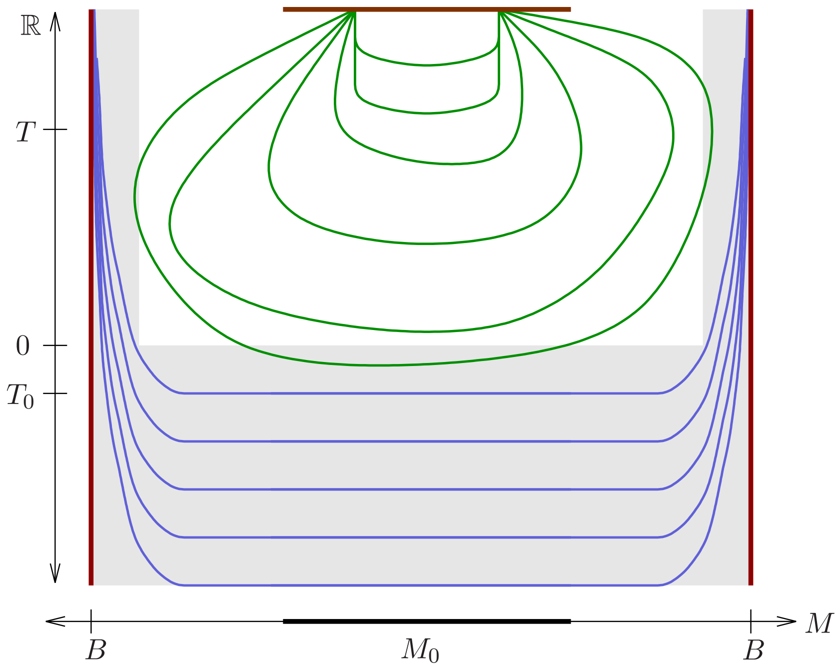

FIGURE 17. The symplectic cobordism used in the proof of Theorem 4, with the negative end "walled off" by holomorphic pages of an open book. The almost complex structure in the shaded region is a non-generic one for which holomorphic open books always exist.

the same Reeb orbits at their boundaries. In other words, M_0 is a symmetric summed open book, which contradicts the definition of a planar torsion domain. □

Proof of Theorem 4. The idea is much the same as in the proof of Theorem 1, but instead of working in the compact context of a symplectic filling, we work in a noncompact symplectic cobordism diffeomorphic to $\mathbb{R} \times M$, in which the negative end is "walled off" so that curves in $\mathcal{M}_0(J)$ cannot reach it. This wall is created by a family of holomorphic curves, namely a subset of the generally non-generic family arising from an open book decomposition (see Figure 17).

Specifically, suppose $\pi : M \setminus B \to S^1$ is an open book decomposition. Recall from Prop. 3.9 that there is a symplectic cobordism $(W, \Omega) = ([0,1] \times M, \Omega)$ where Ω has the form $\omega + d(t\lambda_0)$ near $\{0\} \times M$, $d(e^t\lambda)$ near $\{1\} \times M$ and $d(\varphi(t)\lambda_0)$ in a neighborhood of $[0,1] \times B$ for some positive increasing function φ. Here $\mathcal{H}_\epsilon = (\lambda_\epsilon, \omega)$ is a family of stable Hamiltonian structures adapted to the open book, so $\xi_\epsilon = \ker \lambda_\epsilon$ for some small $\epsilon > 0$ is a supported contact structure and λ is a contact form for ξ_ϵ.

Arguing by contradiction, assume (M, ξ_ϵ) contains a planar torsion domain M_0 that is disjoint from B. We can then find a neighborhood $\mathcal{U} \subset M$ of B such that $M_0 \subset M \setminus \mathcal{U}$ and $\Omega = d(\varphi(t)\lambda_0)$ on $[0,1] \times \mathcal{U}$. Extend W to a noncompact symplectic manifold as follows: first attach to $\{1\} \times M$ a positive cylindrical end that contains a half-symplectization of the form

$$([T \times \infty) \times M, d(e^t\alpha)).$$

Note that since $\{1\} \times M$ is a convex boundary component of (W, Ω), we are free here to choose α as any contact form with $\ker \alpha = \xi_\epsilon$: in particular on M_0 we can assume it is the special Morse-Bott contact form provided by Theorem 7, and since $M_0 \cap \mathcal{U} = \emptyset$, we can also assume $\alpha = \lambda_0$ in $\mathcal{U}$ and $\Omega = d(e^t \lambda_0)$ on $[1, \infty) \times \mathcal{U}$. Secondly, attach to $\{0\} \times M$ a negative cylindrical end of the form

$$((-\infty, 0] \times M, \omega + d(\psi(t)\lambda_0)),$$

where $\psi : (-\infty, 0] \to \mathbb{R}$ is an increasing function with sufficiently small magnitude to make the form symplectic. Denote the resulting noncompact symplectic manifold by (W^∞, ω).

Recall the special almost complex structure $J_0 \in \mathcal{J}(\mathcal{H}_0)$ constructed in §3.2, for which all the pages of π admit J_0-holomorphic lifts in $\mathbb{R} \times M$. We now can choose an almost complex structure J on (W^∞, ω) that has the following properties:

(1) J is everywhere compatible with ω
(2) $J = J_0$ on both $\mathbb{R} \times \mathcal{U}$ and $(-\infty, 0] \times M$
(3) On $[T, \infty) \times M$, J is the special almost complex structure compatible with α provided by Theorem 7.

Now the moduli space $\mathcal{M}_0(J)$ of J-holomorphic curves emerging from M_0 in the positive end can be defined as in the previous proof. The important new feature is that we also have J-holomorphic curves in W^∞ coming from the J_0-holomorphic lifts of pages of the open book: in fact for some $T_0 \in \mathbb{R}$ sufficiently close to $-\infty$, every point in $(-\infty, T_0] \times M$ is contained in such a curve (see Figure 17). The leaves of the foliation in $[T, \infty) \times M_0$ obviously do not intersect these curves, so positivity of intersections implies that no curve in $\mathcal{M}_0(J)$ may intersect them. It follows that the curves in $\mathcal{M}_0(J)$ can never enter $(-\infty, T_0] \times M$, so the compactness result Theorem 8 applies, and we conclude as before that $\mathcal{M}_0(J)$ fills an open and closed subset of W^∞ outside a subset of comdimension 2. But this forces some curve in $\mathcal{M}_0(J)$ to enter the negative end eventually, and we have a contradiction. □

Remark 4.1. For an arguably easier proof of Theorem 4, one can present it as a corollary of Theorem 1 by showing that whenever (M, ξ) is supported by an open book $\pi : M \setminus B \to S^1$ and $\mathcal{U} \subset M$ is a neighborhood of the binding, $(M \setminus \mathcal{U}, \xi)$ can be embedded into a strongly fillable contact manifold. This can be constructed by a doubling trick using the binding sum: if (M', ξ') is supported by an open book that has the same page P as π but inverse monodromy, then one can construct a larger contact manifold by summing every binding component in M to a binding component in M'. The result is a symmetric summed open book which has a strong symplectic filling homeomorphic to $[0, 1] \times S^1 \times P$, in which the natural projection to $[0, 1] \times S^1$ forms a symplectic fibration. The details of this construction are carried out in [LVW]; see also the appendix of [BV].

4.2. **Embedded Contact Homology.** Our goal in this section is to prove Theorems 2, 2′, 6 and 6′. We begin with a quick review of the essential definitions of Embedded Contact Homology, mainly following the discussions in [HS06, §11] and [Tau10b].

4.2.1. *Review of twisted and untwisted ECH.* Assume (M, ξ) is a closed contact 3-manifold with nondegenerate contact form λ, and J is a generic almost complex structure on $\mathbb{R} \times M$ compatible with λ. We will refer to Reeb orbits as **even** or **odd** depending on the parity of their Conley-Zehnder indices: in dynamical terms, an even orbit is always hyperbolic, while an odd orbit can be either elliptic or hyperbolic, the latter if and only if its double cover is even. In §3.1 we defined the notion of an *orbit set* $\boldsymbol{\gamma} = \{(\gamma_1, m_1), \ldots, (\gamma_N, m_N)\}$, and we say that $\boldsymbol{\gamma}$ is **admissible** if $m_i = 1$ whenever γ_i is hyperbolic. Given $h \in H_1(M)$, choose a *reference cycle*, i.e. a 1-cycle $\boldsymbol{\rho}_h$ in M with $[\boldsymbol{\rho}_h] = h$; without loss of generality we can assume $\boldsymbol{\rho}_h$ is represented by an embedded oriented knot in M that is not contained in any closed Reeb orbit. Then adapting the definition of $H_2(M, \boldsymbol{\gamma}^+ - \boldsymbol{\gamma}^-)$ from §3.1, it makes sense to speak of relative homology classes in $H_2(M, \boldsymbol{\rho}_h - \boldsymbol{\gamma})$ for any orbit set $\boldsymbol{\gamma}$ with $[\boldsymbol{\gamma}] = h$.

Given two orbit sets $\boldsymbol{\gamma}^{\pm} = \{(\gamma_1^{\pm}, m_1^{\pm}), \ldots, (\gamma_{N_{\pm}}^{\pm}, m_{N_{\pm}}^{\pm})\}$ and a relative homology class $A \in H_2(M, \boldsymbol{\gamma}^+ - \boldsymbol{\gamma}^-)$ one defines the **ECH index** $I(A) \in \mathbb{Z}$ by choosing any trivialization Φ of ξ along the orbits in $\boldsymbol{\gamma}^{\pm}$ and setting

$$I(A) = c_1^{\Phi}(\xi|_A) + A \bullet_{\Phi} A + \sum_{i=1}^{N_+} \sum_{k=1}^{m_i^+} \mu_{\mathrm{CZ}}^{\Phi}(k\gamma_i^+) - \sum_{i=1}^{N_-} \sum_{k=1}^{m_i^-} \mu_{\mathrm{CZ}}^{\Phi}(k\gamma_i^-), \tag{4.1}$$

where the various symbols are to be interpreted as follows:

- $c_1^{\Phi}(\xi|_A)$ is the relative first Chern number $c_1^{\Phi}(u^*\xi)$ for any asymptotically cylindrical map u representing A,
- $A \bullet_{\Phi} A$ is the *relative self-intersection number*, computed as an algebraic count of intersections of some asymptotically cylindrical representative u with a generic push-off of u that is pushed in the direction of Φ at the cylindrical ends,
- $k\gamma$ denotes the k-fold cover of a Reeb orbit γ.

One can check that this expression does not depend on the choice of trivializations Φ. Since every finite energy J-holomorphic curve u in $\mathbb{R} \times M$ represents a relative homology class, we can define the ECH index of u as $I(u) := I([u])$.

Definition 4.2. A (possibly disconnected) finite energy J-holomorphic curve $u : \dot{\Sigma} \to \mathbb{R} \times M$ is called a **flow line** if it is a disjoint union of two curves u_0 and C, where u_0 is embedded, and C is any collection of trivial cylinders that do not intersect u_0.

Hutchings [Hut02] has shown that for generic J, a flow line u always satisfies $1 \le \mathrm{ind}(u) \le I(u)$. Embedded Contact Homology is defined by counting specifically the flow lines for which this inequality is an equality. For any subgroup $G \subset H_2(M)$, define

$$\widetilde{C}_*(M, \lambda; h, G)$$

to be the free $\mathbb{Z}$-module generated by symbols of the form $e^A\boldsymbol{\gamma}$, where $\boldsymbol{\gamma}$ is an admissible orbit set with $[\boldsymbol{\gamma}] = h$ and $A \in H_2(M, \boldsymbol{\rho}_h - \boldsymbol{\gamma})/G$, meaning $A \sim A'$ whenever $A - A' \in G$. A differential $\partial : \widetilde{C}_*(M, \lambda; h, G) \to \widetilde{C}_{*-1}(M, \lambda; h, G)$ is defined by

$$\partial\left(e^A\boldsymbol{\gamma}\right) = \sum_{\boldsymbol{\gamma}', A'} \# \left(\frac{\mathcal{M}^1_{\mathrm{emb}}(\boldsymbol{\gamma}, \boldsymbol{\gamma}', A')}{\mathbb{R}}\right) e^{A+A'}\boldsymbol{\gamma}',$$

where the sum ranges over all admissible orbit sets $\boldsymbol{\gamma}'$ and $A' \in H_2(M, \boldsymbol{\gamma} - \boldsymbol{\gamma}')/G$, and $\mathcal{M}^1_{\mathrm{emb}}(\boldsymbol{\gamma}, \boldsymbol{\gamma}', A') \subset \mathcal{M}(J)$ is the oriented 1-manifold of (possibly disconnected) finite energy J-holomorphic curves $u : \dot{\Sigma} \to \mathbb{R} \times M$ satisfying the following conditions:

(i) $I(u) = 1$,
(ii) $[u] \sim A'$ in $H_2(M, \boldsymbol{\gamma} - \boldsymbol{\gamma}')/G$,
(iii) u is a flow line in the sense of Definition 4.2.

The orientation of $\mathcal{M}^1_{\mathrm{emb}}(\boldsymbol{\gamma}, \boldsymbol{\gamma}', A')$ is chosen in accordance with [BM04], which requires first choosing an ordering for all the even orbits in M, then ordering the punctures of any $u \in \mathcal{M}^1_{\mathrm{emb}}(\boldsymbol{\gamma}, \boldsymbol{\gamma}', A')$ accordingly. The signed count above is then finite due to the index inequality and compactness theorem in [Hut02].[8] These same results together with the gluing construction of [HT07, HT09] imply that $\partial^2 = 0$, and the resulting homology is denoted by $\widetilde{\mathrm{ECH}}_*(M, \lambda, J; h, G)$. We have two natural choices for the subgroup G: if $G = H_2(M)$, then the terms e^A are all trivial and we obtain the usual *untwisted* Embedded Contact Homology,

$$\mathrm{ECH}_*(M, \lambda, J; h) := \widetilde{\mathrm{ECH}}_*(M, \lambda, J; h, H_2(M)).$$

At the other end of the spectrum, taking G to be the trivial subgroup leads to the *fully twisted* variant of ECH,

$$\widetilde{\mathrm{ECH}}_*(M, \lambda, J; h) := \widetilde{\mathrm{ECH}}_*(M, \lambda, J; h, \{0\}).$$

Since every nontrivial finite energy J-holomorphic curve in $\mathbb{R} \times M$ has at least one positive puncture, the empty orbit set $\emptyset$ always satisfies $\partial\emptyset = 0$, and thus represents a homology class which we call the (untwisted) **contact class**,

$$c(\lambda, J) = [\emptyset] \in \mathrm{ECH}_*(M, \lambda, J; 0).$$

To define the twisted contact class, we note that for $h = 0$ there is a canonical choice of reference cycle $\boldsymbol{\rho}_0$, namely the empty set, so $H_2(M, \boldsymbol{\rho}_0 - \emptyset) = H_2(M)$ and it is natural to define

$$\tilde{c}(\lambda, J) = [e^0\emptyset] \in \widetilde{\mathrm{ECH}}_*(M, \lambda, J; 0).$$

A chain map $U : \widetilde{C}_*(M, \lambda; h, G) \to \widetilde{C}_{*-2}(M, \lambda; h, G)$ can be defined by choosing a generic point $p \in M$ and counting index 2 holomorphic curves that pass through the point $(0, p)$, that is

$$U\left(e^A\boldsymbol{\gamma}\right) = \sum_{\boldsymbol{\gamma}', A'} \#\left(\mathcal{M}^2_{\mathrm{emb}}(\boldsymbol{\gamma}, \boldsymbol{\gamma}', A'; p)\right) e^{A+A'}\boldsymbol{\gamma}',$$

where $\mathcal{M}^2_{\mathrm{emb}}(\boldsymbol{\gamma}, \boldsymbol{\gamma}', A'; p)$ consists of J-holomorphic flow lines u with $I(u) = 2$ and one marked point which is mapped to the point $(0, p)$. We denote by

$$U : \mathrm{ECH}_*(M, \lambda, J; h) \to \mathrm{ECH}_{*-2}(M, \lambda, J; h)$$

and

$$\widetilde{U} : \widetilde{\mathrm{ECH}}_*(M, \lambda, J; h) \to \widetilde{\mathrm{ECH}}_{*-2}(M, \lambda, J; h)$$

respectively the untwisted and fully twisted variants of the resulting map on homology.

It follows from Taubes's isomorphism [Tau10a, Tau10b] that none of the above depends on the choice of λ and J, and the U-map also does not depend on the choice of generic point $p \in M$.

4.2.2. *Proof of the vanishing theorems.* We now prove Theorems 2 and 2′. Assume (M, ξ) contains a planar k-torsion domain M_0 with planar piece $M_0^P \subset M_0$. Note that for some planar torsion domains, there may be multiple subsets of M_0 that could sensibly be called the planar piece (e.g. M_0 could contain multiple planar open books summed together as in Figure 18), so whenever such an ambiguity exists, we choose M_0^P to make k as small as possible. Let λ and J denote the special Morse-Bott contact form and compatible Fredholm regular almost complex structure provided by Theorem 7. Then ∂M_0^P is a non-empty union of tori

$$\partial M_0^P = T_1 \cup \ldots \cup T_n$$

[8] The results in [Hut02] are stated only for a very special class of stable Hamiltonian structures arising from mapping tori, but they extend to the contact case due to the relative asymptotic formulas of Siefring [Sie08].

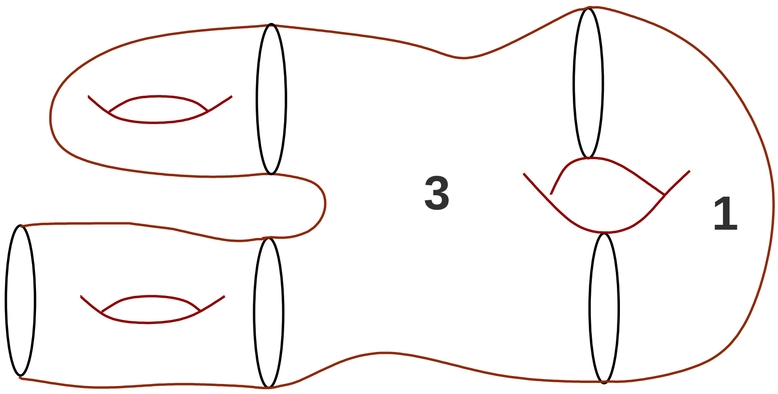

FIGURE 18. A planar torsion domain for which the order is not uniquely defined: depending on the choice of planar piece, the order could be either 1 or 3.

which are Morse-Bott families of Reeb orbits, and the interior of M_0^P may also contain interface tori, which we denote by

$$\mathcal{I}_0 = T_{n+1} \cup \ldots \cup T_{n+r},$$

and binding circles

$$B_0 = \beta_1 \cup \ldots \cup \beta_m.$$

The planar pages in M_0^P have embedded J-holomorphic lifts to $\mathbb{R} \times M$, forming a family of curves,

$$u_{\sigma,\tau} \in \mathcal{M}(J), \qquad (\sigma, \tau) \in \mathbb{R} \times S^1,$$

which have no negative punctures and $m+n+2r$ positive punctures, each asymptotic to simply covered orbits in $B_0 \cup \mathcal{I}_0 \cup \partial M_0^P$, exactly one in each connected component of $B_0 \cup \partial M_0^P$ and two in each component of $\mathcal{I}_0$. Moreover, other than these curves and the obvious trivial cylinders, there is no other connected finite energy J-holomorphic curve in $\mathbb{R} \times M$ with its positive ends approaching any subcollection of the asymptotic orbits of $u_{\sigma,\tau}$.

We now perturb λ to a nondegenerate contact form λ' by the scheme described in [Bou02], so that each of the original Morse-Bott tori $T_j \subset \mathcal{I}_0 \cup \partial M_0^P$ contains exactly two nondegenerate Reeb orbits, one elliptic and one hyperbolic,

$$\gamma_j^e \cup \gamma_j^h \subset T_j.$$

Denoting by Φ_0 the natural trivialization along these orbits determined by the (θ, ρ, ϕ)-coordinates, they satisfy $\mu_{\mathrm{CZ}}^{\Phi_0}(\gamma_j^e) = 1$ and $\mu_{\mathrm{CZ}}^{\Phi_0}(\gamma_j^h) = 0$, and for any number $k_0 \in \mathbb{N}$ we can also arrange that $\mu_{\mathrm{CZ}}^{\Phi_0}(k\gamma_j^e) = 1$ for all $k \leq k_0$. Perturbing J to a generic J' compatible with λ', the family of curves $u_{\sigma,\tau}$ gives rise to embedded J'-holomorphic curves (Figure 19) asymptotic to various combinations of these orbits and the components of B_0. If $u : \dot{\Sigma} \to \mathbb{R} \times M$ is such a curve, then genericity implies $\mathrm{ind}(u) \geq 1$, so we deduce from the index formula that such curves come in two types:

- $\mathrm{ind}(u) = 2$ if all ends approaching $\mathcal{I}_0 \cup \partial M_0^P$ approach elliptic orbits,
- $\mathrm{ind}(u) = 1$ if u has exactly one end approaching a hyperbolic orbit in $\mathcal{I}_0 \cup \partial M_0^P$.

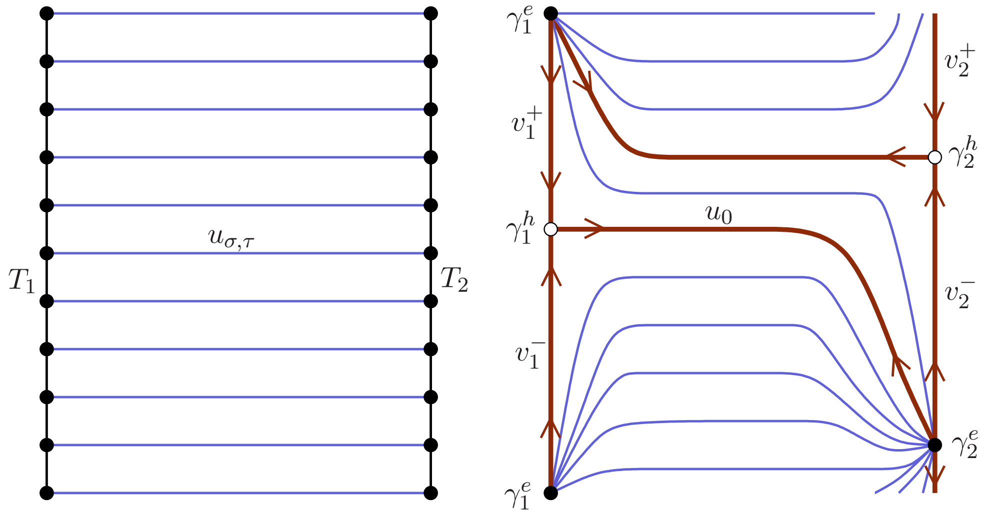

FIGURE 19. The perturbation from Morse-Bott (left) to nondegenerate (right), shown here in the simple case where $u_{\sigma,\tau}$ is a family of cylinders asymptotic to two Morse-Bott tori. All the orbits in the picture point along an S^1-factor through the page, and the top and bottom are identified. Arrows indicate the signs of the ends of the rigid curves in the nondegenerate picture: an end is positive if and only if the arrow points *away from* the orbit.

All of these curves also have genus zero and satisfy $u \bullet_{\Phi_0} u = 0$ and $c_N(u) = c_1^{\Phi_0}(u) - \chi(\dot{\Sigma}) = 0$, so one can then deduce from (4.1) and the index formula (3.7) that $I(u) = \operatorname{ind}(u)$.

Up to $\mathbb{R}$-translation there is now exactly one J'-holomorphic flow line $u_0 : \dot{\Sigma} \to \mathbb{R} \times M$ with all punctures positive and asymptotic to the orbits

$$\gamma_1^h, \gamma_2^e, \dots, \gamma_n^e, \gamma_{n+1}^e, \gamma_{n+1}^e, \dots, \gamma_{n+r}^e, \gamma_{n+r}^e, \beta_1, \dots, \beta_m.$$

Let us therefore define the orbit set

$$\boldsymbol{\gamma}_0 = \{(\gamma_1^h, 1), (\gamma_2^e, 1), \dots, (\gamma_n^e, 1), (\gamma_{n+1}^e, 2), \dots, (\gamma_{n+r}^e, 2), (\beta_1, 1), \dots, (\beta_m, 1)\},$$

for which $[\boldsymbol{\gamma}_0] = 0$, and define also the relative homology class

$$A_0 = -[u_0] \in H_2(M, \boldsymbol{\rho}_0 - \boldsymbol{\gamma}_0).$$

The perturbation from J to J' creates some additional J'-holomorphic cylinders which arise from gradient flow lines along the Morse-Bott families of orbits, as described in [Bou02]. Namely for each $j = 1, \dots, n + r$, there are two embedded cylinders

$$v_j^+, v_j^- : \mathbb{R} \times S^1 \to \mathbb{R} \times M,$$

each with positive end at γ_j^e and negative end at γ_j^h; the images of these cylinders in M are the two connected components of $T_j \setminus (\gamma_j^e \cup \gamma_j^h)$, thus after choosing the labels appropriately, we can assume their relative homology classes are related by

$$[v_j^+] - [v_j^-] = [T_j] \in H_2(M).$$

These cylinders satisfy $\operatorname{ind}(v_j^\pm) = I(v_j^\pm) = 1$.

Now in the twisted ECH complex, the only curves other than u_0 counted by $\partial\left(e^{A_0}\boldsymbol{\gamma}\right)$ are the disjoint unions of $v_j^{\pm}$ with collections of trivial cylinders for $j = 2, \dots, n+r$. The negative ends of such a disjoint union give rise to the orbit set

$$\begin{aligned}\boldsymbol{\gamma}_j := \{&(\gamma_1^h, 1), (\gamma_2^e, 1), \dots, (\gamma_{j-1}^e, 1), (\gamma_j^h, 1), (\gamma_{j+1}^e, 1), \dots, (\gamma_n^e, 1), \\ &(\gamma_{n+1}^e, 2), \dots, (\gamma_{n+r}^e, 2), (\beta_1, 1), \dots, (\beta_m, 1)\}\end{aligned}$$

for $j = 1, \dots, n$, and a similar expression for $j = n+1, \dots, n+r$ which will appear twice due to the multiplicity attached to γ_j^e. Choosing appropriate coherent orientations and adding all this together, we find

$$\begin{aligned}\partial\left(e^{A_0}\boldsymbol{\gamma}_0\right) = e^0\boldsymbol{\emptyset} &+ \sum_{j=2}^{n} e^{A_0+[v_j^-]}\left(e^{[T_j]} - 1\right)\boldsymbol{\gamma}_j \\ &+ \sum_{j=n+1}^{n+r} 2e^{A_0+[v_j^-]}\left(e^{[T_j]} - 1\right)\boldsymbol{\gamma}_j.\end{aligned}$$

We thus have $\partial\left(e^{A_0}\boldsymbol{\gamma}_0\right) = e^0\boldsymbol{\emptyset}$ whenever $[T_j] = 0 \in H_2(M)$ for all $j = 2, \dots, n+r$, which proves Theorem 2′. For untwisted coefficients, we divide the entire calculation by $H_2(M)$ so that $e^{[T_j]} - 1 = 0$ always, thus $\partial\boldsymbol{\gamma}_0 = \boldsymbol{\emptyset}$ holds with no need for any topological condition. With that, the proof of Theorem 2 is complete.

4.2.3. *The U-map.* The proof of Theorems 6 and 6′ is a minor variation on the argument given above. Assume (M, ξ) contains a partially planar domain M_0 with planar piece $M_0^P \subset M_0$, and choose the Morse-Bott data λ, J and nondegenerate perturbation λ', J' exactly as described in the previous section, but adding the following condition: for any given $d \in \mathbb{N}$, Theorem 7 allows us to choose λ so that the uniqueness statement for holomorphic curves subordinate to the planar piece up to multiplicity k holds for any $k \leq d$.

Now consider the J'-holomorphic curves of index 2 with positive ends asymptotic to the elliptic orbits,

$$\gamma_1^e, \dots, \gamma_n^e, \gamma_{n+1}^e, \gamma_{n+1}^e, \dots, \gamma_{n+r}^e, \gamma_{n+r}^e, \beta_1, \dots, \beta_m.$$

These curves have embedded projections to M which foliate an open subset of M_0^P, thus if we choose p in this open subset, there is exactly one curve with the given asymptotics that passes through $(0, p)$. Denote this curve by u_p, and for any $k \in \{1, \dots, d\}$, define the orbit set

$$\boldsymbol{\gamma}^{(k)} = \{(\gamma_1^e, k), \dots, (\gamma_n^e, k), (\gamma_{n+1}^e, 2k), \dots, (\gamma_{n+r}^e, 2k), (\beta_1, k), \dots, (\beta_m, k)\}$$

with $[\boldsymbol{\gamma}^{(k)}] = 0$, and the relative homology class

$$kA_p = -k[u_p] \in H_2(M, \boldsymbol{\rho}_0 - \boldsymbol{\gamma}^{(k)}).$$

The uniqueness statement in Theorem 7 for curves subordinate to the planar piece up to multiplicity d now implies that $\partial\left(e^{kA_p}\boldsymbol{\gamma}^{(k)}\right)$ counts only the disjoint unions of the embedded index 1 cylinders $v_j^{\pm}$ with trivial cylinders. As in the previous section, the contributions from v_j^+ and v_j^- cancel each other out in the untwisted theory, and also in the twisted theory if $[T_j] = 0 \in H_2(M)$, so we conclude in either case that $e^{kA_p}\boldsymbol{\gamma}^{(k)}$ is a cycle in the chain complex. The uniqueness result also implies that there is exactly one curve counted by $U\left(e^{kA_p}\boldsymbol{\gamma}^{(k)}\right)$, namely the disjoint union of u_p with a collection of trivial cylinders. We thus find,

$$U\left(e^{kA_p}\boldsymbol{\gamma}^{(k)}\right) = e^{(k-1)A_p}\boldsymbol{\gamma}^{(k-1)}$$

for each $k \in \{2, \dots, d\}$, and for $k = 1$,

$$U\left(e^{A_p}\boldsymbol{\gamma}^{(1)}\right) = e^0\boldsymbol{\emptyset}.$$

Since the ECH does not depend on the choice of contact form, this shows that for all $d \in \mathbb{N}$ the homology contains an element whose image under d iterations of the U-map is the contact class. The proof of Theorems 6 and 6′ is thus complete.

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