OPEN BOOK DECOMPOSITIONS AND STABLE HAMILTONIAN STRUCTURES

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ABSTRACT. We show that every open book decomposition of a contact 3–manifold can be represented (up to isotopy) by a smooth $\mathbb{R}$–invariant family of pseudoholomorphic curves on its symplectization with respect to a suitable stable Hamiltonian structure. In the planar case, this family survives small perturbations, and thus gives a concrete construction of a stable finite energy foliation that has been used in various applications to planar contact manifolds, including the Weinstein conjecture [ACH05] and the equivalence of strong and Stein fillability [Wenb].

1. INTRODUCTION

The subject of this note is a correspondence between open book decompositions on contact manifolds and J–holomorphic curves in their symplectizations. We will assume throughout that (M, ξ) is a closed 3–manifold with a positive, cooriented contact structure. An *open book decomposition* of M is a fibration

$$\pi : M \setminus B \to S^1,$$

where $B \subset M$ is a link called the *binding*, and the fibers are called *pages*: these are open surfaces whose closures have boundary equal to B. An open book is called *planar* if the pages have genus zero, and it is said to *support* a contact structure ξ if the latter can be written as $\ker \lambda$ for some contact form λ (a *Giroux form*) such that $d\lambda$ is positive on the pages and λ is positive on the binding (oriented as the boundary of the pages). In this case the Reeb vector field X_λ defined by λ is transverse to the pages and parallel to the binding, so in particular the binding is a union of periodic orbits. A picture of a simple open book on the tight 3–sphere is shown in Figure 1.

We say that an almost complex structure J on $\mathbb{R} \times M$ is *compatible with* λ if it is invariant under the natural $\mathbb{R}$–action, maps the unit vector in the $\mathbb{R}$–direction to X_λ and restricts to ξ as a complex structure compatible with $d\lambda|_\xi$. One then considers J–holomorphic curves

$$u : \dot{\Sigma} \to \mathbb{R} \times M,$$

where the domain is a closed Riemann surface with finitely many punctures, and u satisfies a finite energy condition (see [BEH$^+$03]), so that it has “asymptotically cylindrical” behavior at the punctures, approaching closed orbits of X_λ at $\{\pm\infty\} \times M$. Note that whenever the projection of u into M is embedded, it is also transverse to X_λ, a property that is shared by the pages of supporting open books with their Giroux forms. Thus it is

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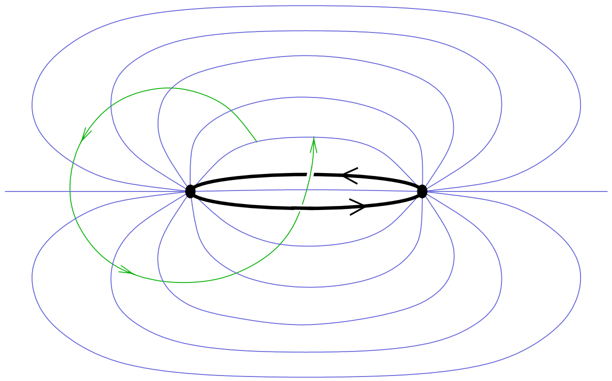

FIGURE 1. An open book decomposition of the tight 3–sphere with one binding orbit and disk-like pages, which are transverse to the Reeb vector field.

natural to ask whether the pages of an open book can in general be presented as projections of holomorphic curves: such a family of holomorphic curves is referred to as a *holomorphic open book*, and is a special case of a *finite energy foliation* (see [HWZ03, Wen08]). We refer to [Etn06] for further details on the rich relationship between open books and contact structures, and [Hof00] for some applications of holomorphic curves in this context to dynamics.

Our main goal is to prove the following.

Main Theorem. *Suppose $\pi : M \setminus B \to S^1$ is a planar open book decomposition on M that supports ξ. Then after an isotopy of π, it admits a nondegenerate Giroux form λ with a compatible almost complex structure J on the symplectization $\mathbb{R}\times M$, and a smooth 2–dimensional $\mathbb{R}$–invariant family of embedded, finite energy J–holomorphic curves in $\mathbb{R} \times M$ with index 2, whose projections to M give an S^1–family of embeddings parametrizing the pages of π.*

Remark 1. It will be clear from the construction that one can choose the Giroux form λ in this theorem so that the binding orbits have arbitrarily small periods compared with all other Reeb orbits in M. This assumption is sometimes useful for compactness arguments, and is exploited e.g. in [ABW].

This result has been used in the literature for various applications, including Abbas-Cieliebak-Hofer’s proof of the Weinstein conjecture for planar contact manifolds [ACH05], and the author’s theorem that strong symplectic fillings of such manifolds are always blowups of Stein fillings [Wenb]. A construction of holomorphic open books was sketched in [ACH05] without many details. The construction explained below is based on a completely different idea, and has the advantage of producing a (usually non-stable) finite energy foliation out of *any* open book, with arbitrary genus. The catch is that this construction requires a choice of J which is not compatible with λ in the sense described above, but is instead compatible with a *stable Hamiltonian structure*, which can be seen as a limit of λ as the contact structure degenerates to a confoliation. The idea is then to recover the contact case by a perturbation argument, but for analytical reasons, this can only be done with a planar open book.

The trouble with the non-planar case is that holomorphic curves of higher genus with the desired intersection theoretic properties never have positive index, and thus generically cannot exist. This problem has an analogue in the study of closed symplectic 4–manifolds, namely in McDuff's classification [McD90] of manifolds that admit nonnegative symplectic spheres—there is no corresponding result for higher genus symplectic surfaces because the dimension of the moduli space of higher genus holomorphic curves is generally too small. In the contact setting, a potential remedy was proposed by Hofer in [Hof00], who suggested considering a more general elliptic problem in which a harmonic 1–form is introduced to raise the index. The study of this problem is a large project in progress by C. Abbas [Abb] and Abbas-Hofer-Lisi [AHL], in which punctured holomorphic curves of genus zero are treated as an easy special case: this would be a necessary ingredient to generalize the approach in [ACH05] to the Weinstein conjecture in dimension three.[1] For other applications however, it is already helpful to know that any open book can be viewed as a family of J–holomorphic curves for some non-generic choice of J. This idea is exploited for instance in [Wenc] to compute certain algebraic invariants of contact manifolds based on holomorphic curves, and in [Wend] to define previously unknown obstructions to symplectic filling.

Our construction rests on the notion of an *abstract open book*, which is defined by the data (P, ψ), where P is a compact oriented surface with boundary representing the *page*, and ψ is a diffeomorphism that fixes the boundary, called the *monodromy map*. Without loss of generality, we can assume that ψ is the identity in a neighborhood of ∂P. Let P_ψ denote the *mapping torus* of ψ, which is the smooth 3–manifold with boundary,

$$P_\psi = (\mathbb{R} \times P)/\sim$$

where $(t+1, p) \sim (t, \psi(p))$. This comes with a natural fibration $P_\psi \to S^1 := \mathbb{R}/\mathbb{Z}$, so that the tangent spaces to P define a 2–plane distribution in TP_ψ, called the *vertical distribution*.

Proposition 2. *Given an abstract open book (P, ψ), let M denote the closed 3–manifold obtained by gluing solid tori to P_ψ so that (P, ψ) defines an open book decomposition of M. Then the vertical distribution on P_ψ can be extended to M as a confoliation ξ_0, such that a C^∞–small perturbation of ξ_0 defines a contact structure ξ_ϵ supported by the open book, and each is compatible with stable Hamiltonian structures $\mathcal{H}_0 = (\xi_0, X_0, \omega_0)$ and $\mathcal{H}_\epsilon = (\xi_\epsilon, X_\epsilon, \omega_\epsilon)$ such that $\mathcal{H}_\epsilon$ is C^∞–close to $\mathcal{H}_0$.*

[1]In the mean time, Taubes [Tau07] has produced a proof of the Weinstein conjecture in dimension three based on Seiberg-Witten theory. It is generally believed that a proof based on holomorphic curves should also be possible, but none has yet appeared.

We will prove this via a concrete construction in the next section, after recalling precisely what a stable Hamiltonian structure $\mathcal{H} = (\xi, X, \omega)$ is, and how it defines a special class of almost complex structures on $\mathbb{R} \times M$. The next step is the following:

Proposition 3. *Given (P, ψ), M and $\mathcal{H}_0 = (\xi_0, X_0, \omega_0)$ as in Proposition 2, there exists an almost complex structure J_0 compatible with $\mathcal{H}_0$ such that the pages of the open book on M lift to embedded J_0–holomorphic curves in $\mathbb{R} \times M$, with positive ends and index $2 - 2g$, where g is the genus of P.*

We will prove this in §3, and recall the definition of the *index* of a J–holomorphic curve and its significance. We'll then show that the index 2 curves obtained in the case $g = 0$ survive as a holomorphic open book under the small perturbation from $\mathcal{H}_0$ to $\mathcal{H}_\epsilon$, thus proving the main theorem.

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2. Stable Hamiltonian structures

Stable Hamiltonian structures were introduced in [BEH+03] (although the name came somewhat later, cf. [EKP06, Eli07]) as a generalized setting for holomorphic curves in symplectizations that accomodates both contact geometry and Floer homology, among other things. Given a closed oriented 3–manifold M, a stable Hamiltonian structure $\mathcal{H} = (\xi, X, \omega)$ is defined by

(1) a coorientable 2–plane distribution $\xi \subset TM$,
(2) a vector field X (the *Reeb vector field*) that is everywhere transverse to ξ and has flow preserving ξ,
(3) a closed 2–form ω (the *taming form*) such that $\omega|_\xi > 0$ and $\iota_X \omega \equiv 0$.

One can associate to $\mathcal{H} = (\xi, X, \omega)$ the unique 1–form λ such that $\ker \lambda = \xi$ and $\lambda(X) \equiv 1$, which then automatically satisfies $d\lambda(X, \cdot) \equiv 0$. Moreover, $\iota_X \omega \equiv 0$ implies that the flow of X preserves not only ξ but also its symplectic structure defined by $\omega|_\xi$. Now, given $\mathcal{H}$, the so-called *symplectization* $\mathbb{R} \times M$ inherits a natural splitting $T(\mathbb{R} \times M) = \mathbb{R} \oplus \mathbb{R}X \oplus \xi$, and we use this to define a special class of almost complex structures $\mathcal{J}(\mathcal{H})$ on $\mathbb{R} \times M$, so that for every $J \in \mathcal{J}(\mathcal{H})$,

(1) J is invariant under the natural $\mathbb{R}$–action on $\mathbb{R} \times M$,
(2) $J\partial_a = X$, where ∂_a denotes the unit vector in the $\mathbb{R}$–direction,
(3) $J(\xi) = \xi$ and $J|_\xi$ is compatible with the symplectic structure $\omega|_\xi$.

Notice that the definition of $\mathcal{J}(\mathcal{H})$ depends on ω only up to the *conformally* symplectic structure that it induces on ξ. Thus one can always replace $\mathcal{H} = (\xi, X, \omega)$ by $\mathcal{H}' = (\xi, X, f\omega)$ for any smooth function $f : M \to (0, \infty)$ with $df \wedge \omega = 0$; then $\mathcal{J}(\mathcal{H}) = \mathcal{J}(\mathcal{H}')$, and the notions of finite energy J–holomorphic curves defined via $\mathcal{H}$ and $\mathcal{H}'$ coincide.

If the distribution ξ in $\mathcal{H} = (\xi, X, \omega)$ is a contact structure, then any choice of λ with $\ker \lambda = \xi$ defines X uniquely: in standard contact geometric terms, it is the Reeb vector field determined by the contact form λ. In this case one can also take $d\lambda$ as a natural taming form, though as mentioned above, it is not the *only* choice.

Symplectic fibrations over S^1 provide another natural source of stable Hamiltonian structures. Suppose $\pi : M \to S^1$ is a locally trivial symplectic fibration whose standard fiber is a symplectic surface (S, σ), possibly with boundary, and denote the coordinate on the base by t. The vertical subspaces form an integrable distribution $\xi \subset TM$, and any symplectic connection can be defined so that parallel transport is the flow of a vector field X on M with $\pi_* X = \partial_t$. There is then a unique 2–form ω on M such that $\omega|_\xi = \sigma$ and $\omega(X, \cdot) \equiv 0$, and we claim that $\mathcal{H} := (\xi, X, \omega)$ is a stable Hamiltonian structure on M. One only has to verify that ω is closed; to see this, identify a neighborhood of any point in M with $(-\epsilon, \epsilon) \times S$ via a symplectic local trivialization and denote the real coordinate by t. Then X can be written on $(-\epsilon, \epsilon) \times S$ in the form

$$X(t, p) = \partial_t + V(t, p)$$

for some t–dependent locally Hamiltonian vector field $V_t = V(t, \cdot)$ on S, and σ defines a 2–form on $(-\epsilon, \epsilon) \times S$ with $\partial_t \in \ker \sigma$. One can then check that ω has the form $\sigma + \iota_V \sigma \wedge dt$, which is closed because $\iota_{V_t}\sigma$ is a closed 1–form on S for every t. An important special case of this construction is the mapping torus S_φ for a symplectomorphism $\varphi \in \mathrm{Symp}(S, \sigma)$: then the Floer homology of (S, φ) can be viewed as a special case of symplectic field theory on $(S_\varphi, \mathcal{H})$.

We shall now prove Proposition 2 by constructing a stable Hamiltonian structure that combines both of the examples above. The resulting distribution ξ will be a *confoliation* (cf. [ET98]), which means that the associated 1–form satisfies $\lambda \wedge d\lambda \ge 0$; it is a contact structure wherever this inequality is strict, and is a symplectic fibration everywhere else.

Suppose (P, ψ) is an abstract open book, and $\phi : P_\psi \to S^1$ is its mapping torus, regarded as a fibration over S^1, with the vertical distribution denoted by $\xi_0 \subset TP_\psi$. For some neighborhood $\mathcal{U}$ of each component of ∂P, choose $\delta > 0$ small and identify $\mathcal{U}$ with $[1 - \delta, 1 + \delta) \times S^1$ by a diffeomorphism

$$(\rho, \theta) : \mathcal{U} \to [1 - \delta, 1 + \delta) \times S^1$$

such that $d\theta \wedge d\rho$ defines the positive orientation of P. We can assume without loss of generality that ψ is the identity on $\mathcal{U}$, so a neighborhood of each boundary component of P_ψ now looks like $S^1 \times [1 - \delta, 1 + \delta) \times S^1$ with coordinates (ϕ, ρ, θ).

We will also use the symbols (θ, ρ, ϕ) to denote coordinates on the solid torus $S^1 \times \mathbb{D}$, where θ is assigned to the first factor and $(\rho, \phi) \in [0, 1] \times S^1$ are polar coordinates on the closed unit disk $\mathbb{D} \subset \mathbb{R}^2$. Then there is a closed manifold

$$M := P_\psi \cup_{\partial P_\psi} \left(\bigcup S^1 \times \mathbb{D} \right)$$

defined by gluing a copy of $S^1 \times \mathbb{D}$ to each boundary component of P_ψ, with attaching maps defined to be the identity in the coordinates $(\theta, \rho, \phi) \in S^1 \times [1 - \delta, 1] \times S^1$. Denoting the union of all the loops $\{\rho = 0\}$ by B, we now have a natural fibration $\pi : M \setminus B \to S^1$ defined by the ϕ–coordinate.

Define $\lambda_0|_{P_\psi} = d\phi$, so $\xi_0 = \ker \lambda_0$. Then we can extend ξ_0 over M as a confoliation by defining λ_0 for $\rho < 1 + \delta$ as

$$\lambda_0 = f(\rho)\, d\theta + g(\rho)\, d\phi$$

for some pair of smooth functions $f, g : [0, 1 + \delta] \to \mathbb{R}$ such that

(1) The path $\rho \mapsto (f(\rho), g(\rho)) \in \mathbb{R}^2$ moves through the first quadrant from $(f(0), g(0)) = (c, 0)$ for some $c > 0$ to $(f(1+\delta), g(1+\delta)) = (0, 1)$ and is constant for $\rho \in [1 - \delta, 1 + \delta]$.
(2) The function
$$D(\rho) := f(\rho) g'(\rho) - f'(\rho) g(\rho)$$
is positive on $(0, 1 - \delta)$, and $g''(0) > 0$.
(3) There is a small number $\delta' > \delta$ such that $g(\rho) = 1$ for all $\rho \in [1 - \delta', 1 + \delta)$.
(4) The maps $\mathbb{D} \to \mathbb{R}$ defined by $(\rho, \phi) \mapsto f(\rho)$ and $(\rho, \phi) \mapsto g(\rho)/\rho^2$ are smooth at the origin.

The last condition requires $f'(0) = g'(0) = 0$, and it ensures that λ_0 is well defined and smooth at the coordinate singularity $\rho = 0$. The second guarantees that λ_0 is contact for $\rho < 1 - \delta$, and the significance of the third can be seen by computing the Reeb vector field in this region: we find

$$X_0(\theta, \rho, \phi) = \frac{g'(\rho)}{D(\rho)} \partial_\theta - \frac{f'(\rho)}{D(\rho)} \partial_\phi, \tag{2.1}$$

which is identically equal to ∂_ϕ for $\rho \in [1 - \delta', 1 - \delta)$.

It follows from a fundamental theorem of Giroux [Gir] that every closed contact 3–manifold is isomorphic to (M, ξ_ϵ), where ξ_ϵ is a small perturbation of a confoliation ξ_0 as constructed above. Let us make this perturbation explicit: following [Etn06], choose a smooth 1–form η on P such that $\eta = (2 - \rho)\, d\theta$ near the boundary and $d\eta > 0$ everywhere. Then if $\tau : [0, 1] \to [0, 1]$ is a smooth function that equals 0 for t on a neighborhood of 0 and 1 for t on a neighborhood of 1, we define a 1–form on $[0, 1] \times P$ by

$$\alpha = \tau(\phi) \psi^* \eta + [1 - \tau(\phi)]\, \eta,$$

where ϕ is now the coordinate on $[0, 1]$. This extends to $\mathbb{R} \times P$ and then descends to a smooth 1–form on P_ψ such that $\alpha = (2 - \rho)\, d\theta$ near ∂P_ψ and $d\alpha|_{\xi_0} > 0$. Then for sufficiently small $\epsilon > 0$,

$$\lambda_\epsilon := d\phi + \epsilon \alpha$$

is a contact form on P_ψ: indeed, $d\lambda_\epsilon = \epsilon\, d\alpha$ is positive on ξ_0, and thus also on the C^∞–close perturbation $\xi_\epsilon := \ker \lambda_\epsilon$. In the region $\rho \in [1 - \delta, 1 + \delta)$ near any component of ∂P_ψ, we now have $\lambda_\epsilon = \epsilon(2 - \rho)\, d\theta + d\phi$, thus we can extend λ_ϵ to a contact form on M close to λ_0 by choosing C^∞–small perturbations (f_ϵ, g_ϵ) of (f, g) such that

(1) $(f_\epsilon(\rho), g_\epsilon(\rho)) = (\epsilon(2 - \rho), 1)$ for $\rho \in [1 - \delta, 1 + \delta)$,
(2) $g_\epsilon(\rho) = 1$ and $f'_\epsilon(\rho) < 0$ for all $\rho \in [1 - \delta', 1 + \delta)$,
(3) $(f_\epsilon(\rho), g_\epsilon(\rho)) = (f(\rho), g(\rho))$ for $\rho \in [0, 1 - \delta']$.

Now if X_ϵ denotes the Reeb vector field determined by λ_ϵ, we have $X_\epsilon \equiv X_0$ on $\{\rho < 1 - \delta\}$; in particular this equals ∂_ϕ for $\rho \in [1 - \delta', 1 - \delta)$.

Lemma 4. *X_0 extends over M as the C^∞–limit of X_ϵ as $\epsilon \to 0$.*

Proof. On P_ψ, the direction of X_ϵ is determined by $\ker d\lambda_\epsilon = \ker d\alpha$, and is therefore independent of ϵ, so X_ϵ converges as $\epsilon \to 0$ to the unique vector field that spans $\ker d\alpha$ and satisfies $d\phi(X_0) \equiv 1$. In a neighborhood of ∂P_ψ this is simply ∂_ϕ, so it fits together smoothly with (2.1). $\square$

To complete the proof of Proposition 2, choose a smooth function $h : [1-\delta', 1-\delta] \to (0,\infty)$ which equals $-f'(\rho)$ for ρ near $1-\delta'$ and 1 for ρ near $1-\delta$. Then a taming form for ξ_0 and X_0 can be defined by

$$\omega_0 = \begin{cases} d\alpha & \text{on } P_\psi, \\ h(\rho)\ d\theta \wedge d\rho & \text{for } \rho \in [1-\delta', 1-\delta), \\ d\lambda_0 & \text{for } \rho < 1-\delta', \end{cases}$$

making $\mathcal{H}_0 := (\xi_0, X_0, \omega_0)$ into a stable Hamiltonian structure. Since X_ϵ and X_0 are everywhere colinear and ξ_ϵ is assumed close to ξ_0, ω_0 also furnishes a taming form for ξ_ϵ and X_ϵ, defining $\mathcal{H}_\epsilon := (\xi_\epsilon, X_\epsilon, \omega_0)$. Observe that $\omega_0 = F_\epsilon\ d\lambda_\epsilon$ for a function F_ϵ with $dF_\epsilon \wedge d\lambda_\epsilon = 0$.

Remark 5. The data $\mathcal{H}_0 = (\xi_0, X_0, \omega_0)$ give P_ψ the structure of a symplectic fibration, where the symplectic form on the fibers is $d\alpha|_{\xi_0}$, and X_0 defines a symplectic connection.

Remark 6. The taming form ω_0 will not play any role in the arguments to follow, but it is important in further applications, cf. [Wenc]. In particular, one needs it to obtain compactness results for a sequence of J_ϵ–holomorphic curves with $J_\epsilon \in \mathcal{J}(\mathcal{H}_\epsilon)$ as $\epsilon \to 0$.

3. Finite energy foliations

Let us now apply Proposition 2 to prove Proposition 3 and the main theorem. We begin by choosing an appropriate $J_0 \in \mathcal{J}(\mathcal{H}_0)$ and constructing a foliation of $\mathbb{R} \times M$ by J_0–holomorphic curves. On P_ψ this is easy: ξ_0 is tangent to the fibers and is preserved by any admissible complex structure, thus for any fiber $F \subset P_\psi$ of the mapping torus, $\{\text{const}\} \times F \subset \mathbb{R} \times P_\psi$ is an embedded holomorphic curve for any J_0. The task is therefore to find a foliation by holomorphic curves in $\mathbb{R} \times (S^1 \times \mathbb{D})$, which have a puncture asymptotic to the orbit at $\{\rho = 0\}$ and fit together smoothly with the fibers $\{\text{const}\} \times F$. For a sufficiently symmetric choice of the data, this is merely a matter of writing down the Cauchy-Riemann equations and solving them: we have $\lambda_0 = f\ d\theta + g\ d\phi$ and $X_0 = \frac{g'}{D}\partial_\theta - \frac{f'}{D}\partial_\phi$, which reduce to $d\phi$ and ∂_ϕ respectively near $\partial(S^1 \times \mathbb{D})$. Choose vector fields

$$v_1 = \partial_\rho, \qquad v_2 = -g(\rho)\partial_\theta + f(\rho)\partial_\phi$$

to span ξ_0, along with a smooth function $\beta(\rho) > 0$, and define $J_0 \in \mathcal{J}(\mathcal{H}_0)$ at $(\theta, \rho, \phi) \in S^1 \times \mathbb{D}$ by the condition $J_0 v_1 = \beta(\rho) v_2$. The function β can be chosen so that this definition of J_0 extends smoothly to $\rho = 0$, and we shall assume $\beta(\rho) = 1$ for ρ outside a neighborhood of 0. Then one can compute (cf. [Wen08, §4.2]) that in conformal coordinates (s,t), a map

$$u(s,t) = (a(s,t), \theta(s,t), \rho(s,t), \phi(s,t))$$

with $\rho(s,t) < 1-\delta$ is J_0–holomorphic if and only if it satisfies the equations

$$a_s = f\theta_t + g\phi_t \qquad \rho_s = \frac{1}{\beta D}(f'\theta_t + g'\phi_t)$$
$$a_t = -f\theta_s - g\phi_s \qquad \rho_t = -\frac{1}{\beta D}(f'\theta_s + g'\phi_s)$$

where f, g, D and β are all functions of $\rho(s,t)$. If $\rho(s,t) \ge 1-\delta'$, then $g'(\rho) = 0$ and $g(\rho) = 1 = \beta(\rho)$, so the two equations on the right become

$$\rho_s = -\theta_t, \qquad \rho_t = \theta_s.$$

There are then solutions of the form

$$u : [0,\infty) \times S^1 \to \mathbb{R} \times (S^1 \times \mathbb{D}) : (s,t) \mapsto (a(s), t, \rho(s), \phi_0)$$

for any constant $\phi_0 \in S^1$, where $a(s)$ and $\rho(s)$ solve the ordinary differential equations

$$\frac{da}{ds} = f(\rho), \qquad \frac{d\rho}{ds} = \begin{cases} \frac{f'(\rho)}{\beta(\rho)D(\rho)} & \text{if } \rho < 1-\delta, \\ -1 & \text{if } \rho \ge 1-\delta'. \end{cases} \tag{3.1}$$

Let us now add the following standing assumptions for f and g:

(1) $f'(\rho) < 0$ for all $\rho \in (0, 1-\delta)$.
(2) $f'(\rho)/g'(\rho)$ is a constant irrational number close to zero for sufficiently small $\rho > 0$.

Observe that one can impose these conditions and in addition require $f(0) > 0$ to be arbitrarily small, the latter being the period of the Reeb orbit at $\rho = 0$ (see Remark 1). Now by a straightforward computation of the linearized Reeb flow, the second assumption ensures that this orbit and all its multiple covers will be nondegenerate, and these are the only periodic orbits in some neighborhood. By the first assumption, the unique solution to (3.1) with $\rho(0) = 1$ and any given value of $a(0) \in \mathbb{R}$ yields a J_0–holomorphic half-cylinder u which is positively asymptotic as $s \to \infty$ to the embedded orbit at $\{\rho = 0\}$ and has $a(s,t)$ and $\phi(s,t)$ both constant near the boundary. The image can therefore be attached smoothly to the holomorphic fiber $\{\text{const}\} \times F \subset \mathbb{R} \times P_\psi$, giving an extension of the latter to an embedded J_0–holomorphic curve in $\mathbb{R} \times M$, with no boundary and with punctures asymptotic to the binding orbits of the open book; the collection of all these curves defines the $\mathbb{R}$–invariant foliation of $\mathbb{R} \times M$ shown in Figure 2. This proves Proposition 3 except for the index calculation (see Equation (3.2) below, and the ensuing discussion).

Having constructed a foliation by J_0–holomorphic curves for the data $\mathcal{H}_0 = (\xi_0, X_0, \omega_0)$, we now wish to deform the foliation as $\mathcal{H}_0$ is perturbed to the contact data $\mathcal{H}_\epsilon = (\xi_\epsilon, X_\epsilon, \omega_\epsilon)$ for sufficiently small ϵ. The crucial consequence of Proposition 2 is that we can pick an almost complex structure $J_\epsilon \in \mathcal{J}(\mathcal{H}_\epsilon)$ that is C^∞–close to J_0. By the construction of X_ϵ and ξ_ϵ, we can also assume J_ϵ equals J_0 on a neighborhood of $\mathbb{R} \times B \subset \mathbb{R} \times M$, thus the perturbation of our foliation will be a straightforward application of the implicit function theorem for holomorphic curves. This is however the point where we'll need the open book to be *planar*, as otherwise the virtual dimension of the moduli space we've constructed turns out to be too small.

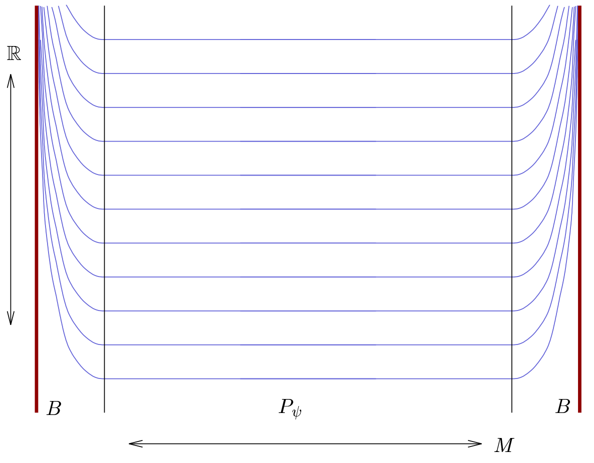

FIGURE 2. The J_0–holomorphic curves that foliate $\mathbb{R} \times M$ and project to any given open book decomposition of M. The $\mathbb{R}$–component of each curve is constant for the part within the mapping torus P_ψ, and approaches $+\infty$ near the binding.

Let us compute this dimension. Our assumptions on f and g for ρ close to zero imply that each of the binding orbits $\gamma \subset B$ has Conley-Zehnder index $\mu_{\mathrm{CZ}}(\gamma) = 1$ with respect to the natural trivialization defined by the coordinates. Let $u : \dot{\Sigma} \to \mathbb{R} \times M$ denote one of the holomorphic curves in our foliation. Since it is embedded, it sits in a moduli space whose virtual dimension (the *index* of u) equals the Fredholm index of the linearized normal Cauchy-Riemann operator, cf. [Wena]. This index is

$$\mathrm{ind}(u) = \chi(\dot{\Sigma}) + 2c_1(N_u) + \sum_{\gamma \subset B} \mu_{\mathrm{CZ}}(\gamma), \tag{3.2}$$

where $c_1(N_u)$ denotes the *relative* first Chern number of the normal bundle of u with respect to the natural trivializations of ξ_0 (which equals N_u at the asymptotic limits) defined by our coordinates near B. Assume the pages (and thus also $\dot{\Sigma}$) have genus g. We claim now that $c_1(N_u) = 0$, and thus (3.2) implies

$$\mathrm{ind}(u) = 2 - 2g.$$

Indeed, since u is always transverse to the subspaces spanned by ∂_a and X_0, one can define the normal bundle to consist of these spaces in P_ψ, and extend it into the neighborhood of B so that it always contains ∂_ϕ: there is thus a nonzero section of N_u which looks like X_0 over P_ψ and ∂_ϕ near the ends, and the latter is constant in the asymptotic trivialization.

Assume from now on that $g = 0$, so the J_0–holomorphic curves constructed above have index 2. We will apply the following strong version of the implicit function theorem, which is valid only for a special class of

punctured holomorphic spheres in dimension four; proofs of the following (in slightly more general versions) may be found in [Wen05, Wene], and a special case appeared already in [HWZ99].

Proposition 7 (Strong implicit function theorem)**.** *Assume M is any closed 3–manifold with stable Hamiltonian structure $\mathcal{H} = (\xi, X, \omega)$, $J \in \mathcal{J}(\mathcal{H})$, and*

$$u = \left(u^{\mathbb{R}}, u^M\right) : \dot{\Sigma} \to \mathbb{R} \times M$$

is a punctured J–holomorphic curve with the following properties:

(1) *u is embedded, and asymptotic to distinct simply covered periodic orbits at each puncture.*
(2) *$\dot{\Sigma}$ has genus zero.*
(3) *All asymptotic orbits are nondegenerate with odd Conley-Zehnder index.*
(4) $\mathrm{ind}(u) = 2$.

Then u is Fredholm regular and belongs to a smooth 2–parameter family of embedded curves

$$u_{(\sigma,\tau)} = \left(u_\tau^{\mathbb{R}} + \sigma, u_\tau^M\right) : \dot{\Sigma} \to \mathbb{R} \times M, \qquad (\sigma, \tau) \in \mathbb{R} \times (-1, 1)$$

with $u_{(0,0)} = u$, whose images foliate an open neighborhood of $u(\dot{\Sigma})$ in $\mathbb{R} \times M$. Moreover, the maps $u_\tau^M : \dot{\Sigma} \to M$ are all embedded and foliate an open neighborhood of $u^M(\dot{\Sigma})$ in M.

Here, *Fredholm regular*[2] means the moduli space of (unparametrized) J–holomorphic curves near u can be described as the zero set of a nonlinear Cauchy-Riemann type operator whose linearization at u is a surjective Fredholm operator. The usual implicit function theorem in a Banach manifold setting then implies that this moduli space is a smooth 2–manifold near u, thus the 2–parameter family obtained in the proposition is unique up to changes of parametrization. The reason for the nice geometric structure of the family is that if u is embedded, then all nearby curves can be described via sections of the normal bundle N_u which must satisfy a linear Cauchy-Riemann type equation. Since $\dim M = 3$, N_u is a complex line bundle, so the zeroes of its sections (with prescribed asymptotic behavior) can be counted and related to the same homotopy invariant quantities that figure into the index formula. Notably, the integer $c_1(N_u)$, defined as a relative Chern number with respect to certain special asymptotic trivializations (cf. [Wena]), satisfies the relation

$$2c_1(N_u) = \mathrm{ind}(u) - 2 + 2g + \#\Gamma_0,$$

where g is the genus of $\dot{\Sigma}$ and Γ_0 is the set of punctures at which the asymptotic orbit has even Conley-Zehnder index. Thus in the present case, this number vanishes and implies that nontrivial sections satisfying the relevant Cauchy-Riemann type equation must be zero free. It follows that this solution set can have dimension at most 2, so the linearized operator has no codimension, and all nearby curves are push-offs of zero free sections and hence disjoint. The corresponding statement about the projected maps

[2]The term *unobstructed* also appears often in the literature as a synonym.

$u_\tau^M : \dot\Sigma \to M$ follows easily from this due to $\mathbb{R}$–invariance. We should note one more important property of Fredholm regularity which will be useful presently: it allows one to apply the implicit function theorem to deform the family $u_{(\sigma,\tau)}$ under small perturbations of J.

We apply the above machinery as follows. Let $\mathcal{M}_0$ denote the connected 2–dimensional moduli space of curves that form the J_0–holomorphic open book; dividing by the natural $\mathbb{R}$–action, we have $\mathcal{M}_0/\mathbb{R} \cong S^1$. For some small $\epsilon_0 > 0$, assume $\{J_\tau\}_{\tau\in(-\epsilon_0,\epsilon_0)}$ is a smooth family of almost complex structures such that $J_\tau = J_0$ for $\tau \le 0$, $J_\tau \in \mathcal{J}(\mathcal{H}_\tau)$ for $\tau > 0$ and $J_\tau \equiv J_0$ in a neighborhood of $\mathbb{R}\times B$. Now define the moduli space

$$\widehat{\mathcal{M}} = \{(\tau, u) \mid \tau \in (-\epsilon_0, \epsilon_0),\ u \text{ is a finite energy } J_\tau\text{–holomorphic curve}\},$$

let $\mathcal{M}$ denote the connected component of $\widehat{\mathcal{M}}$ containing $\{0\}\times\mathcal{M}_0$, and for each $\tau \in (-\epsilon_0, \epsilon_0)$ define the subset $\mathcal{M}_\tau := \{(\tau, u) \in \mathcal{M}\}$. An argument by positivity of intersections as in [HWZ95, ACH05] shows that all curves $u \in \mathcal{M}_\tau$ are embedded, and no two curves in $\mathcal{M}_\tau$ (for fixed τ) can intersect. Moreover, they all have index 2 and genus zero, and have asymptotic orbits with exclusively *odd* Conley-Zehnder index. Proposition 7 now implies that each $\mathcal{M}_\tau$ is a smooth 2–manifold and $\mathcal{M}_\tau/\mathbb{R}$ is a 1–manifold that locally foliates $M \setminus B$. It also follows that $\mathcal{M}$ is a smooth 3–manifold.

We claim that for $\tau > 0$ sufficiently small, $\mathcal{M}_\tau/\mathbb{R}$ is diffeomorphic to S^1 and hits every point in $M \setminus B$. To see this, pick a loop $\ell \subset M \setminus B$ that passes once transversely through the projection of every curve in $\mathcal{M}_0$. This defines an evaluation map

$$\text{ev} : \mathcal{M}_0/\mathbb{R} \to \ell,$$

which is a diffeomorphism. Moreover since every curve in $\mathcal{M}$ can be compactified to a surface with boundary in B, the algebraic intersection number of $u \in \mathcal{M}$ with ℓ is invariant, thus every curve in $\mathcal{M}$ must intersect ℓ, and the aforementioned positivity of intersections argument implies that for any given $p \in \ell$ and $\tau \in (-\epsilon_0, \epsilon_0)$, there is at most one element of $\mathcal{M}_\tau/\mathbb{R}$ with p in its image. Now for every $u \in \mathcal{M}_0/\mathbb{R}$, pick an open neighborhood $(0, u) \in \mathcal{U}_u \subset \mathcal{M}/\mathbb{R}$ sufficiently small so that all curves in $\mathcal{U}_u$ hit ℓ transversely, exactly once. The evaluation map $\text{ev} : \mathcal{U}_u \to \ell$ is therefore well defined, and writing the projection $\pi_1 : \mathcal{M}/\mathbb{R} \to (-\epsilon_0, \epsilon_0) : (\tau, u) \mapsto \tau$, the map

$$\pi_1 \times \text{ev} : \mathcal{U}_u \to (-\epsilon_0, \epsilon_0) \times \ell$$

has nonsingular derivative at $(0, u)$, hence we can find an open neighborhood $\ell_u \subset \ell$ of $\text{ev}(u)$ and a number $\epsilon_u \in (0, \epsilon_0)$ such that $(-\epsilon_u, \epsilon_u) \times \ell_u$ is in the image of $\mathcal{U}_u$ under $\pi_1 \times \text{ev}$. The sets ℓ_u for all $u \in \mathcal{M}_0/\mathbb{R}$ now form an open covering of the compact set ℓ, so we can pick finitely many curves $u_1, \ldots, u_N \in \mathcal{M}_0$ such that $\ell = \ell_{u_1} \cup \ldots \cup \ell_{u_N}$, and set $\epsilon := \min_j\{\epsilon_{u_j}\} > 0$. Now the image of $\mathcal{U}_{u_1} \cup \ldots \cup \mathcal{U}_{u_N}$ under $\pi_1 \times \text{ev}$ contains $(-\epsilon, \epsilon) \times \ell$, hence for any $\tau \in (0, \epsilon)$ and $p \in \ell$, there is a curve in $\mathcal{M}_\tau/\mathbb{R}$ passing transversely through ℓ at p. It follows that the evaluation map extends to $\mathcal{M}_\tau/\mathbb{R}$ as a diffeomorphism $\text{ev} : \mathcal{M}_\tau/\mathbb{R} \to \ell$. Finally, we observe from Proposition 7 that the set

$$\{p \in M \setminus B \mid p \text{ is in the image of some } u \in \mathcal{M}_\tau/\mathbb{R}\}$$

is open, and it is also closed since $\mathcal{M}_\tau/\mathbb{R}$ is compact, thus it is all of $M \setminus B$.

One can perturb λ_ϵ further so that it becomes nondegenerate, and repeating the argument above then gives the desired foliation by holomorphic curves for a nondegenerate contact form.

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