GENERIC TRANSVERSALITY FOR UNBRANCHED COVERS OF CLOSED PSEUDOHOLOMORPHIC CURVES

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Abstract. We prove that in closed almost complex manifolds of any dimension, generic perturbations of the almost complex structure suffice to achieve transversality for all unbranched multiple covers of simple pseudoholomorphic curves with deformation index zero. A corollary is that the Gromov-Witten invariants (without descendants) of symplectic 4-manifolds can always be computed as a signed and weighted count of honest J-holomorphic curves for generic tame J: in particular, each such invariant is an integer divided by a weighting factor that depends only on the divisibility of the corresponding homology class. The transversality proof is based on an analytic perturbation technique, originally due to Taubes.

Contents

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1. Introduction

The Gromov-Witten invariants of closed symplectic manifolds are defined in principle by counting J-holomorphic curves for generic tame almost complex structures J. One of the main technical hurdles in this definition is that moduli spaces of J-holomorphic curves are not generally manifolds of the "expected" dimension unless multiply covered curves can be excluded; thus in practice, the definition usually requires more sophisticated techniques such as virtual cycles, abstract multivalued perturbations, or stabilizing divisors, see e.g. [FO99, LT98, Rua99, Sie, CM07, IPa, HWZ].

It is nonetheless interesting to ask under what circumstances the "classical" technique of perturbing J generically suffices for a complete description of moduli spaces of multiply covered curves. Results of this nature are desirable for several reasons: one is that the resulting definition of the Gromov-Witten invariants is simpler to understand and to apply. Another is that the relationship between simple curves and their multiple covers can reveal nontrivial relations among Gromov-Witten invariants that cannot be seen by more abstract techniques; one example of this phenomenon is the Gopakumar-Vafa conjecture on symplectic Calabi-Yau 3-folds, see [GV, BP01, BP08, IPb]. While moduli spaces of multiply covered curves cannot generally achieve regularity in the usual sense, it is sometimes enough to show that they are *as regular as possible.* A simple J-holomorphic curve u with deformation index 0 is called "super-rigid" if, roughly speaking, the set of all covers of u is an open subset in the moduli space of all J-holomorphic curves (see §1.1 for a more precise definition), so in particular, no sequence of curves geometrically distinct from u can converge to any cover of u. The index relations between simple J-holomorphic curves and their multiple covers make the following conjecture plausible:[1]

Conjecture 1.1. *On any closed symplectic manifold (M, ω) of real dimension at least four, there exists a Baire subset $\mathcal{J}_{\text{reg}}$ in the space of smooth ω-tame almost complex structures such that for all $J \in \mathcal{J}_{\text{reg}}$, every closed, connected and simple J-holomorphic curve with deformation index 0 is super-rigid.*

Some special cases of this conjecture have been proved previously by Lee-Parker [LP07, LP12] and Eftekhary [Eft16]. The techniques used in the present paper are related to those of [LP07, LP12], which also play a role in the announced solution by Ionel and Parker to the Gopakumar-Vafa conjecture [IPb].

For an *unbranched* cover of a simple curve, the super-rigidity condition is equivalent to the usual notion of *Fredholm regularity*, and our main result (stated as Theorem 1.3 below) is that this can always be achieved by choosing J generically. This may be seen as an initial step toward a proof of Conjecture 1.1 in full generality. While the result holds in all dimensions, its consequences are especially interesting in dimension four: as we will show in §1.2, it implies that Gromov-Witten invariants without descendants in this setting can be computed without the aid of domain-dependent or inhomogeneous perturbations, and they therefore satisfy integrality conditions that are not apparent from the more general definitions; see Theorem 1.8 and Corollary 1.9.

[1] After this article was submitted for publication, the second author produced a preprint [Wenb] that proves Conjecture 1.1 in all dimensions greater than four, together with a substantial generalization of Theorem 1.3, using different techniques based on the Sard-Smale theorem and representation theory.

Our proof is quite different from the methods that symplectic topologists typically use to establish transversality: it does not involve the Sard-Smale theorem, but is instead based on an analytic perturbation theory technique introduced by Taubes in his definition of the Gromov invariants of symplectic 4-manifolds [Tau96b]. It works in the symplectic category in all dimensions greater than two, but it does not work in the algebraic or complex category, i.e. if we start with an integrable complex structure J, then our perturbation to achieve regularity will *always* make J nonintegrable (see Remark 2.1). The method also is not strictly limited to unbranched covers: for any given cover of a simple curve with index 0, we will show how to perturb J such that the super-rigidity condition is achieved for the given cover. Since spaces of unbranched covers do not have moduli, this suffices to prove our main result, and it also lends hope that similar methods could be used to prove Conjecture 1.1 in full generality, though at present it is not clear whether the kind of perturbation we define can achieve super-rigidity for all branched covers at once in a space with nontrivial moduli.[2]

We aim in future work to prove similar results for covers of finite-energy punctured J-holomorphic curves in symplectic cobordisms, which should have interesting applications in Symplectic Field Theory [EGH00] and Embedded Contact Homology [Hut14]. A few special cases of super-rigidity in the punctured case have previously been observed by the second author [Wen10], as well as work of Fabert [Fab13], and unpublished work of Hutchings [Hut]; those examples were restricted to dimension four, but the methods introduced in the present paper have no such restrictions.

1.1. **The main result.** Assume (M, J_{fix}) is an almost complex manifold of dimension $2n \geq 4$, $\mathcal{U} \subset M$ is an open subset with compact closure, and

$$\mathcal{J}(M\,;\mathcal{U}, J_{\text{fix}})$$

denotes the space of smooth almost complex structures on M that match J_{fix} outside of $\mathcal{U}$, with its natural C^∞-topology. If M also carries a symplectic structure ω for which J_{fix} is ω-tame or ω-compatible, we will denote the corresponding spaces of tame/compatible almost complex structures matching J_{fix} outside $\mathcal{U}$ by

$$\mathcal{J}^{\text{tame}}(M, \omega\,;\mathcal{U}, J_{\text{fix}}),\ \ \mathcal{J}^{\text{comp}}(M, \omega\,;\mathcal{U}, J_{\text{fix}}) \subset \mathcal{J}(M\,;\mathcal{U}, J_{\text{fix}}).$$

Remark 1.2. The existence of a symplectic form on M is not required for any of the arguments in this paper, but since it is important in applications, we will generally assume at least that (M, ω) is symplectic and all almost complex structures under consideration are ω-tame. Note that $\mathcal{J}^{\text{tame}}(M, \omega\,;\mathcal{U}, J_{\text{fix}})$ is an open subset of $\mathcal{J}(M\,;\mathcal{U}, J_{\text{fix}})$, thus all statements made about $\mathcal{J}^{\text{tame}}(M, \omega\,;\mathcal{U}, J_{\text{fix}})$ will have obvious analogues for $\mathcal{J}(M\,;\mathcal{U}, J_{\text{fix}})$.

With Remark 1.2 in mind, from now on we fix a symplectic form ω on M and assume J_{fix} is ω-tame. Given $J \in \mathcal{J}^{\text{tame}}(M, \omega\,;\mathcal{U}, J_{\text{fix}})$, a closed connected Riemann surface (Σ, j)

[2] A preliminary version of this paper (under a different title) claimed a proof of Conjecture 1.1 using similar techniques, but this argument had gaps that we have thus far been unable to fill. See Remark 2.7.

and a J-holomorphic curve[3] $u : (\Sigma, j) \to (M, J)$, the **index** of u is the integer

$$\operatorname{ind}(u) = (n-3)\chi(\Sigma) + 2c_1(u), \tag{1.1}$$

where we abbreviate $c_1(u) := \langle c_1(TM, J), [u]\rangle$, $[u] := u_*[\Sigma] \in H_2(M)$. A closed and connected J-holomorphic curve $\tilde{u} : (\widetilde{\Sigma}, \tilde{j}) \to (M, J)$ is said to be a (d-fold) **multiple cover** of u if $\tilde{u} = u \circ \varphi$ for some holomorphic map $\varphi : (\widetilde{\Sigma}, \tilde{j}) \to (\Sigma, j)$ of degree $d \geq 2$, and u is called **simple** if it is nonconstant and is not a multiple cover of any other curve. The map $\varphi : \widetilde{\Sigma} \to \Sigma$ is generally a branched cover, and we call it **unbranched** (and $\tilde{u}$ an *unbranched cover of* u) if it is an honest covering map, meaning its set of branch points is empty.

We say that the curve $u : \Sigma \to M$ is **Fredholm regular** if a neighborhood of u in the moduli space of unparametrized J-holomorphic curves is cut out transversely, see e.g. [Wena, §4.3]. In this paper we will mainly deal with immersed curves, for which a precise definition of regularity is easier to state: suppose $u : \Sigma \to M$ is immersed and denote its complex normal bundle by $N_u \to \Sigma$. The linearized Cauchy-Riemann operator associated to u is the real-linear first-order differential operator

$$\mathbf{D}_u : \Gamma(u^*TM) \to \Omega^{0,1}(\Sigma, u^*TM) : \eta \mapsto \nabla\eta + J(u) \circ \nabla\eta \circ j + (\nabla_\eta J) \circ Tu \circ j, \tag{1.2}$$

where ∇ is any choice of symmetric connection on M. We define the **normal Cauchy-Riemann operator** at u as the restriction of $\mathbf{D}_u$ to sections of N_u, composed with the projection $\pi_N : u^*TM \to N_u$, hence

$$\mathbf{D}_u^N = \pi_N \circ \mathbf{D}_u|_{\Gamma(N_u)} : \Gamma(N_u) \to \Omega^{0,1}(\Sigma, N_u).$$

This is also a Cauchy-Riemann type operator, so its extension to any reasonable Banach space completions such as

$$\mathbf{D}_u^N : W^{k,p}(N_u) \to W^{k-1,p}(\overline{\operatorname{Hom}}_{\mathbb{C}}(T\Sigma, N_u)) \tag{1.3}$$

for $k \in \mathbb{N}$ and $p > 1$ is a Fredholm operator, and elliptic regularity implies that its kernel and cokernel do not depend on the choices k and p. The curve u is then Fredholm regular if and only if the linear map (1.3) is surjective. In the present paper, we will sometimes deal with multiple covers $\tilde{u} = u \circ \varphi$ for which u is immersed but φ may have branch points, in which case $\mathbf{D}_{\tilde{u}}^N$ can naturally be defined as a Cauchy-Riemann type operator on $N_{\tilde{u}} := \varphi^* N_u$. The curve u is then called **super-rigid** if it is immersed with index 0 and $\mathbf{D}_{\tilde{u}}^N$ is injective for every cover $\tilde{u}$ of u. Note that if $\varphi : \widetilde{\Sigma} \to \Sigma$ has degree $d \in \mathbb{N}$ and $Z(d\varphi) \geq 0$ denotes the number of branch points of φ counted with multiplicities, then the Riemann-Hurwitz formula

$$-\chi(\widetilde{\Sigma}) + d\chi(\Sigma) = Z(d\varphi) \tag{1.4}$$

implies

$$\operatorname{ind}(\tilde{u}) = d \cdot \operatorname{ind}(u) - (n-3)Z(d\varphi),$$

[3]When we use the word "curve" to describe $u : (\Sigma, j) \to (M, J)$, we mean that (Σ, j) is a smooth (non-nodal) Riemann surface and u is a smooth map, or in some cases an equivalence class of smooth maps up to parametrization (this will be clear from context). By default this excludes nodal curves, and when we do mean "nodal curve" we will make this explicit. This usage is common in symplectic topology but may differ from conventions in the algebraic geometry literature.

hence unbranched covers of immersed index 0 curves are also immersed with index 0, and super-rigidity for unbranched covers is therefore the same as Fredholm regularity.

Here is our main result.

Theorem 1.3. *Assume (M, ω) is a symplectic manifold*[4] *with tame almost complex structure J_{fix}, and $\mathcal{U}$ is an open subset with compact closure. Then there exists a Baire subset $\mathcal{J}_{\text{reg}} \subset \mathcal{J}^{\text{tame}}(M, \omega\,; \mathcal{U}, J_{\text{fix}})$ such that for every $J \in \mathcal{J}_{\text{reg}}$, all unbranched covers of simple closed J-holomorphic curves of index 0 contained fully in $\mathcal{U}$ are Fredholm regular.*

Moreover, if J_{fix} is ω-compatible, then there is a Baire subset $\mathcal{J}_{\text{reg}} \subset \mathcal{J}^{\text{comp}}(M, \omega\,; \mathcal{U}, J_{\text{fix}})$ such that for every $J \in \mathcal{J}_{\text{reg}}$, all unbranched covers of embedded *closed J-holomorphic curves of index 0 contained fully in $\mathcal{U}$ are Fredholm regular.*

Remark 1.4. We do not know whether the restriction to embedded curves in the ω-compatible case can be relaxed; the reason is explained in Remark 3.3. This is in any case only a restriction in dimension four, since embeddedness is a generic property of holomorphic curves in higher dimensions (see e.g. [Wena, §4.6] or [OZ09]). In the ω-tame case, our argument works for all immersed curves with distinct transverse self-intersections, which is a generic property even in dimension four.

The next two remarks draw attention to generalizations of Theorem 1.3 that might naturally be expected to hold but do *not* follow from our arguments, and in some cases are actually false.

Remark 1.5. The standard transversality results as in [MS04, Wena] for simple J-holomorphic curves have straightforward extensions to generic 1-paramater families $\{J_\tau\}$ of almost complex structures, showing in essence that the space of pairs

$$\{(\tau, u) \mid u \text{ is simple and } J_\tau\text{-holomorphic}\}$$

is a manifold of dimension $\operatorname{ind}(u) + 1$. This means that all simple J_τ-holomorphic curves are regular for almost every τ, but there may be birth-death bifurcations at a discrete set of parameter values. The work of Taubes [Tau96a] shows that when multiple covers are allowed, more general types of bifurcations must be considered, so e.g. the extension of the usual results for simple curves to unbranched covers of index 0 curves is not at all straightforward. We will not prove anything in this paper about generic 1-parameter families of data.

Remark 1.6. The standard results for simple curves do not require the curves to be *fully* contained in the perturbation domain $\mathcal{U}$ in order to achieve transversality; it suffices rather that they should intersect $\mathcal{U}$ *somewhere*, the key point being that there is an injective point mapped into $\mathcal{U}$. Our methods on the other hand work only for curves that are fully contained in $\mathcal{U}$, and we do not know whether this assumption can be weakened. The reason for this is discussed in Remark 2.1. In this sense, Theorem 1.3 seems to represent a fundamentally different phenomenon from the usual transversality results for simple curves.

[4]As indicated in Remark 1.2, the first statement in the theorem could also be stated without reference to any symplectic structure, producing a Baire subset of $\mathcal{J}(M\,; \mathcal{U}, J_{\text{fix}})$.

1.2. **Application to Gromov-Witten theory.** In the results of this section, the words "for generic J..." should be understood to mean that there exists a Baire subset of the appropriate space of almost complex structures for which the statement is true.

Let $\mathcal{M}_{g,m}(A,J)$ denote the moduli space of smooth unparametrized J-holomorphic curves in M with genus g and m marked points in the homology class $A \in H_2(M)$; the precise definition will be recalled in the discussion below. We denote the natural evaluation map by

$$\mathrm{ev} : \mathcal{M}_{g,m}(A,J) \to M^m,$$

and let

$$\mathcal{M}^*_{g,m}(A,J) \subset \mathcal{M}_{g,m}(A,J)$$

denote the open subset consisting of simple curves. For any integer $m \geq 0$, the m-point Gromov-Witten invariant

$$\mathrm{GW}^{(M,\omega)}_{g,m,A} : H^*(M)^{\otimes m} \to \mathbb{Q}$$

is defined morally by counting intersections of the evaluation map with cycles in M^m determined by an m-tuple of cohomology classes. The standard definition of these invariants in [RT97] for semipositive symplectic manifolds (which includes all symplectic 4-manifolds) requires generic inhomogeneous perturbations to the nonlinear Cauchy-Riemann equation, thus breaking the symmetry inherent in multiply covered curves. We will now show that when $\dim_{\mathbb{R}} M = 4$, these invariants can also be computed by simpler means that do not break the symmetry. Recall from [MS04, §6.5] that for any subset $\mathcal{M}^* \subset \mathcal{M}_{g,m}(A,J)$, the restriction $\mathrm{ev} : \mathcal{M}^* \to M^m$ is said to be a **pseudocycle of dimension** $d \geq 0$ if $\mathcal{M}^*$ is a smooth d-dimensional manifold and $\overline{\mathcal{M}}_{g,m}(A,J) \setminus \mathcal{M}^*$ can be covered by subsets on which ev factors through a smooth map to M^m from a manifold of dimension at most $d-2$. In this case one can define integer-valued intersection products of ev with homology classes in M^m. The following proposition for the case $m \geq 1$ is presumably not a new result, but we are not aware of any proof of it in the current literature; ours will require only the standard transversality results for simple curves.

Proposition 1.7. *Assume (M,ω) is a closed symplectic 4-manifold. Then for generic ω-compatible or tame almost complex structures J and for every $A \in H_2(M)$ and every pair of nonnegative integers (g,m) satisfying $-(2-2g)+2c_1(A) > 0$ and $m \geq 1$, the evaluation map $\mathrm{ev} : \mathcal{M}^*_{g,m}(A,J) \to M^m$ on the set of simple curves is a pseudocycle of dimension $-(2-2g)+2c_1(A)+2m$. The corresponding m-point Gromov-Witten invariant can thus be computed as an intersection number*

$$\mathrm{GW}^{(M,\omega)}_{g,m,A}(\alpha_1,\ldots,\alpha_m) = \left[\mathrm{ev}|_{\mathcal{M}^*_{g,m}(A,J)}\right] \cdot \left(\mathrm{PD}(\alpha_1) \times \ldots \times \mathrm{PD}(\alpha_m)\right),$$

and in particular, its values are always integers.

The picture for the 0-point invariants with $g \geq 1$ is somewhat different, as it turns out that multiply covered curves cannot be avoided in this case, but only *unbranched* covers need be considered. The arguments behind Proposition 1.7 thus combine with Theorem 1.3 to give the following more novel result.

Theorem 1.8. *For generic ω-tame almost complex structures J on a closed symplectic 4-manifold (M,ω), the set of index 0 curves satisfying any given bound on their genus and area is finite, and all of them are Fredholm regular.*

We should again caution the reader that we do not know whether the generic J in Theorem 1.8 can be chosen to be *compatible* with ω (see Remark 1.4), though one can require this if one is only interested in covers of embedded curves (as in [Tau96a, Tau96b]). Choosing J tame is in any case good enough to compute Gromov-Witten invariants. In order to state the main corollary, we can associate to any integral homology class $A \in H_2(M)$ in a symplectic manifold (M, ω) its **symplectic divisibility**

$$d_\omega(A) \in \mathbb{N},$$

defined as the product of the finite set of integers $k \in \mathbb{N}$ such that $A = kB$ for some primitive class $B \in H_2(M)$ with $\omega(B) > 0$.

Corollary 1.9. *Suppose (M, ω) is a closed symplectic 4-manifold and $A \in H_2(M)$ and $g \in \mathbb{N}$ satisfy $-(2-2g) + 2c_1(A) = 0$. Then the 0-point Gromov-Witten invariant can be computed for generic tame almost complex structures J as a signed and weighted count of finitely many J-holomorphic curves*

$$\mathrm{GW}_{g,0,A}^{(M,\omega)} = \sum_{u \in \mathcal{M}_{g,0}(A,J)} \frac{\sigma(u)}{|\operatorname{Aut}(u)|},$$

where for each curve u, $\sigma(u) \in \{-1, 1\}$ is determined by an orientation of the determinant line bundle, and $\operatorname{Aut}(u)$ denotes the automorphism group of u. In particular, the number $\mathrm{GW}_{0,0,A}^{(M,\omega)}$ is always an integer, while for $g \geq 1$, $d_\omega(A) \cdot \mathrm{GW}_{g,0,A}^{(M,\omega)}$ is an integer.

In order to prepare for the proofs of these results, let us recall the definitions of the relevant moduli spaces. Given integers $g, m \geq 0$ and a homology class $A \in H_2(M)$, the moduli space of **unparametrized J-holomorphic curves** $\mathcal{M}_{g,m}(A, J)$ can be defined as the set of equivalence classes of tuples (Σ, j, Θ, u) where (Σ, j) is a closed connected Riemann surface of genus g, $\Theta \subset \Sigma$ is an ordered set of m distinct points (the **marked points**), and $u : (\Sigma, j) \to (M, J)$ is a J-holomorphic map satisfying $[u] = A$, with equivalence defined by $(\Sigma, j, \Theta, u) \sim (\Sigma', \psi^* j, \psi^{-1}(\Theta), u \circ \psi)$ for diffeomorphisms $\psi : \Sigma' \to \Sigma$. The **automorphism group** $\operatorname{Aut}(u)$ of $[(\Sigma, j, \Theta, u)] \in \mathcal{M}_{g,m}(A, J)$ is the group of biholomorphic diffeomorphisms $\psi : (\Sigma, j) \to (\Sigma, j)$ that fix each of the marked points and satisfy $u = u \circ \psi$; it is always finite, and is trivial whenever u is simple. The **Gromov compactification** of $\mathcal{M}_{g,m}(A, J)$ is the space $\overline{\mathcal{M}}_{g,m}(A, J)$ of (equivalence classes of) **stable nodal curves** $(S, j, \Theta, \Delta, u)$, where now S may be disconnected, and the original data are augmented by an unordered set of distinct points in $S \setminus \Theta$, arranged into unordered pairs

$$\Delta = \{\{\hat{z}_1, \check{z}_1\}, \ldots, \{\hat{z}_r, \check{z}_r\}\},$$

such that $u(\hat{z}_i) = u(\check{z}_i)$ for each $i = 1, \ldots, r$. We call the pairs $\{\hat{z}_i, \check{z}_i\}$ **nodes**, and each individual $\hat{z}_i$ or $\check{z}_i \in S$ a **nodal point**. The curves in $\overline{\mathcal{M}}_{g,m}(A, J)$ are required to have **arithmetic genus** g, which means that the surface obtained from S by performing connected sums at all matched pairs of nodal points is a closed connected surface of genus g. The stability condition requires that any component of $S \setminus (\Theta \cup \Delta)$ on which u is constant should have negative Euler characteristic. With this condition, $\overline{\mathcal{M}}_{g,m}(A, J)$ can be given a natural topology as a metrizable Hausdorff space, and it is compact whenever J is tamed by a symplectic form. A definition of the topology may be found e.g. in [BEH+03]; for sequences in $\mathcal{M}_{g,m}(A, J)$, it amounts to the notion of C^∞-convergence for j and u after a choice of parametrization for which all domains and marked point sets are identified. Curves $[(S, j, \Theta, \Delta, u)] \in \overline{\mathcal{M}}_{g,m}(A, J)$ with $\Delta = \emptyset$ can equivalently be regarded as elements of $\mathcal{M}_{g,m}(A, J)$, and are thus called **smooth** curves to distinguish them from nodal curves. The evaluation map is defined by

$$\mathrm{ev} : \mathcal{M}_{g,m}(A, J) \to M \times \ldots \times M : [(\Sigma, j, (\zeta_1, \ldots, \zeta_m), u)] \mapsto (u(\zeta_1), \ldots, u(\zeta_m)),$$

and it extends to a continuous map on $\overline{\mathcal{M}}_{g,m}(A, J)$.

When there is no danger of confusion, we shall sometimes abuse notation by denoting equivalence classes $[(\Sigma, j, \Theta, u)] \in \mathcal{M}_{g,m}(A, J)$ or $[(S, j, \Theta, \Delta, u)] \in \overline{\mathcal{M}}_{g,m}(A, J)$ simply by $u \in \mathcal{M}_{g,m}(A, J)$ or $u \in \overline{\mathcal{M}}_{g,m}(A, J)$ respectively, and we will refer to the restriction of a nodal curve $[(S, j, \Theta, \Delta, u)]$ to any connected component of its domain S as a **smooth component** of u. Recall that $\mathcal{M}_{g,0}(A, J)$ has **virtual dimension** equal to the index of any curve $u \in \mathcal{M}_{g,0}(A, J)$.

It will be useful to recall certain index relations for degenerating sequences of holomorphic curves. Suppose $\dim_{\mathbb{R}} M = 2n$, and $[(\Sigma, j_k, u_k)] \in \mathcal{M}_{g,0}(A, J)$ is a sequence converging to a stable nodal curve $[(S, j_\infty, \Delta, u_\infty)] \in \overline{\mathcal{M}}_{g,0}(A, J)$ with smooth components

$$\left\{[(S_i, j^i_\infty, u^i_\infty)] \in \mathcal{M}_{g_i}(A_i, J)\right\}_{i=1,\ldots,r}.$$

Then if $N_i := |S_i \cap \Delta| \geq 1$ denotes the number of nodal points on S_i for $i = 1, \ldots, r$, we have $\chi(\Sigma) = \sum_i [\chi(S_i) - N_i]$, so the index formula (1.1) gives

$$\operatorname{ind}(u_k) = \sum_{i=1}^{r} \left[\operatorname{ind}(u^i_\infty) - (n-3)N_i\right]. \tag{1.5}$$

Note that by the stability condition, we have

$$\chi(S_i) - N_i < 0 \quad \text{whenever } A_i = 0. \tag{1.6}$$

If $A_i \neq 0$, then $u^i_\infty = v^i \circ \varphi^i$ for some simple curve v^i and holomorphic map φ^i of degree $d_i \geq 1$ with $Z(d\varphi^i) \geq 0$ branch points, and the Riemann-Hurwitz formula combined with (1.1) gives

$$\operatorname{ind}(u^i_\infty) = d_i \cdot \operatorname{ind}(v^i) - (n-3)Z(d\varphi^i). \tag{1.7}$$

Proof of Proposition 1.7. Assume J is chosen so that all somewhere injective curves are Fredholm regular. Then $\mathcal{M}^*_{g,m}(A, J)$ is a manifold of real dimension $\operatorname{ind}(u) + 2m$ for any $u \in \mathcal{M}^*_{g,m}(A, J)$. The index relations (1.5) and (1.7) imply that if $u_k \in \mathcal{M}^*_{g,m}(A, J)$ is a sequence of simple curves with $\operatorname{ind}(u_k) > 0$ converging to a nodal curve u_∞, then the nonconstant components of u_∞ cover simple curves whose indices add up to at most $\operatorname{ind}(u_k) - 2$. More concretely, if u_∞ has smooth components $u^1_\infty, \ldots, u^r_\infty$, each u^i_∞ having $N_i \geq 1$ nodal points, then the 4-dimensional case of (1.5) together with the stability condition (1.6) implies

$$\operatorname{ind}(u_k) \geq \sum_{\{i \mid u^i_\infty \neq \text{const}\}} \left[\operatorname{ind}(u^i_\infty) + N_i\right], \tag{1.8}$$

with equality if and only if u_∞ has no constant (i.e. "ghost") components. This shows in particular that

$$\operatorname{ind}(u_k) \geq 2 + \sum_{\{i \mid u_\infty^i \neq \text{const}\}} \operatorname{ind}(u_\infty^i). \tag{1.9}$$

Now by (1.7) in the case $n = 2$, we see that if u_∞^i is a d_i-fold cover of a simple curve v^i, then $\operatorname{ind}(u_\infty^i) \geq d_i \operatorname{ind}(v^i)$, with equality if and only if the cover is unbranched. Since $\operatorname{ind}(v^i) \geq 0$ by genericity, this implies that each smooth component u_∞^i has index at least two less than $\operatorname{ind}(u_k)$. On the other hand, if $u_\infty = \lim u_k$ is a smooth curve that is a d-fold cover $v \circ \varphi$ of some simple curve v, then (1.7) gives

$$\operatorname{ind}(u_\infty) = d \cdot \operatorname{ind}(v) + Z(d\varphi) \geq d \cdot \operatorname{ind}(v),$$

and since $\operatorname{ind}(u_\infty) > 0$ by assumption and the index is always even, we conclude $\operatorname{ind}(v) \leq \operatorname{ind}(u_\infty) - 2$ unless $d = 1$. These relations imply the pseudocycle condition. □

Proof of Theorem 1.8 and Corollary 1.9. Applying the index relations as in the proof of Proposition 1.7 above, we find that the worst case scenario for a degenerating sequence of index 0 curves $u_k \to u_\infty$ is that u_∞ is an *unbranched* cover of a simple index 0 curve. For generic tame J, Theorem 1.3 implies that the latter is regular, hence all curves in $\overline{\mathcal{M}}_{g,0}(A, J)$ are smooth and regular, and therefore isolated due to the implicit function theorem. The integrality condition in Corollary 1.9 arises from the observation that whenever $u \in \mathcal{M}_{g,0}(A, J)$ is a d-fold cover of a simple curve $v \in \mathcal{M}_{g',0}(B, J)$, we necessarily have $A = dB$ and $\omega(B) > 0$, and the order of the automorphism group $\operatorname{Aut}(u)$ is an integer dividing d. For $g = 0$ the integrality result is stronger, because the Riemann-Hurwitz formula forbids the existence of unbranched covers with genus 0, hence every curve in $\mathcal{M}_{0,0}(A, J)$ is simple. □

1.3. **Outline of the paper.** The main steps in the proof of Theorem 1.3 will be explained in §2, modulo three technical results concerning (1) the nonlinear problem, (2) the linear problem, and (3) obstruction theory. The remainder of the paper will then be concerned with these three technical results: the nonlinear result in §3, the linear result in §5 and §6, and the obstruction theoretic result (which is only needed for the case $\dim_{\mathbb{R}} M \geq 6$) in §4. These are followed by a brief appendix recalling the essential result from analytic perturbation theory that is needed in §6.

A brief remark on terminology. Since many important objects in this paper do not carry natural complex structures, our formulas for dimensions and Fredholm indices generally give the *real* dimension unless otherwise noted, even in cases where this number is always even. The major exceptions are the bundles u^*TM and N_u associated to a J-holomorphic curve $u : (\Sigma, j) \to (M, J)$; these are naturally complex vector bundles and are described in terms of their *complex* rank.

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2. The main argument

The goal of this section will be to reduce the proof of Theorem 1.3 to a sequence of three technical results to be proved in later sections.

2.1. **Unbranched tori in dimension four.** Before diving into the details on Theorem 1.3, it may be instructive to recall the argument of Taubes which has inspired the present approach to regularity for multiple covers. The Gromov invariants were defined in [Tau96a, Tau96b] as certain counts of holomorphic curves in symplectic 4-manifolds, including both embedded curves and unbranched covers of embedded holomorphic tori with index 0. In order to achieve transversality for the multiple covers, Taubes argued in [Tau96b, §7(b)] as follows. Assume $u : \mathbb{T}^2 \to M$ is an embedded J-holomorphic torus with index 0, $\varphi : \mathbb{T}^2 \to \mathbb{T}^2$ is a holomorphic covering map and $\tilde{u} = u \circ \varphi$. Then the normal Cauchy-Riemann operator for $\tilde{u}$ can be identified with an operator of the form

$$\mathbf{D} = \bar{\partial} + A : C^\infty(\mathbb{T}^2, \mathbb{C}) \to C^\infty(\mathbb{T}^2, \mathbb{C}),$$

where $\bar{\partial} = \partial_s + i\partial_t$ in holomorphic coordinates $s + it$ on $\mathbb{T}^2$ and $A \in C^\infty(\mathbb{T}^2, \mathrm{End}_{\mathbb{R}}(\mathbb{C}))$. Taubes shows that one can always perturb the ambient almost complex structure along u such that $\mathbf{D}$ becomes

$$\mathbf{D}_\tau \eta := \mathbf{D}\eta + \tau\beta\bar{\eta}$$

for some $\beta \in C^\infty(\mathbb{T}^2, \mathbb{C}^*)$ and a small parameter $\tau \in \mathbb{R}$. This perturbation of the linear operator is required to be complex-antilinear, and it must never vanish, but in contrast to the standard transversality arguments as in [MS04], it is allowed to be arbitrarily symmetric, so in particular the fact that $\tilde{u}$ is a multiple cover poses no difficulty here. The main challenge is now to show that this perturbed operator will always be injective for sufficiently small $\tau > 0$. The argument for this involves two main ingredients.

(1) Bochner-Weitzenböck technique: The following argument shows that $\mathbf{D}_\tau$ must be injective for all $\tau \gg 0$. Fix the standard real-valued L^2-inner product on $C^\infty(\mathbb{T}^2, \mathbb{C})$ and let $\mathbf{D}^*$ and $\mathbf{D}_\tau^*$ denote the formal adjoints of $\mathbf{D}$ and $\mathbf{D}_\tau$ respectively; explicitly, we have $\mathbf{D}^* = \partial + A^*$ and $\mathbf{D}_\tau^*\eta = \mathbf{D}^*\eta + \tau\bar{\beta}\bar{\eta}$, where $\partial = \partial_s - i\partial_t$ and $A^* \in C^\infty(\mathbb{T}^2, \mathrm{End}_{\mathbb{R}}(\mathbb{C}))$ denotes the pointwise real-linear transpose of A. From these relations, one obtains a Weitzenböck formula,

$$\mathbf{D}_\tau^*\mathbf{D}_\tau\eta = \mathbf{D}^*\mathbf{D}\eta + \tau L\eta + \tau^2|\beta|^2\eta, \tag{2.1}$$

where $L \in C^\infty(\mathbb{T}^2, \mathrm{End}_{\mathbb{R}}(\mathbb{C}))$ is the zeroth-order real-linear operator $L\eta = \bar{\beta}\overline{A\eta} + A^*\beta\bar{\eta} - (\partial\beta)\bar{\eta}$. The crucial point in (2.1) is that $\mathbf{D}_\tau^*\mathbf{D}_\tau\eta$ and $\mathbf{D}^*\mathbf{D}\eta$ differ only by a zeroth-order term—the complex-*anti*linear nature of the perturbation causes all other derivatives of η

to cancel. For all $\eta \in C^\infty(\mathbb{T}^2, \mathbb{C})$, we then have

$$\begin{aligned} \|\mathbf{D}_\tau \eta\|_{L^2}^2 &= \langle \eta, \mathbf{D}_\tau^* \mathbf{D}_\tau \eta \rangle_{L^2} = \left\langle \eta, \mathbf{D}^*\mathbf{D}\eta + \tau L\eta + \tau^2 |\beta|^2 \eta \right\rangle_{L^2} \\ &= \|\mathbf{D}\eta\|_{L^2}^2 + \tau \langle \eta, L\eta \rangle_{L^2} + \tau^2 \langle \eta, |\beta|^2 \eta \rangle_{L^2} \\ &\geq \|\mathbf{D}\eta\|_{L^2}^2 + (c\tau^2 - c'\tau)\|\eta\|_{L^2}^2 \end{aligned} \tag{2.2}$$

for some constants $c, c' > 0$. Here we have used the fact that β is nowhere zero so that $\langle \eta, |\beta|^2 \eta \rangle_{L^2} \geq c\|\eta\|_{L^2}^2$.

(2) Analytic perturbation theory: Regard $\mathbf{D}_\tau$ as a complex-linear operator $H^1(\mathbb{T}^2, \mathbb{C}) \to L^2(\mathbb{T}^2, \mathbb{C})$, or more accurately on the complexifications of these two spaces. Then $\mathbf{D}_\tau$ depends analytically on the parameter $\tau \in \mathbb{C}$, so the set of all $\tau \in \mathbb{C}$ for which $\mathbf{D}_\tau$ is not an isomorphism looks locally like the zero-set of an analytic function on $\mathbb{C}$, i.e. $\mathbf{D}_\tau$ has nontrivial kernel either for all τ or only for a discrete subset. (A proof of this fact is given in the Appendix.) Step (1) implies that it is the latter, not the former.

Remark 2.1. The first step described above depends crucially on the following two properties of the perturbation, both of which lend a distinctive flavor to our main result:

(1) The perturbation from $\mathbf{D}$ to $\mathbf{D}_\tau$ must be *antilinear*, otherwise the Weitzenböck formula (2.1) does not hold. This implies that, in general, the generic almost complex structures for which our transversality result holds can *never* be expected to be integrable.
(2) The perturbation must also be *nowhere zero* so that $\|\eta\|_{L^2}$ can be bounded below via $\langle \eta|\beta|^2\eta\rangle_{L^2}$ in (2.2). This is why our proof of Theorem 1.3 does not work for curves that only pass through the perturbation domain rather than being fully contained in it (see Remark 1.6).

We will see that both of these features also appear in the general case to be discussed below.

Remark 2.2. A version of the Bochner-Weitzenböck technique described above has also appeared in the work of Lee and Parker on Kähler surfaces with positive geometric genus, see [LP07, Proposition 8.6]. In their more specialized setting, the terms linear in τ vanish for geometric reasons, thus one obtains super-rigidity for all (not necessarily small) perturbations of the type that they consider, without any need to apply analytic perturbation theory.

2.2. **Three technical results for the general case.** We now describe what is required in order to generalize the argument of Taubes sketched above.

The first technical result we will need describes the perturbation of the normal Cauchy-Riemann operator realized by a certain class of perturbations to the almost complex structure. Working under the assumptions of Theorem 1.3, suppose $u : (\Sigma, j) \to (M, J)$ is an immersed J-holomorphic curve with image fully contained in $\mathcal{U}$, choose a tangent/normal splitting $u^*TM = T_u \oplus N_u$ with $T_u = \operatorname{im} du$, and abbreviate the complex vector bundles

$$E := N_u, \qquad F := \overline{\operatorname{Hom}}_{\mathbb{C}}(T\Sigma, N_u) = T^{0,1}\Sigma \otimes E,$$

both of which have rank $m := n - 1$. The normal Cauchy-Riemann operator $\mathbf{D}_u^N$ then maps sections of E to sections of F. Suppose $\{J_\tau \in \mathcal{J}^{\text{tame}}(M, \omega\,; \mathcal{U}, J_{\text{fix}})\}_{\tau \in (-\epsilon, \epsilon)}$ is a smooth 1-parameter family of almost complex structures such that

$$J_0 \equiv J, \quad \text{and} \quad J_\tau|_{T_u} \equiv J|_{T_u} \text{ for all } \tau.$$

Then $u : (\Sigma, j) \to (M, J_\tau)$ is J_τ-holomorphic for all τ, though the previously chosen normal bundle $N_u \subset u^*TM$ may fail to be J_τ-invariant for $\tau \neq 0$. Nonetheless one can always find a smooth 1-parameter family of complex bundle isomorphisms

$$\Phi_\tau : (TM, J) \to (TM, J_\tau)$$

that fix T_u and satisfy $\Phi_0 = \mathbb{1}$, allowing us to define perturbed complex normal bundles $N_{u,\tau} := \Phi_\tau(N_u)$ and normal Cauchy-Riemann operators

$$\mathbf{D}^N_{u,\tau} : \Gamma(N_{u,\tau}) \to \Gamma(\overline{\mathrm{Hom}}_{\mathbb{C}}(T\Sigma, N_{u,\tau})),$$

so that a 1-parameter family of operators $\Gamma(E) \to \Gamma(F)$ can be defined by

$$\Phi_\tau^{-1} \mathbf{D}^N_{u,\tau} \Phi_\tau : \Gamma(E) \to \Gamma(F).$$

We will prove the following result in §3.

Proposition 2.3. *Assume the curve $u : (\Sigma, j) \to (M, J)$ in the above setup is immersed with only transverse double points, such that no point in M is in the image of more than two distinct points of Σ. Then given any real-linear bundle map $B : E \to F$, one can choose the families of ω-tame almost complex structures $\{J_\tau\}$ and complex bundle isomorphisms $\{\Phi_\tau\}$ as above such that*

$$\Phi_\tau^{-1} \mathbf{D}^N_{u,\tau} \Phi_\tau = \mathbf{D}^N_u + \tau B.$$

In particular, for any $p > 1$, this defines a family of Fredholm operators $W^{1,p}(E) \to L^p(F)$ that depends analytically on the parameter τ. If J is ω-compatible and u has no double points, then one can also arrange that $J_\tau \in \mathcal{J}^{\mathrm{comp}}(M, \omega\,; \mathcal{U}, J_{\mathrm{fix}})$ for all τ.

Continuing with the above setup, assume now that $\mathrm{ind}(u) = 0$. Then 0 is also the index of $\mathbf{D}^N_u$, which is $m\chi(\Sigma) + 2c_1(E)$, hence $-c_1(E) = m\chi(\Sigma) + c_1(E) = c_1(F)$, implying the existence of a complex-antilinear bundle isomorphism $B : E \to F$. Let $\langle\ ,\ \rangle$ denote a Hermitian bundle metric on E, and denote its real part by $\langle\ ,\ \rangle_{\mathbb{R}}$; if J is ω-compatible, we may assume that $\langle\ ,\ \rangle_{\mathbb{R}}$ matches the restriction of $\omega(\cdot, J\cdot)$ to N_u. For our linear transversality argument, it will be important to establish the following symmetry property for B, which will be possible due to an obstruction theoretic argument explained in §4. Note that the condition described here is vacuous when E is a line bundle, so this step did not appear in Taubes's argument of §2.1 and is only needed for the higher-dimensional case.

Proposition 2.4. *Every homotopy class of complex-antilinear bundle isomorphisms $B : E \to \overline{\mathrm{Hom}}_{\mathbb{C}}(T\Sigma, E)$ contains one that satisfies the following condition: for all $z \in \Sigma$, $X \in T_z\Sigma$ and $\xi, \eta \in E_z$,*

$$\langle \xi, B\eta(X) \rangle_{\mathbb{R}} = \langle B\xi(X), \eta \rangle_{\mathbb{R}}.$$

The remaining crucial ingredient will be a generalization of Taubes's analytic perturbation theory argument described in §2.1. Fix $B : E \to F$ as given by Proposition 2.4, and assume $\varphi : (\widetilde{\Sigma}, \tilde{j}) \to (\Sigma, j)$ is a holomorphic map of degree $d \geq 1$. The generalized normal bundle of $\tilde{u} := u \circ \varphi$ is then $\widetilde{E} := N_{\tilde{u}} = \varphi^* E$, and we define $\widetilde{F} := \overline{\mathrm{Hom}}_{\mathbb{C}}(T\widetilde{\Sigma}, \widetilde{E})$ so that

$\mathbf{D}_{\tilde{u}}^N$ maps $\Gamma(\widetilde{E})$ to $\Gamma(\widetilde{F})$. If $\{J_\tau\}$ is a 1-parameter family of almost complex structures as in Proposition 2.3 so that $\mathbf{D}_{u,\tau}^N$ for each τ is conjugate to $\mathbf{D}_u^N + \tau B$, then the resulting perturbed normal Cauchy-Riemann operators $\mathbf{D}_{\tilde{u},\tau}^N$ are conjugate to the family

$$\mathbf{D}_{\tilde{u}}^N + \tau B_\varphi, : \Gamma(\widetilde{E}) \to \Gamma(\widetilde{F}),$$

where

$$B_\varphi : \varphi^* E \to \overline{\mathrm{Hom}}_{\mathbb{C}}(T\widetilde{\Sigma}, \varphi^* E) : \eta \mapsto B\eta \circ T\varphi.$$

We will prove the following in §6, using a Weitzenböck formula developed in §5.

Proposition 2.5. *Given any B and φ as described above, the operator $\mathbf{D}_{\tilde{u}}^N + \tau B_\varphi$ is injective for all $\tau \in \mathbb{R}$ outside of a discrete subset.*

2.3. **Proof of Theorem 1.3.** Assuming Propositions 2.3, 2.4 and 2.5, we now prove the main result. The following topological argument is also inspired by ideas of Taubes (cf. [MS04, pp. 52–53] or [Wena, §4.4.2]). We shall carry out the argument first in the setting of embedded holomorphic curves and compatible almost complex structures, and then explain what modifications are needed for the immersed/tame case.

Fix an integer $g \ge 0$, a homology class $A \in H_2(M)$ and a closed connected and oriented surface Σ of genus g. Recall that the Teichmüller space $\mathcal{T}(\Sigma) = \mathcal{J}(\Sigma)/\operatorname{Diff}_0(\Sigma)$ is a smooth manifold diffeomorphic to $\mathbb{C}^N$, with $N = 3g - 3$ for $g \ge 2$ or $N = g$ for $g = 0, 1$. In particular, $\mathcal{T}(\Sigma)$ is contractible, allowing us to fix a smooth family of complex structures

$$\{j_x \in \mathcal{J}(\Sigma)\}_{x \in \mathbb{C}^N}$$

for which the natural projection to $\mathcal{T}(\Sigma)$ is bijective. Fix Riemannian metrics on Σ and M, denoting the resulting distance functions all by $\operatorname{dist}(\ ,\ )$. Now for any $J \in \mathcal{J}(M\,;\mathcal{U}, J_{\mathrm{fix}})$ and $N \in \mathbb{N}$, define

$$\mathcal{M}_g(A, J, N) \subset \mathcal{M}_{g,0}(A, J)$$

to consist of every equivalence class in $\mathcal{M}_{g,0}(A, J)$ admitting a representative of the form (Σ, j_x, u) such that the following conditions are satisfied:

(1) j_x is "not close to degenerating":
$$|x| \le N$$

(2) u is "not close to bubbling":
$$|du(z)| \le N \quad \text{for all } z \in \Sigma;$$

(3) u is "not close to being non-embedded":
$$\min_{z \in \Sigma} |du(z)| \ge \frac{1}{N}, \quad \text{and} \quad \inf_{z, \zeta \in \Sigma,\ z \ne \zeta} \frac{\operatorname{dist}(u(z), u(\zeta))}{\operatorname{dist}(z, \zeta)} \ge \frac{1}{N};$$

(4) u is "not close to escaping $\mathcal{U}$":
$$\operatorname{dist}\left(u(\Sigma), M \setminus \mathcal{U}\right) \ge \frac{1}{N}.$$

The union of the subsets $\mathcal{M}_g(A, J, N)$ for all $N \in \mathbb{N}$ consists precisely of all curves in $\mathcal{M}_{g,0}(A, J)$ that are embedded and contained in $\mathcal{U}$. We claim that for any fixed $N \in \mathbb{N}$, $\mathcal{M}_g(A, J, N)$ is compact—in fact:

Lemma 2.6. *For any $N \in \mathbb{N}$ and any convergent sequence $J_k \to J \in \mathcal{J}(M\,;\mathcal{U}, J_{\text{fix}})$, every sequence $u_k \in \mathcal{M}_g(A, J_k, N)$ has a subsequence converging to an element of $\mathcal{M}_g(A, J, N)$.*

Proof. By assumption, the given sequence admits representatives of the form (Σ, j_{x_k}, u_k) that each satisfy the four conditions listed above. Condition (1) implies $|x_k| \le N$ for all k, so we can take a subsequence for which the complex structures j_{x_k} converge to some j_x with $|x| \le N$. The second condition then implies via elliptic regularity that after passing to a further subsequence, the maps u_k converge in C^∞ to a pseudoholomorphic map $u : (\Sigma, j_x) \to (M, J)$ with $|du| \le N$ everywhere. Given this convergence, (3) and (4) are both closed conditions and are thus also satisfied by u, so (Σ, j_x, u) represents an element of $\mathcal{M}_g(A, J, N)$. $\square$

Now for each $N \in \mathbb{N}$, define

$$\mathcal{J}_{\text{reg}}(N) \subset \mathcal{J}^{\text{comp}}(M, \omega\,;\mathcal{U}, J_{\text{fix}})$$

to consist of all $J \in \mathcal{J}^{\text{comp}}(M, \omega\,;\mathcal{U}, J_{\text{fix}})$ with the property that for every index 0 curve $[(\Sigma, j, u)] \in \mathcal{M}_g(A, J, N)$ and every unbranched holomorphic cover $\varphi : (\widetilde{\Sigma}, \tilde{\jmath}) \to (\Sigma, j)$ of degree at most N, the curve $\tilde{u} = u \circ \varphi$ is Fredholm regular.

We claim that $\mathcal{J}_{\text{reg}}(N)$ is open. If this is not the case, then there exists a sequence $J_k \in \mathcal{J}^{\text{comp}}(M, \omega\,;\mathcal{U}, J_{\text{fix}})$ converging to $J \in \mathcal{J}_{\text{reg}}(N)$, together with a sequence $[(\Sigma, j_k, u_k)] \in \mathcal{M}_g(A, J_k, N)$ and unbranched covers $\varphi_k : (\widetilde{\Sigma}_k, \tilde{\jmath}_k) \to (\Sigma, j_k)$ with $\deg(\varphi_k) \le N$ for which $\operatorname{ind}(u_k) = 0$ but $u_k \circ \varphi_k$ is not regular. But then $[(\Sigma, j_k, u_k)]$ has a subsequence converging to an element $[(\Sigma, j, u)] \in \mathcal{M}_g(A, J, N)$, and since each (Σ, j_k) has only finitely many unbranched covers of degree at most N up to biholomorphic equivalence, we may also assume after reparametrization that a subsequence of φ_k converges to another unbranched cover $\varphi : (\widetilde{\Sigma}, \tilde{\jmath}) \to (\Sigma, j)$ of degree at most N. Since $J \in \mathcal{J}_{\text{reg}}(N)$, $u \circ \varphi$ is regular, but this condition is open and thus gives a contradiction.

We claim next that $\mathcal{J}_{\text{reg}}(N)$ is dense. To see this, note first that by the standard transversality theory as in [MS04], any $J \in \mathcal{J}^{\text{comp}}(M, \omega\,;\mathcal{U}, J_{\text{fix}})$ has a perturbation $J' \in \mathcal{J}^{\text{comp}}(M, \omega\,;\mathcal{U}, J_{\text{fix}})$ for which all curves in $\mathcal{M}_g(A, J', N)$ are Fredholm regular, as all of them have injective points mapped into $\mathcal{U}$. Since $\mathcal{M}_g(A, J', N)$ is compact, the set of index 0 curves in $\mathcal{M}_g(A, J', N)$ is now finite. For each individual such curve $[(\Sigma, j, u)]$ and each unbranched cover $\varphi : (\widetilde{\Sigma}, \tilde{\jmath}) \to (\Sigma, j)$, the combination of Propositions 2.3, 2.4 and 2.5 provides a 1-parameter family of perturbed almost complex structures $\{J_\tau \in \mathcal{J}^{\text{comp}}(M, \omega\,;\mathcal{U}, J_{\text{fix}})\}$ with $J_0 = J'$ such that the normal Cauchy-Riemann operator of $u \circ \varphi$ becomes injective for sufficiently small $\tau > 0$. Note that by the implicit function theorem, there is a natural bijective correspondence between the sets of index 0 curves in $\mathcal{M}_g(A, J', N)$ and $\mathcal{M}_g(A, J_\tau, N)$ for τ sufficiently small. Now since the set of covers $u \circ \varphi$ with $u \in \mathcal{M}_g(A, J', N)$, $\operatorname{ind}(u) = 0$ and $\deg(\varphi) \le N$ is finite up to biholomorphic equivalence, one can repeat this procedure finitely many times to obtain an arbitrarily small perturbation J'' of J' for which all such covers become regular, meaning $J'' \in \mathcal{J}_{\text{reg}}(N)$.

Finally, the desired Baire subset can be defined as the countable intersection of the sets $\mathcal{J}_{\text{reg}}(N)$ for all possible $N \in \mathbb{N}$, $g \ge 0$ and $A \in H_2(M)$, thus concluding the proof of Theorem 1.3 for embedded curves.

Remark 2.7. The difficulty in using this method to prove super-rigidity for *branched* covers is that for a given (Σ, j) and $N \in \mathbb{N}$, the set of inequivalent branched covers of (Σ, j) with degree at most N is generally uncountable, so there is no guarantee that any single perturbation J_τ could make the normal operator injective for all of them at once. The analytic perturbation trick unfortunately provides no obvious control over the function

$$\varphi \mapsto \sup \left\{ \tau_0 > 0 \mid \mathbf{D}^N_{u\circ\varphi} \text{ defined with respect to } J_\tau \text{ is injective for all } \tau \in (0, \tau_0] \right\},$$

e.g. it could vary discontinuously as φ moves in the moduli space of branched covers.

The above argument could also be repeated verbatim to find corresponding Baire subsets of $\mathcal{J}(M\,;\mathcal{U}, J_{\text{fix}})$ and $\mathcal{J}^{\text{tame}}(M, \omega\,;\mathcal{U}, J_{\text{fix}})$ that establish regularity for unbranched covers of embedded curves. This means *all* simple curves without loss of generality if $\dim_{\mathbb{R}} M \geq 6$, but a modified argument is needed in dimension four to handle curves with self-intersections. If $\dim_{\mathbb{R}} M = 4$, we modify the definition of $\mathcal{M}_g(A, J, N)$ as follows. For any simple curve $u \in \mathcal{M}_{g,0}(A, J)$, define the integer $d(u) \geq 0$ by

$$2d(u) = \left| \{(z, \zeta) \in \Sigma \times \Sigma \mid u(z) = u(\zeta) \text{ and } z \neq \zeta\} \right|.$$

Recall that by the adjunction inequality, this number satisfies

$$A \cdot A \geq 2d(u) + c_1(A) - (2 - 2g),$$

with equality if and only if u is immersed with only transverse double points. With this in mind, define

$$d(A, g) := \frac{1}{2}\left(A \cdot A - c_1(A)\right) + 1 - g,$$

and define $\mathcal{M}_g(A, J, N)$ via conditions (1), (2) and (4) above, plus the following replacement of condition (3):

(3a) $\min_{z\in\Sigma} |du(z)| \geq \dfrac{1}{N}$;

(3b) There exists a point $z_0 \in \Sigma$ such that

$$\inf_{z\in\Sigma\setminus\{z_0\}} \frac{\operatorname{dist}(u(z_0), u(z))}{\operatorname{dist}(z_0, z)} \geq \frac{1}{N};$$

(3c) M contains $d := d(A, g)$ distinct points $p_1, \ldots, p_d \in M$ at which $|u^{-1}(p_j)| > 1$, and

$$\operatorname{dist}\left((p_1, \ldots, p_d), \Delta\right) \geq \frac{1}{N},$$

where $\Delta \subset M^d$ denotes the set of tuples $(x_1, \ldots, x_d)$ for which at least two of the points coincide.

The adjunction inequality implies that every curve in $u \in \mathcal{M}_g(A, J, N)$ is immersed with transverse double points, all at distinct points in the image, and $\bigcup_{N\in\mathbb{N}} \mathcal{M}_g(A, J, N)$ now consists of all curves in $\mathcal{M}_{g,0}(A, J)$ that have these properties. The only other modification needed from the embedded case is in the proof that $\mathcal{J}_{\text{reg}}(N)$ is dense. This is where we need to allow $J \in \mathcal{J}^{\text{tame}}(M, \omega\,;\mathcal{U}, J_{\text{fix}})$ instead of $\mathcal{J}^{\text{comp}}(M, \omega\,;\mathcal{U}, J_{\text{fix}})$, as Proposition 2.3 does not provide an ω-compatible perturbation if u has double points. Note however that after a small perturbation of any given J, we are free to assume that all simple index 0 curves are immersed with transverse double points at separate points in the image (see e.g. [Wena, Exercise 4.65 and §4.6]), in which case Propositions 2.3 and 2.5 can be used to find an ω-tame perturbation in $\mathcal{J}_{\mathrm{reg}}(N)$. With this established, the rest of the proof goes through as before. $\square$

3. Normal perturbations of almost complex structures

The purpose of this section is to prove Proposition 2.3. Fix a tame almost complex structure $J \in \mathcal{J}^{\mathrm{tame}}(M, \omega\,; \mathcal{U}, J_{\mathrm{fix}})$ and a closed J-holomorphic curve $u : (\Sigma, j) \to (M, J)$ that has image in $\mathcal{U}$ and is immersed with at most finitely many double points, all transverse and at distinct points in the image. Note that if $\dim_{\mathbb{R}} M \geq 6$, this assumption means u is embedded.

Choose a complex subbundle $N_u \subset u^*TM$ such that $u^*TM = T_u \oplus N_u$, where $T_u := \operatorname{im} du$. In the 4-dimensional case, our assumption about double points implies that we can also arrange

$$(T_u)_z = (N_u)_\zeta \quad \text{and} \quad (T_u)_\zeta = (N_u)_z$$

whenever $u(z) = u(\zeta)$ with $z \neq \zeta$. To construct a suitable perturbation of J, fix $Y \in \Gamma(\overline{\mathrm{End}}_{\mathbb{C}}(TM, J))$ with support in $\overline{\mathcal{U}}$ and let

$$\Phi := \mathbb{1} + \frac{1}{2} JY \in \Gamma(\mathrm{End}_{\mathbb{R}}(TM)).$$

We shall always assume that Y is C^0-small enough for Φ to be everywhere invertible, in which case

$$J' := \Phi J \Phi^{-1}$$

defines an almost complex structure that is close to J and therefore tame if Y is sufficiently small. We shall make use of the splitting $u^*TM = T_u \oplus N_u$ and restrict Y by assuming that along u, it takes the block form

$$Y(u(z)) = \begin{pmatrix} 0 & Y^{NT}(z) \\ 0 & 0 \end{pmatrix} \in \overline{\mathrm{End}}_{\mathbb{C}}(T_u \oplus N_u) \quad \text{for all } z \in \Sigma, \tag{3.1}$$

where Y^{NT} is a (necessarily complex-antilinear) bundle map $N_u \to T_u$. Note that if u has any double points, then this condition requires Y to vanish at the images of those points. Writing the tangent and normal parts of J along u as $J^T : T_u \to T_u$ and $J^N : N_u \to N_u$ respectively, we now have

$$\Phi(u(z)) = \begin{pmatrix} \mathbb{1} & \frac{1}{2} J^T(z) Y^{NT}(z) \\ 0 & \mathbb{1} \end{pmatrix} \quad \text{for all } z \in \Sigma, \tag{3.2}$$

and thus

$$J'(u(z)) = \begin{pmatrix} J^T(z) & Y^{NT}(z) \\ 0 & J^N(z) \end{pmatrix} \quad \text{for all } z \in \Sigma. \tag{3.3}$$

This shows that $J'|_{T_u} = J|_{T_u}$, so u is also J'-holomorpic. We can now define a J'-invariant normal bundle along u by

$$N'_u := \Phi(N_u) \subset u^*TM,$$

so $\Phi|_{N_u} : (N_u, J) \to (N'_u, J')$ is a complex bundle isomorphism by construction. Let $\pi_{N'} : u^*TM = T_u \oplus N'_u \to N'_u$ denote the resulting normal projection, which gives rise to a perturbed normal Cauchy-Riemann operator

$$\mathbf{D}_u^{N'} = \pi_{N'} \circ \mathbf{D}'_u\big|_{\Gamma(N'_u)} : \Gamma(N'_u) \to \Omega^{0,1}(\Sigma, N'_u),$$

where $\mathbf{D}'_u$ denotes the linearized Cauchy-Riemann operator for u as a J'-holomorphic curve. Conjugating this with the bundle isomorphism gives an operator

$$\Phi^{-1} \circ \mathbf{D}_u^{N'} \circ \Phi : \Gamma(N_u) \to \Omega^{0,1}(\Sigma, N_u).$$

Lemma 3.1. *There exists a smooth bundle map* $A : N_u \to \overline{\mathrm{Hom}}_{\mathbb{C}}(T\Sigma, N_u)$ *such that* $\Phi^{-1} \circ \mathbf{D}_u^{N'} \circ \Phi = \mathbf{D}_u^N + A$. *For any connection* ∇ *on* TM, A *is given by the formula*

$$A\eta = \pi_N \circ \nabla_\eta Y \circ Tu \circ j.$$

Remark 3.2. Implicit in the above statement is that the expression on the right hand side of the formula does not depend on the choice of connection. This will follow from a direct calculation in the proof, but the intuitive reason for it is that under the block decomposition of $\nabla_\eta Y$ given by the splitting $u^*TM = T_u \oplus N_u$, only the lower-left block (mapping T_u to N_u) is relevant in the above expression, while the corresponding block of Y itself has been assumed to vanish along u.

Proof of Lemma 3.1. In terms of the splitting $u^*TM = T_u \oplus N_u$, the perturbed normal projection $u^*TM \to N'_u$ is given in block form by

$$\pi_{N'} = \begin{pmatrix} 0 & \frac{1}{2} J^T Y^{NT} \\ 0 & \mathbb{1} \end{pmatrix},$$

so using (3.2) to write $\Phi^{-1}(u(z)) = \begin{pmatrix} \mathbb{1} & -\frac{1}{2} J^T(z) Y^{NT}(z) \\ 0 & \mathbb{1} \end{pmatrix}$, we find

$$\Phi^{-1} \circ \pi_{N'} = \pi_N.$$

Recall now from [Wen10, Lemma 3.8] that $\mathbf{D}_u$ maps sections of T_u to $(0,1)$-forms valued in u^*TM with vanishing normal component. The same applies to $\mathbf{D}'_u$, hence for $\eta \in \Gamma(N_u)$, we have $\Phi\eta - \eta \in \Gamma(T_u)$ and thus

$$\left(\Phi^{-1} \circ \mathbf{D}_u^{N'} \circ \Phi\right) \eta = (\Phi^{-1} \circ \pi_{N'}) \mathbf{D}'_u(\Phi\eta) = \pi_N(\mathbf{D}'_u \eta).$$

To compute $\mathbf{D}'_u\eta$, choose any smooth 1-parameter family of maps $u_\rho : \Sigma \to M$ for $\rho \in (-\epsilon, \epsilon)$ with $u_0 = u$ and $\partial_\rho u_\rho|_{\rho=0} = \eta$. Then for any connection ∇ on TM and any holomorphic local coordinate system (s,t) on some open subset in Σ, the $(0,1)$-form $\mathbf{D}'_u\eta$ is given locally by

$$\begin{aligned}
(\mathbf{D}'_u\eta)\partial_s &= \nabla_\rho \left(\partial_s u_\rho + J'(u_\rho)\, \partial_t u_\rho\right)\big|_{\rho=0} \\
&= \nabla_\rho \left(\partial_s u_\rho + J(u_\rho)\, \partial_t u_\rho + \left[J'(u_\rho) - J(u_\rho)\right] \partial_t u_\rho\right)\big|_{\rho=0} \\
&= (\mathbf{D}_u\eta)\partial_s + \nabla_\rho \left(\left[J'(u_\rho) - J(u_\rho)\right] \partial_t u_\rho\right)\big|_{\rho=0} \\
&= (\mathbf{D}_u\eta)\partial_s + \left[\nabla_\eta (J' - J)\right] \partial_t u + \left[J'(u) - J(u)\right] \nabla_\rho \partial_t u_\rho\big|_{\rho=0}.
\end{aligned} \tag{3.4}$$

By (3.3), the image of $J' - J$ has vanishing normal component everywhere along u, so the third term on the right hand side of (3.4) does not contribute to $\pi_N(\mathbf{D}'_u\eta)$. Removing the local coordinates, we thus obtain the global expression

$$\left(\Phi^{-1} \circ \mathbf{D}_u^{N'} \circ \Phi\right) \eta = \mathbf{D}_u^N \eta + \pi_N \circ \nabla_\eta (J' - J) \circ Tu \circ j.$$

To simplify the last term, observe that since $J' = \Phi J \Phi^{-1}$ with $\Phi = \mathbb{1} + \frac{1}{2}JY$, $JY = -YJ$ and $J^2 = -\mathbb{1}$, we have

$$(J' - J)\Phi = \Phi J - J\Phi = \left(\mathbb{1} + \frac{1}{2}JY\right) J - J\left(\mathbb{1} + \frac{1}{2}JY\right) = \frac{1}{2}JYJ + \frac{1}{2}Y = Y,$$

hence $J' - J = Y\Phi^{-1}$, and therefore

$$\nabla_\eta(J' - J) = (\nabla_\eta Y)\Phi^{-1} + Y(\nabla_\eta \Phi^{-1}).$$

Composing the second of these two terms with $Tu \circ j$ produces a section with vanishing normal component due to (3.1), so it does not contribute. In the remaining expression, Φ^{-1} can be omitted since it acts trivially on the tangential component, and this produces the formula that was claimed. □

Proof of Proposition 2.3. Given a bundle map $B : N_u \to \overline{\mathrm{Hom}}_{\mathbb{C}}(T\Sigma, N_u)$, it will suffice to carry out the construction in Lemma 3.1 with Φ replaced by the 1-parameter family of bundle isomorphisms $\Phi_\tau = \mathbb{1} + \frac{1}{2}\tau JY$, as long as $Y \in \Gamma(\overline{\mathrm{End}}_{\mathbb{C}}(TM, J))$ can be chosen to match a block expression of the form (3.1) along u, with normal derivative along u satisfying

$$\pi_N \circ \nabla_\eta Y \circ Tu \circ j = B\eta \quad \text{for all } \eta \in N_u. \tag{3.5}$$

Since $Tu \circ j : T\Sigma \to T_u$ is a complex-linear bundle isomorphism, this is clearly possible if u is embedded, as one can then assume $Y = 0$ along u and choose its normal derivative to satisfy (3.5). Note that if J is ω-compatible, then J_τ will also be ω-compatible if and only if Y is everywhere symmetric with respect to the metric $\omega(\cdot, J\cdot)$, and this can also be achieved in the absence of double points since (3.5) only constrains the lower-left block of $\nabla_\eta Y$ with respect to the splitting $u^*TM = T_u \oplus N_u$.

We must be a bit more careful if $\dim_{\mathbb{R}} M = 4$ and u has double points. Assume $u(z) = u(\zeta) = p$, with $(T_u)_z = (N_u)_\zeta$ and vice versa. We can choose local coordinates $(z_1, z_2) \in \mathbb{C}^2$ near p that identify p with the origin, while the images of u near z and ζ are identified with subsets of $\mathbb{C} \times \{0\}$ and $\{0\} \times \mathbb{C}$ respectively. In this neighborhood, choose a complex local trivialization of (TM, J) identifying the normal subspaces along $\mathbb{C} \times \{0\}$ with $\{0\} \oplus \mathbb{C}$ and those along $\{0\} \times \mathbb{C}$ with $\mathbb{C} \oplus \{0\}$, and let ∇ be the trivial connection with respect to this trivialization. We claim that in this trivialization near p, a suitable Y can be written in the form

$$Y(z_1, z_2) = \begin{pmatrix} 0 & Y_{12}(z_1, z_2) \\ Y_{21}(z_1, z_2) & 0 \end{pmatrix}$$

for some functions Y_{12} and Y_{21} valued in $\overline{\mathrm{End}}_{\mathbb{C}}(\mathbb{C})$. Indeed, the condition (3.1) now becomes

$$\begin{aligned} Y_{21}(z_1, 0) &= 0 \quad \text{for all } z_1, \\ Y_{12}(0, z_2) &= 0 \quad \text{for all } z_2, \end{aligned}$$

while (3.5) specifies the normal derivatives of Y_{21} along $\mathbb{C} \times \{0\}$ and Y_{12} along $\{0\} \times \mathbb{C}$. After choosing Y_{12} and Y_{21} to satisfy these conditions, we can then also arrange $Y_{21}(0, z_2) = Y_{12}(z_1, 0) = 0$ for all z_1, z_2 ouside some small neighborhood of 0, hence Y vanishes along u outside a neighborhood of p, and the previous argument for the embedded case can then be used to extend Y globally. □

Remark 3.3. If J is ω-compatible and u has double points, then the above proof fails to provide ω-compatible perturbations J_τ: in a neighborhood of a double point, the last step in the construction generally forces the upper-right block of (3.1) to take nonzero values, thus violating the symmetry condition required for ω-compatibility. This is why the statement of Theorem 1.3 in the compatible case is limited to embedded curves.

4. Symmetric bundle isomorphisms

We now state and prove a result that implies Proposition 2.4.

Proposition 4.1. *Suppose $E \to \Sigma$ is a Hermitian vector bundle, let $\langle\ ,\ \rangle_{\mathbb{R}}$ denote the real part of its bundle metric, and suppose $L \to \Sigma$ is a complex line bundle. Then every homotopy class of complex-antilinear bundle isomorphisms $B : E \to \overline{\mathrm{Hom}}_{\mathbb{C}}(L, E)$ contains one that satisfies the condition*

$$\langle \xi, B\eta(X)\rangle_{\mathbb{R}} = \langle B\xi(X), \eta\rangle_{\mathbb{R}} \quad \textit{for all } (X, \xi, \eta) \in L \oplus E \oplus E.$$

Observe first that a choice of complex-antilinear isomorphism $B : E \to \overline{\mathrm{Hom}}_{\mathbb{C}}(L, E)$ is equivalent via the correspondence $B\eta(X) = \widehat{B}X(\eta)$ to a choice of complex-antilinear bundle map

$$\widehat{B} : L \to \overline{\mathrm{End}}_{\mathbb{C}}(E)$$

with the property that for all nonzero $X \in L$, $\widehat{B}(X)$ is invertible. Proposition 4.1 is then equivalent to showing that every homotopy class of bundle maps $\widehat{B}$ with the above property contains one for which $\widehat{B}(X)$ is always symmetric. This is clearly true for the restriction of $\widehat{B}$ to the 0-skeleton of Σ, since the space of antilinear isomorphisms on any complex vector space is connected and contains one that is symmetric. Extending this to the 1-skeleton and then the 2-skeleton of Σ is possible due to Proposition 4.2 below.

Identify $\mathbb{C}^m$ with $\mathbb{R}^{2m}$ so that $\mathrm{End}_{\mathbb{C}}(\mathbb{C}^m)$ is regarded as the real subspace of $\mathrm{End}_{\mathbb{R}}(\mathbb{R}^{2m}) = \mathrm{End}_{\mathbb{R}}(\mathbb{C}^m)$ consisting of linear maps that commute with the standard complex structure $i \in \mathrm{GL}(2m, \mathbb{R})$. We then denote

$$\begin{aligned}
\overline{\mathrm{Aut}}_{\mathbb{C}}(\mathbb{C}^m) &:= \overline{\mathrm{End}}_{\mathbb{C}}(\mathbb{C}^m) \cap \mathrm{GL}(2m, \mathbb{R}),\\
\overline{\mathrm{Aut}}^S_{\mathbb{C}}(\mathbb{C}^m) &:= \left\{ A \in \overline{\mathrm{Aut}}_{\mathbb{C}}(\mathbb{C}^m) \mid A = A^T \right\},
\end{aligned}$$

where A^T means the usual transpose of real $2m$-by-$2m$ matrices.

Proposition 4.2. *We have*

$$\pi_1\left(\overline{\mathrm{Aut}}_{\mathbb{C}}(\mathbb{C}^m), \overline{\mathrm{Aut}}^S_{\mathbb{C}}(\mathbb{C}^m)\right) = \pi_2\left(\overline{\mathrm{Aut}}_{\mathbb{C}}(\mathbb{C}^m), \overline{\mathrm{Aut}}^S_{\mathbb{C}}(\mathbb{C}^m)\right) = 0.$$

The proof of the proposition occupies the remainder of this section. Observe first that composition with the real-linear isomorphism

$$\mathbb{C}^m \to \mathbb{C}^m : v \mapsto \bar{v}$$

identifies $\overline{\mathrm{Aut}}_{\mathbb{C}}(\mathbb{C}^m)$ with $\mathrm{GL}(m, \mathbb{C}) \subset \mathrm{GL}(2m, \mathbb{R})$ and $\overline{\mathrm{Aut}}^S_{\mathbb{C}}(\mathbb{C}^m)$ with

$$\mathrm{GL}^S(m, \mathbb{C}) := \left\{ A \in \mathrm{GL}(m, \mathbb{C}) \mid A = A^T \right\},$$

where in the latter case A^T denotes the transpose (not the adjoint!) of the m-by-m complex matrix A, i.e. $A^T = \overline{A}^\dagger$. The proposition is therefore equivalent to the computation

$$\pi_1\left(\mathrm{GL}(m,\mathbb{C}), \mathrm{GL}^S(m,\mathbb{C})\right) = \pi_2\left(\mathrm{GL}(m,\mathbb{C}), \mathrm{GL}^S(m,\mathbb{C})\right) = 0. \tag{4.1}$$

We prove this in five steps.

Step 1. Consider the map

$$Q : \mathrm{GL}(m,\mathbb{C})/\,\mathrm{O}(m,\mathbb{C}) \to \mathrm{GL}^S(m,\mathbb{C}) : A \mapsto A^T A, \tag{4.2}$$

where $\mathrm{O}(m,\mathbb{C})$ denotes the complex orthogonal group $\{A \in \mathrm{GL}(m,\mathbb{C}) \mid A^T A = \mathbb{1}\}$. We claim that Q is a bijection. Injectivity is easy to check; surjectivity follows from the fact that every $A \in \mathrm{GL}^S(m,\mathbb{C})$ defines a symmetric nondegenerate complex bilinear form

$$(v,w) \mapsto v^T A w,$$

and all such forms are equivalent up to a choice of basis. Since $\mathrm{GL}(m,\mathbb{C})$ is connected, it follows that $\mathrm{GL}^S(m,\mathbb{C})$ is connected.

Step 2. We claim that for all $m \in \mathbb{N}$, $\mathrm{O}(m,\mathbb{C})$ has exactly two connected components. It is clear that there are at least two, as every $A \in \mathrm{O}(m,\mathbb{C})$ has $\det A = \pm 1$. It suffices therefore to prove that $\mathrm{SO}(m,\mathbb{C}) := \{A \in \mathrm{O}(m,\mathbb{C}) \mid \det A = 1\}$ is connected. This is true for $m = 1$ since $\mathrm{SO}(1,\mathbb{C})$ is the trivial group. The claim then follows by induction using the fibration

$$\mathrm{SO}(m-1,\mathbb{C}) \hookrightarrow \mathrm{SO}(m,\mathbb{C}) \xrightarrow{\pi} H^{m-1},$$

where $H^{m-1} := \{v \in \mathbb{C}^m \mid v^T v = 1\}$ and $\pi(A)$ is defined as the first column of A. The fact that π is surjective can be proved using the same argument that is used in diagonalizing quadratic forms: it reduces to the fact that any given $v_1 \in H^{m-1}$ can be extended to a complex basis $v_1, \dots, v_m \in H^{m-1}$ of $\mathbb{C}^m$ such that $v_i^T v_j = \delta_{ij}$.

Step 3. We claim that $\pi_1(\mathrm{GL}(m,\mathbb{C})/\,\mathrm{O}(m,\mathbb{C})) \cong \mathbb{Z}$ is generated by the projection to $\mathrm{GL}(m,\mathbb{C})/\,\mathrm{O}(m,\mathbb{C})$ of the path

$$\gamma : [0,1] \to \mathrm{GL}(m,\mathbb{C}) : t \mapsto \begin{pmatrix} e^{\pi i t} & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}.$$

To see this, consider the long exact sequence of the fibration $\mathrm{O}(m,\mathbb{C}) \overset{\iota}{\hookrightarrow} \mathrm{GL}(m,\mathbb{C}) \xrightarrow{p} GL(m,\mathbb{C})/\,\mathrm{O}(m,\mathbb{C})$:

$$\begin{aligned} \ldots \longrightarrow \pi_1(\mathrm{GL}(m,\mathbb{C})) \xrightarrow{p_*} \pi_1(\mathrm{GL}(m,\mathbb{C})/\,\mathrm{O}(m,\mathbb{C})) \xrightarrow{\partial} \\ \pi_0(\mathrm{O}(m,\mathbb{C})) \longrightarrow \pi_0(\mathrm{GL}(m,\mathbb{C})) = 0. \end{aligned}$$

Any loop in $\mathrm{GL}(m,\mathbb{C})/\,\mathrm{O}(m,\mathbb{C})$ can be represented as a path $\beta : [0,1] \to \mathrm{GL}(m,\mathbb{C})$ with $\beta(0) = \mathbb{1}$ and $\beta(1) \in \mathrm{O}(m,\mathbb{C})$, and the map ∂ can then be written as

$$\partial[\beta] = \det \beta(1) \in \{1, -1\} = \pi_0(\mathrm{O}(m,\mathbb{C})),$$

applying the result of Step 2. Since $\ker \partial = \operatorname{im} p_*$, any such path β with $\det \beta(1) = 1$ is equivalent in $\pi_1(\mathrm{GL}(m,\mathbb{C})/\,\mathrm{O}(m,\mathbb{C}))$ to a loop in $\mathrm{GL}(m,\mathbb{C})$, and using the standard computation of $\pi_1(\mathrm{GL}(m,\mathbb{C})) = \pi_1(\mathrm{U}(m))$, any such loop is homotopic to

$$S^1 \to \mathrm{GL}(m,\mathbb{C}) : t \mapsto \begin{pmatrix} e^{2\pi k i t} & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}$$

for some $k \in \mathbb{Z}$. Thus any such element of $\pi_1(\mathrm{GL}(m,\mathbb{C})/\,\mathrm{O}(m,\mathbb{C}))$ is an even power of γ. If on the other hand $\det \beta(1) = -1$, then we can concatenate β with the loop $t \mapsto [\beta(1)\gamma(t)]$ in $\mathrm{GL}(m,\mathbb{C})/\,\mathrm{O}(m,\mathbb{C})$, whose determinant at $t = 1$ is positive, implying that $\beta \cdot \gamma \in \pi_1(\mathrm{GL}(m,\mathbb{C})/\,\mathrm{O}(m,\mathbb{C}))$ is an even power of γ, so this proves the claim.

Step 4. We claim that the composition of the map Q in (4.2) with the inclusion $\mathrm{GL}^S(m,\mathbb{C}) \hookrightarrow \mathrm{GL}(m,\mathbb{C})$ induces an isomorphism

$$\pi_1\left(\mathrm{GL}(m,\mathbb{C})/\,\mathrm{O}(m,\mathbb{C})\right) = \pi_1(\mathrm{GL}(m,\mathbb{C})).$$

This follows by computing the action of this map on the generator of $\pi_1\left(\mathrm{GL}(m,\mathbb{C})/\,\mathrm{O}(m,\mathbb{C})\right)$ as described in Step 3.

Step 5. Consider the homotopy exact sequence for $(\mathrm{GL}(m,\mathbb{C}), \mathrm{O}(m,\mathbb{C}))$:

$$\begin{aligned} \ldots \longrightarrow & \pi_2(\mathrm{GL}(m,\mathbb{C})) \xrightarrow{\alpha_2} \pi_2\left(\mathrm{GL}(m,\mathbb{C}), \mathrm{GL}^S(m,\mathbb{C})\right) \xrightarrow{\partial_2} \\ & \pi_1\left(\mathrm{GL}^S(m,\mathbb{C})\right) \xrightarrow{\iota_*} \pi_1(\mathrm{GL}(m,\mathbb{C})) \xrightarrow{\alpha_1} \pi_1\left(\mathrm{GL}(m,\mathbb{C}), \mathrm{GL}^S(m,\mathbb{C})\right) \xrightarrow{\partial_1} \\ & \pi_0\left(\mathrm{GL}^S(m,\mathbb{C})\right) = 0. \end{aligned}$$

We showed in Step 4 that ι_* is an isomorphism, thus $\alpha_1 = 0$, implying that ∂_1 is injective and thus

$$\pi_1\left(\mathrm{GL}(m,\mathbb{C}), \mathrm{GL}^S(m,\mathbb{C})\right) = 0.$$

Moreover, the injectivity of ι_* implies $\partial_2 = 0$, so α_2 is surjective and, since $\pi_2(\mathrm{GL}(m,\mathbb{C})) = \pi_2(\mathrm{U}(m)) = 0$,

$$\pi_2\left(\mathrm{GL}(m,\mathbb{C}), \mathrm{GL}^S(m,\mathbb{C})\right) = 0.$$

This completes the proof of Proposition 4.2 and hence, by standard obstruction theory as in [Ste51], Proposition 4.1.

5. A Weitzenböck formula for antilinear perturbations

In preparation for the proof of Proposition 2.5, we now explain a generalization of the Weitzenböck formula that was derived in §2.1 for trivial bundles on the torus.

Throughout this section, we assume (Σ, j) is a closed connected Riemann surface and $(E, J) \to (\Sigma, j)$ is a complex vector bundle of rank $m \in \mathbb{N}$ with Hermitian structure $\langle\ ,\ \rangle_E$. Fix also a j-invariant Riemannian metric on Σ, which is the real part of a Hermitian structure $\langle\ ,\ \rangle_\Sigma$ on $T\Sigma$, and denote the induced volume form on Σ by $d\mathrm{vol}$. This choice determines a complex-linear bundle isomorphism[5]

$$T\Sigma \to \Lambda^{0,1}T^*\Sigma : X \mapsto X^{0,1} := \langle \cdot, X \rangle_\Sigma \tag{5.1}$$

[5]We are using the convention that Hermitian bundle metrics are antilinear in the first and linear in the second argument.

and consequently a global trivialization

$$\Lambda^{1,0}T^*\Sigma \otimes \Lambda^{0,1}T^*\Sigma \to \mathbb{C} : \lambda \otimes X^{0,1} \mapsto \lambda(X). \tag{5.2}$$

Moreover, the rank m complex bundle

$$F := \Lambda^{0,1}T^*\Sigma \otimes E$$

inherits from $\langle\ ,\ \rangle_\Sigma$ and $\langle\ ,\ \rangle_E$ a Hermitian bundle metric $\langle\ ,\ \rangle_F$, and we shall define real-valued L^2-pairings for sections of E and F by

$$\langle \eta, \xi \rangle_{L^2(E)} := \operatorname{Re} \int_\Sigma \langle \eta, \xi \rangle_E \, d\,\mathrm{vol}, \quad \text{for} \quad \eta, \xi \in \Gamma(E),$$
$$\langle \alpha, \lambda \rangle_{L^2(F)} := \operatorname{Re} \int_\Sigma \langle \alpha, \lambda \rangle_F \, d\,\mathrm{vol}, \quad \text{for} \quad \alpha, \lambda \in \Gamma(F).$$

Given any real-linear map $\mathbf{D} : \Gamma(E) \to \Gamma(F)$, the **formal adjoint** $\mathbf{D}^* : \Gamma(F) \to \Gamma(E)$ is defined via the relation

$$\langle \lambda, \mathbf{D}\eta \rangle_{L^2(F)} = \langle \mathbf{D}^*\lambda, \eta \rangle_{L^2(E)} \quad \text{for all} \quad \eta \in \Gamma(E),\ \lambda \in \Gamma(F).$$

Recall that $\mathbf{D} : \Gamma(E) \to \Omega^{0,1}(\Sigma, E) = \Gamma(F)$ is called a **Cauchy-Riemann type** operator on E if it satisfies the Leibniz rule

$$\mathbf{D}(f\eta) = (\bar\partial f)\eta + f\,\mathbf{D}\eta \quad \text{for all} \quad f \in C^\infty(\Sigma, \mathbb{R}),\ \eta \in \Gamma(E),$$

where $\bar\partial f := df + i\,df \circ j$. Similarly, we will say that $\mathbf{D} : E \to \Omega^{1,0}(\Sigma, E) = \Gamma(\Lambda^{1,0}T^*\Sigma \otimes E)$ is an **anti-Cauchy-Riemann type** operator on E if it satisfies

$$\mathbf{D}(f\eta) = (\partial f)\eta + f\,\mathbf{D}\eta \quad \text{for all} \quad f \in C^\infty(\Sigma, \mathbb{R}),\ \eta \in \Gamma(E), \tag{5.3}$$

with $\partial f := df - i\,df \circ j$. If $\mathbf{D}$ is of Cauchy-Riemann type, then it is well known that $\mathbf{D}^*$ is conjugate via real-linear bundle isomorphisms to another Cauchy-Riemann type operator; more precisely, the natural complex bundle isomorphism

$$\Lambda^{1,0}T^*\Sigma \otimes F = \Lambda^{1,0}T^*\Sigma \otimes \Lambda^{0,1}T^*\Sigma \otimes E = E \tag{5.4}$$

defined via (5.2) identifies $-\mathbf{D}^*$ with an anti-Cauchy-Riemann type operator

$$-\mathbf{D}^* : \Gamma(F) \to \Gamma(E) = \Gamma(\Lambda^{1,0}T^*\Sigma \otimes F) = \Omega^{1,0}(\Sigma, F).$$

Proposition 5.1. *Suppose $\mathbf{D} : \Gamma(E) \to \Gamma(F)$ is a real-linear Cauchy-Riemann type operator, $B : E \to F$ is a complex-antilinear bundle map satisfying the symmetry condition*

$$\operatorname{Re}\langle \eta, B\xi(X) \rangle_E = \operatorname{Re}\langle B\eta(X), \xi \rangle_E \quad \textit{for all} \quad (X, \eta, \xi) \in T\Sigma \oplus E \oplus E, \tag{5.5}$$

and $\mathbf{D}_B := \mathbf{D} + B$. Then the complex vector bundle[6] *$\overline{\mathrm{Hom}}_\mathbb{C}(E, F)$ admits a real-linear anti-Cauchy-Riemann type operator ∂_H such that for all $\eta \in \Gamma(E)$,*

$$\mathbf{D}_B^*\mathbf{D}_B\eta = \mathbf{D}^*\mathbf{D}\eta + B^*B\eta - (\partial_H B)\eta.$$

[6]We define the complex structure on $\mathrm{Hom}_\mathbb{R}(E, F)$ and its subbundles such as $\overline{\mathrm{Hom}}_\mathbb{C}(E, F)$ via the complex structure of F, i.e. $B \mapsto J \circ B$.

Remark 5.2. In the above formula, the product of $\partial_H B \in \Omega^{1,0}(\Sigma, \overline{\mathrm{Hom}}_{\mathbb{C}}(E,F))$ with $\eta \in \Gamma(E)$ is interpreted as a section of E via the product pairing

$$\big(\Lambda^{1,0}T^*\Sigma \otimes \overline{\mathrm{Hom}}_{\mathbb{C}}(E,F)\big) \otimes E \to \Lambda^{1,0}T^*\Sigma \otimes F$$

and the isomorphism (5.4).

The proof of Proposition 5.1 will rely mainly on a few basic observations about anti-Cauchy-Riemann operators. Recall that a complex-valued function f on an open subset of Σ is called **antiholomorphic** if it satisfies $\partial f \equiv 0$. The composition of a holomorphic and an antiholomorphic function is antiholomorphic, and the product of two antiholomorphic functions is also antiholomorphic, thus it makes sense to speak of *antiholomorphic vector bundles* over Σ. Anti-Cauchy-Riemann type operators have several properties analogous to Cauchy-Riemann type operators, notably:

(1) The difference between two anti-Cauchy-Riemann type operators on the same bundle is a zeroth-order operator.
(2) The complex-linear part of any real-linear anti-Cauchy-Riemann type operator is also an anti-Cauchy-Riemann type operator.
(3) Every antiholomorphic vector bundle carries a natural complex-linear anti-Cauchy-Riemann operator that annihilates local antiholomorphic sections, and conversely, every complex-linear anti-Cauchy-Riemann operator on $(E,J) \to (\Sigma, j)$ induces an antiholomorphic bundle structure in this way.

The first two statements are easy consequences of the Leibniz rule (5.3). The third is non-trivial, but is equivalent to the corresponding fact about Cauchy-Riemann type operators and holomorphic bundles over Riemann surfaces.

Lemma 5.3. *Suppose E_1 and E_2 are complex vector bundles over (Σ, j) endowed with anti-Cauchy-Riemann type operators $\mathbf{D}_1$ and $\mathbf{D}_2$ respectively. Then* $\mathrm{Hom}_{\mathbb{C}}(E_1, E_2)$ *admits an anti-Cauchy-Riemann type operator $\mathbf{D}_{12}$ such that for all $\Phi \in \Gamma(\mathrm{Hom}_{\mathbb{C}}(E_1,E_2))$ and $\eta \in \Gamma(E_1)$,*

$$\mathbf{D}_2(\Phi\eta) = (\mathbf{D}_{12}\Phi)\eta + \Phi(\mathbf{D}_1\eta).$$

Proof. Write $\mathbf{D}_1 = \mathbf{D}_1^{\mathbb{C}} + A$ and $\mathbf{D}_2 = \mathbf{D}_2^{\mathbb{C}} + B$, where $\mathbf{D}_1^{\mathbb{C}}$ and $\mathbf{D}_2^{\mathbb{C}}$ are complex-linear anti-Cauchy-Riemann type operators (e.g. the complex-linear parts of $\mathbf{D}_1$ and $\mathbf{D}_2$ respectively), so

$$A : E_1 \to \Lambda^{1,0}T^*\Sigma \otimes E_1 \quad \text{and} \quad B : E_2 \to \Lambda^{1,0}T^*\Sigma \otimes E_2$$

are zeroth-order terms. Then $\mathbf{D}_1^{\mathbb{C}}$ and $\mathbf{D}_2^{\mathbb{C}}$ induce antiholomorphic bundle structures on E_1 and E_2, and $\mathrm{Hom}_{\mathbb{C}}(E_1,E_2)$ therefore inherits local trivializations with transition maps that are products of antiholomorphic functions, giving rise to an antiholomorphic structure and a corresponding complex-linear anti-Cauchy-Riemann operator $\mathbf{D}_{12}^{\mathbb{C}}$ that satisfies

$$\mathbf{D}_2^{\mathbb{C}}(\Phi\eta) = (\mathbf{D}_{12}^{\mathbb{C}}\Phi)\eta + \Phi(\mathbf{D}_1^{\mathbb{C}}\eta)$$

for all $\Phi \in \Gamma(\mathrm{Hom}_{\mathbb{C}}(E_1,E_2))$ and $\eta \in \Gamma(E_1)$. The desired operator can then be defined as $\mathbf{D}_{12} = \mathbf{D}_{12}^{\mathbb{C}} + C$, where $C : \mathrm{Hom}_{\mathbb{C}}(E_1,E_2) \to \Lambda^{1,0}T^*\Sigma \otimes \mathrm{Hom}_{\mathbb{C}}(E_1,E_2)$ is a bundle map taking the form

$$(C\Phi)\eta = B(\Phi\eta) - \Phi(A\eta) \in \Lambda^{1,0}T^*\Sigma \otimes E_2$$

for $(\Phi,\eta) \in \mathrm{Hom}_{\mathbb{C}}(E_1,E_2) \oplus E_1$. $\square$

For any vector bundle (E_1, J_1) over Σ, let $E_1^{\mathbf{c}}$ denote its **conjugate bundle**, defined as the same real vector bundle but with complex structure $-J_1$. The identity map gives a natural complex-antilinear bundle isomorphism

$$E_1 \to E_1^{\mathbf{c}} : v \mapsto \bar{v},$$

and if E_1 carries a Hermitian bundle metric $\langle\ ,\ \rangle_{E_1}$, its conjugate inherits a Hermitian structure defined by

$$\langle \bar{v}, \bar{w} \rangle_{E_1^{\mathbf{c}}} = \langle w, v \rangle_{E_1}.$$

There are canonical complex-linear bundle isomorphisms

$$(E_1 \otimes E_2)^{\mathbf{c}} = E_1^{\mathbf{c}} \otimes E_2^{\mathbf{c}}, \quad \operatorname{Hom}_{\mathbb{C}}(E_1, E_2)^{\mathbf{c}} = \operatorname{Hom}_{\mathbb{C}}(E_1^{\mathbf{c}}, E_2^{\mathbf{c}}), \quad \operatorname{Hom}_{\mathbb{C}}(E_1^{\mathbf{c}}, E_2) = \overline{\operatorname{Hom}}_{\mathbb{C}}(E_1, E_2),$$

where the third of these identifies $\beta \in \operatorname{Hom}_{\mathbb{C}}(E_1^{\mathbf{c}}, E_2)$ with the antilinear map

$$B : E_1 \to E_2 : \eta \mapsto \beta\bar{\eta}.$$

The metric on Σ determines a complex-linear isomorphism

$$(T\Sigma)^{\mathbf{c}} \to \Lambda^{1,0}T^*\Sigma : \bar{X} \mapsto X^{1,0} := \langle X, \cdot \rangle_{\Sigma},$$

so together with (5.1), this identifies $\Lambda^{1,0}T^*\Sigma$ and $\Lambda^{0,1}T^*\Sigma$ with each other's conjugate bundles. Observe now that if $\mathbf{D} : \Gamma(E) \to \Gamma(F)$ is a Cauchy-Riemann type operator, then

$$\mathbf{D}^{\mathbf{c}}\bar{\eta} := \overline{\mathbf{D}\eta}$$

defines an anti-Cauchy-Riemann type operator

$$\mathbf{D}^{\mathbf{c}} : \Gamma(E^{\mathbf{c}}) \to \Gamma(F^{\mathbf{c}}) = \Gamma\big((\Lambda^{0,1}T^*\Sigma \otimes E)^{\mathbf{c}}\big) = \Gamma(\Lambda^{1,0}T^*\Sigma \otimes E^{\mathbf{c}}) = \Omega^{1,0}(\Sigma, E^{\mathbf{c}}).$$

Given an antilinear bundle map $B : E \to F$, let $\beta : E^{\mathbf{c}} \to F$ denote the corresponding complex-linear bundle map such that

$$B\eta = \beta\bar{\eta},$$

and let $\beta^\dagger : F \to E^{\mathbf{c}}$ denote the adjoint of β with respect to the Hermitian structures on $E^{\mathbf{c}}$ and F, i.e.

$$\langle \lambda, \beta\bar{\eta} \rangle_F = \langle \beta^\dagger \lambda, \bar{\eta} \rangle_{E^{\mathbf{c}}} \quad \text{for all} \quad (\bar{\eta}, \lambda) \in E^{\mathbf{c}} \oplus F.$$

Conjugating this then gives a bundle map

$$\overline{\beta^\dagger} = \bar{\beta}^\dagger : F^{\mathbf{c}} \to E.$$

We claim that $\beta : E^{\mathbf{c}} \to F$ can also be regarded as a bundle map $F^{\mathbf{c}} \to E$. Indeed, using the isomorphism

$$F^{\mathbf{c}} = (\Lambda^{0,1}T^*\Sigma \otimes E)^{\mathbf{c}} = \Lambda^{1,0}T^*\Sigma \otimes E^{\mathbf{c}},$$

we obtain from $\beta : E^{\mathbf{c}} \to F$ a bundle map

$$F^{\mathbf{c}} = \Lambda^{1,0}T^*\Sigma \otimes E^{\mathbf{c}} \xrightarrow{\mathbb{1} \otimes \beta} \Lambda^{1,0}T^*\Sigma \otimes F,$$

where the target can be identified with E via (5.4).

Lemma 5.4. *Fix a complex-linear bundle map $\beta : E^{\mathbf{c}} \to F$ and let $B : E \to F : \eta \mapsto \beta\bar{\eta}$. Then B satisfies the symmetry condition* (5.5) *if and only if β and $\bar{\beta}^\dagger$ define identical bundle maps $F^{\mathbf{c}} \to E$.*

Proof. It will suffice to show that (5.5) holds if and only if for every $z \in \Sigma$, $\eta \in E_z$ and $\bar{\lambda} \in F_z^{\mathbf{c}}$,

$$\mathrm{Re}\langle \beta\bar{\lambda}, \eta\rangle_E = \mathrm{Re}\langle \bar{\beta}^\dagger \bar{\lambda}, \eta\rangle_E.$$

Choose any nonzero vector $X \in T_z\Sigma$; we can then write $\lambda = X^{0,1} \otimes \xi \in \Lambda^{0,1}T_z^*\Sigma \otimes E_z = F_z$ where $\xi := \lambda(X)/|X|^2_\Sigma \in E_z$. Similarly, $\beta\bar{\eta} = B\eta = X^{0,1} \otimes \theta$, where $\theta := B\eta(X)/|X|^2_\Sigma \in E_z$. Then

$$\begin{aligned}\langle \bar{\beta}^\dagger \bar{\lambda}, \eta\rangle_E &= \langle \bar{\lambda}, \bar{\beta}\eta\rangle_{F^{\mathbf{c}}} = \langle \beta\bar{\eta}, \lambda\rangle_F = \langle X^{0,1} \otimes \theta, X^{0,1} \otimes \xi\rangle_F = \langle X, X\rangle_\Sigma \langle \theta, \xi\rangle_E \\ &= \langle B\eta(X), \xi\rangle_E.\end{aligned}$$

Likewise, writing $\beta\bar{\xi} = X^{0,1} \otimes \zeta$ for $\zeta := B\xi(X)/|X|^2_\Sigma \in E_z$, we use the natural isomorphisms (5.2), (5.4) and

$$(\Lambda^{0,1}T^*\Sigma)^{\mathbf{c}} \to \Lambda^{1,0}T^*\Sigma : \overline{X^{0,1}} \mapsto X^{1,0}$$

to obtain

$$\begin{aligned}\langle \beta\bar{\lambda}, \eta\rangle_E &= \langle \beta(X^{1,0} \otimes \bar{\xi}), \eta\rangle_E = \langle X^{1,0} \otimes \beta\bar{\xi}, \eta\rangle_E = \langle X^{1,0} \otimes X^{0,1} \otimes \zeta, \eta\rangle_E \\ &= \left\langle \langle X, X\rangle_\Sigma \frac{1}{|X|^2_\Sigma} B\xi(X), \eta \right\rangle_E = \langle B\xi(X), \eta\rangle_E.\end{aligned}$$

□

Proof of Proposition 5.1. Writing $\mathbf{D}_B^* = \mathbf{D}^* + B^*$, we first expand

$$\mathbf{D}_B^*\mathbf{D}_B\eta = (\mathbf{D}^* + B^*)(\mathbf{D} + B)\eta = \mathbf{D}^*\mathbf{D}\eta + B^*B\eta + \mathbf{D}^*(B\eta) + B^*(\mathbf{D}\eta).$$

We will see that all derivatives of η cancel in the sum of the last two terms. Write $B\eta = \beta\bar{\eta}$, where $\beta \in \Gamma(\mathrm{Hom}_{\mathbb{C}}(E^{\mathbf{c}}, F))$. To understand $\mathbf{D}^*(B\eta) = \mathbf{D}^*(\beta\bar{\eta})$, we can view $-\mathbf{D}^*$ as an anti-Cauchy-Riemann type operator on F, and since $\mathbf{D}^{\mathbf{c}}$ is likewise an anti-Cauchy-Riemann type operator on $E^{\mathbf{c}}$, Lemma 5.3 provides an anti-Cauchy-Riemann type operator ∂_H on $\mathrm{Hom}_{\mathbb{C}}(E^{\mathbf{c}}, F)$ such that

$$-\mathbf{D}^*(\beta\bar{\eta}) = (\partial_H\beta)\bar{\eta} + \beta\,\mathbf{D}^{\mathbf{c}}\bar{\eta}. \tag{5.6}$$

For the final term in the expansion, observe that for any $z \in \Sigma$, $\xi \in E_z$ and $\lambda \in F_z$,

$$\mathrm{Re}\langle \lambda, B\eta\rangle_F = \mathrm{Re}\langle \lambda, \beta\bar{\eta}\rangle_F = \mathrm{Re}\langle \beta^\dagger\lambda, \bar{\eta}\rangle_{E^{\mathbf{c}}} = \mathrm{Re}\langle \eta, \bar{\beta}^\dagger\bar{\lambda}\rangle_E = \mathrm{Re}\langle \bar{\beta}^\dagger\bar{\lambda}, \eta\rangle_E,$$

which gives the formula $B^*\lambda = \bar{\beta}^\dagger\bar{\lambda}$, hence

$$B^*(\mathbf{D}\eta) = \bar{\beta}^\dagger\,\mathbf{D}^{\mathbf{c}}\bar{\eta}. \tag{5.7}$$

Putting (5.6) and (5.7) together and applying Lemma 5.4, we have

$$\mathbf{D}^*(B\eta) + B^*(\mathbf{D}\eta) = -(\partial_H\beta)\bar{\eta} + (\bar{\beta}^\dagger - \beta)\,\mathbf{D}^{\mathbf{c}}\bar{\eta} = -(\partial_H\beta)\bar{\eta},$$

and the stated formula follows by using the natural identification of $\mathrm{Hom}_{\mathbb{C}}(E^{\mathbf{c}}, F)$ with $\overline{\mathrm{Hom}}_{\mathbb{C}}(E, F)$ to view ∂_H as an anti-Cauchy-Riemann type operator on the latter. □

Suppose next that $(\widetilde{\Sigma}, \tilde{\jmath})$ is another closed connected Riemann surface.

Definition 5.5. Given a nonconstant holomorphic map $\varphi : (\widetilde{\Sigma}, \tilde{\jmath}) \to (\Sigma, j)$ and a Cauchy-Riemann type operator $\mathbf{D}$ on E, define $\varphi^*\mathbf{D}$ to be the unique Cauchy-Riemann type operator on φ^*E that satisfies

$$(\varphi^*\mathbf{D})(\eta \circ \varphi) = \varphi^*(\mathbf{D}\eta) \quad \text{for all} \quad \eta \in \Gamma(E). \tag{5.8}$$

The uniqueness of $\varphi^*\mathbf{D}$ is clear from (5.8). To see that such an operator always exists, write $\mathbf{D} = \mathbf{D}^{\mathbb{C}} + A$ where $\mathbf{D}^{\mathbb{C}}$ is a complex-linear Cauchy-Riemann type operator and $A : E \to F$ is a real-linear bundle map, which we can view equivalently as a $(0,1)$-form valued in $\mathrm{End}_{\mathbb{R}}(E)$. Then $\mathbf{D}^{\mathbb{C}}$ induces a holomorphic bundle structure on E, which pulls back to define a holomorphic structure on φ^*E and consequently a Cauchy-Riemann type operator $\varphi^*\mathbf{D}^{\mathbb{C}}$. The operator $\varphi^*\mathbf{D}^{\mathbb{C}} + \varphi^*A$ then satisfies (5.8).

Example 5.6. If $u : (\Sigma, j) \to (M, J)$ is an immersed J-holomorphic curve and $\tilde{u} = u \circ \varphi$, then $\mathbf{D}^N_{\tilde{u}} = \varphi^*\mathbf{D}^N_u$.

The next lemma is only interesting when φ has branch points and is thus not needed for the proof of Theorem 1.3, but the general case of Proposition 2.5 requires it. Given $\mathbf{D}$ and B as in Proposition 5.1 and a nonconstant holomorphic map $\varphi : (\widetilde{\Sigma}, \tilde{\jmath}) \to (\Sigma, j)$, let us abbreviate

$$\widetilde{E} = \varphi^*E, \qquad \widetilde{F} = \Lambda^{0,1}T^*\widetilde{\Sigma} \otimes \widetilde{E}, \qquad \widetilde{\mathbf{D}} = \varphi^*\mathbf{D} : \Gamma(\widetilde{E}) \to \Gamma(\widetilde{F}).$$

Viewing B as an $\overline{\mathrm{End}}_{\mathbb{C}}(E)$-valued $(0,1)$-form on Σ, we can then define

$$\widetilde{B} = \varphi^*B \in \Omega^{0,1}(\widetilde{\Sigma}, \overline{\mathrm{End}}_{\mathbb{C}}(\widetilde{E})), \qquad \widetilde{\mathbf{D}}_B = \widetilde{\mathbf{D}} + \widetilde{B} : \Gamma(\widetilde{E}) \to \Gamma(\widetilde{F}).$$

Choose a Hermitian structure $\langle\ ,\ \rangle_{\widetilde{\Sigma}}$ on $T\widetilde{\Sigma}$, whose real part is then a $\tilde{\jmath}$-invariant Riemannian metric on $\widetilde{\Sigma}$. The bundles $\widetilde{E}$ and $\widetilde{F}$ now inherit natural Hermitian structures, the former as the pullback of E and the latter as the tensor product $\Lambda^{0,1}T^*\widetilde{\Sigma} \otimes \widetilde{E}$, and these determine formal adjoint operators $\widetilde{\mathbf{D}}^*$ and $\widetilde{\mathbf{D}}^*_B$. The symmetry assumption (5.5) on B implies that $\widetilde{B}$ also satisfies this condition, so that Proposition 5.1 gives a Weitzenböck formula over $\widetilde{\Sigma}$ in the form

$$\widetilde{\mathbf{D}}^*_B\widetilde{\mathbf{D}}_B\eta = \widetilde{\mathbf{D}}^*\widetilde{\mathbf{D}}\eta + \widetilde{B}^*\widetilde{B}\eta - (\tilde{\partial}_H\widetilde{B})\eta$$

for some anti-Cauchy-Riemann type operator $\tilde{\partial}_H$ on $\overline{\mathrm{Hom}}_{\mathbb{C}}(\widetilde{E}, \widetilde{F})$.

Lemma 5.7. *Assume the Riemannian metric* $\mathrm{Re}\langle\ ,\ \rangle_{\widetilde{\Sigma}}$ *on* $\widetilde{\Sigma}$ *is flat near all critical points of* φ. *Then there exists a constant* $c > 0$ *such that*

$$\left|\tilde{\partial}_H\widetilde{B}(z)\right| \le c|d\varphi(z)|^2 \quad \textit{for all} \quad z \in \widetilde{\Sigma}.$$

Proof. Recall from the proof of Proposition 5.1 that after identifying $\overline{\mathrm{Hom}}_{\mathbb{C}}(\widetilde{E}, \widetilde{F})$ with $\mathrm{Hom}_{\mathbb{C}}(\widetilde{E}^{\mathbf{c}}, \widetilde{F})$ by writing $\widetilde{B}\eta = \tilde{\beta}\bar{\eta}$ for $\tilde{\beta} \in \Gamma(\mathrm{Hom}_{\mathbb{C}}(\widetilde{E}^{\mathbf{c}}, \widetilde{F}))$, the operator $\tilde{\partial}_H$ is determined by the two anti-Cauchy-Riemann type operators $\widetilde{\mathbf{D}}^{\mathbf{c}}$ and $-\widetilde{\mathbf{D}}^*$ via a Leibniz rule. It will suffice to check that $|\tilde{\partial}_H\tilde{\beta}| \le c|d\varphi|^2$ holds in suitable local trivializations in a neighborhood of each branch point $z_0 \in \widetilde{\Sigma}$. Since the metric on $\widetilde{\Sigma}$ is assumed flat near z_0 and induces the same conformal structure as $\tilde{\jmath}$, we can find holomorphic coordinates $z = s + it$ on some neighorhood $\widetilde{\mathcal{U}} \subset \widetilde{\Sigma}$ of z_0 in which the area form determined by the metric is $ds \wedge dt$, and the induced bundle metric on $\Lambda^{0,1}T^*\widetilde{\Sigma}|_{\widetilde{\mathcal{U}}}$ satisfies $|d\bar{z}|_{\widetilde{\Sigma}} = 1$. Choose holomorphic coordinates also on a neighborhood $\mathcal{U} \subset \Sigma$ of $\varphi(z_0)$ and assume without loss of generality that $\varphi(\widetilde{\mathcal{U}}) = \mathcal{U}$. Next, fix a unitary trivialization of $E|_{\mathcal{U}}$, pull it back to define a trivialization of $\widetilde{E}|_{\widetilde{\mathcal{U}}}$, and use this together with the frame $d\bar{z}$ to trivialize $\widetilde{F} = \Lambda^{0,1}T^*\widetilde{\Sigma} \otimes \widetilde{E}$ over $\widetilde{\mathcal{U}}$. These trivializations identify $\mathbf{D}$ and $\widetilde{\mathbf{D}}$ locally with operators of the form

$$\mathbf{D} = \bar{\partial} + A, \qquad \widetilde{\mathbf{D}} = \bar{\partial} + \tilde{A},$$

where $\bar{\partial} = \partial_s + i\partial_t$, $A : \mathcal{U} \to \mathrm{End}_{\mathbb{R}}(\mathbb{C}^m)$ and $\tilde{A} : \widetilde{\mathcal{U}} \to \mathrm{End}_{\mathbb{R}}(\mathbb{C}^m)$. Using the natural trivialization induced on $\widetilde{E}^{\mathbf{c}}|_{\widetilde{\mathcal{U}}}$ for which the canonical antilinear isomorphism $\widetilde{E} \to \widetilde{E}^{\mathbf{c}}$ appears as complex conjugation, $\widetilde{\mathbf{D}}^{\mathbf{c}}$ can now be written as

$$\widetilde{\mathbf{D}}^{\mathbf{c}} = \partial + \tilde{A}^{\mathbf{c}},$$

where $\tilde{A}^{\mathbf{c}} : \widetilde{\mathcal{U}} \to \mathrm{End}_{\mathbb{R}}(\mathbb{C}^m)$ is defined by $\tilde{A}^{\mathbf{c}}\bar{\eta} = \overline{\tilde{A}\eta}$. Observe now that our trivializations of $\widetilde{E}$ and $\widetilde{F}$ over $\widetilde{\mathcal{U}}$ are both unitary, and since the area form $\widetilde{\mathcal{U}}$ is also standard in coordinates, the formal adjoint of $\widetilde{\mathbf{D}}$ takes the form

$$\widetilde{\mathbf{D}}^* = -\partial + \tilde{A}^{\mathrm{T}}.$$

From these expressions and the Leibniz rule (cf. the proof of Lemma 5.3), one derives a function $\widetilde{C} : \widetilde{\mathcal{U}} \to \mathrm{End}_{\mathbb{R}}(\mathrm{End}_{\mathbb{C}}(\mathbb{C}^m))$ such that the local formula for $\tilde{\partial}_H$ as a differential operator on $\mathrm{End}_{\mathbb{C}}(\mathbb{C}^m)$-valued functions is

$$\tilde{\partial}_H = \partial + \widetilde{C} \quad \text{where} \quad (\widetilde{C}\Phi)\bar{\eta} = -\tilde{A}^{\mathrm{T}}(\Phi\bar{\eta}) - \Phi(\tilde{A}^{\mathbf{c}}\bar{\eta}). \tag{5.9}$$

Recall now that since $\widetilde{\mathbf{D}} = \varphi^*\mathbf{D}$, A and $\tilde{A}$ represent elements of $\Omega^{0,1}(\Sigma, \mathrm{End}_{\mathbb{R}}(E))$ and $\Omega^{0,1}(\widetilde{\Sigma}, \mathrm{End}_{\mathbb{R}}(\widetilde{E}))$ respectively, with the latter being the pullback of the former via φ. To make this explicit, the function $A : \mathcal{U} \to \mathrm{End}_{\mathbb{R}}(\mathbb{C}^m)$ represents a $(0,1)$-form that corresponds under our trivialization of $E|_{\mathcal{U}}$ to $d\bar{z} \otimes A \in \Omega^{0,1}(\mathrm{End}_{\mathbb{R}}(\mathbb{C}^m))$, and $\tilde{A}$ then corresponds to the pullback $\varphi^*(d\bar{z} \otimes A) = d\bar{\varphi} \otimes (A \circ \varphi) = d\bar{z} \otimes \overline{\varphi'} \cdot (A \circ \varphi)$, giving the relation

$$\tilde{A}(z) = \overline{\varphi'(z)} A(\varphi(z)).$$

This implies an estimate of the form $|\tilde{A}(z)| \le c|\varphi'(z)|$ and, by (5.9), a similar estimate for $|\widetilde{C}(z)|$. Finally, viewing $\tilde{\beta}$ as a $(0,1)$-form valued in $\mathrm{Hom}_{\mathbb{C}}(\widetilde{E}^{\mathbf{c}}, \widetilde{E})$, it is also the pullback of a $\mathrm{Hom}_{\mathbb{C}}(E^{\mathbf{c}}, E)$-valued $(0,1)$-form and is thus similarly represented in trivializations by a function $\tilde{\beta} : \widetilde{\mathcal{U}} \to \mathrm{End}_{\mathbb{C}}(\mathbb{C}^m)$ that satisfies

$$\tilde{\beta}(z) = \overline{\varphi'(z)} \beta(\varphi(z))$$

for some function $\beta : \mathcal{U} \to \mathrm{End}_{\mathbb{C}}(\mathbb{C}^m)$. The estimate $|\tilde{\partial}_H \tilde{\beta}| = |\partial\tilde{\beta} + \widetilde{C}\tilde{\beta}| \le c|\varphi'|^2$ now follows by a short calculation: indeed, $|\widetilde{C}\tilde{\beta}| \le |\widetilde{C}| \cdot |\tilde{\beta}| \le c|\varphi'|^2$ for some $c > 0$, and since $\overline{\varphi'}$ is antiholomorphic, $\partial\tilde{\beta} = \partial\left(\overline{\varphi'} \cdot (\beta \circ \varphi)\right) = \overline{\varphi'}(\partial\beta \circ \varphi)\varphi'$ similarly satisfies $|\partial\tilde{\beta}| \le c|\varphi'|^2$. ☐

6. Regularity for the linearized operator

We now state and prove a linear perturbation result that implies Proposition 2.5. The result is a higher-dimensional generalization of results for complex line bundles that were proved by Taubes [Tau96a, Tau96b], and similar results stated in [Rau04].

Assume (Σ, j) and $(\widetilde{\Sigma}, \tilde{\jmath})$ are closed connected Riemann surfaces, $\varphi : (\widetilde{\Sigma}, \tilde{\jmath}) \to (\Sigma, j)$ is a holomorphic map of degree $d \geq 1$, $(E, J) \to (\Sigma, j)$ is a complex vector bundle of rank $m \geq 1$, and $\mathbf{D} : \Gamma(E) \to \Omega^{0,1}(\Sigma, E)$ is a real-linear Cauchy-Riemann type operator. As in the previous section, we shall abbreviate

$$\widetilde{E} = \varphi^* E, \qquad \widetilde{\mathbf{D}} = \varphi^* \mathbf{D},$$

where $\varphi^* \mathbf{D} : \Gamma(\varphi^* E) \to \Omega^{0,1}(\widetilde{\Sigma}, \varphi^* E)$ denotes the induced Cauchy-Riemann type operator on the pullback (see Definition 5.5).

Now assume $\operatorname{ind}(\mathbf{D}) = 0$. By the Riemann-Roch formula, this means

$$-c_1(E) = m\chi(\Sigma) + c_1(E) = c_1(\overline{\operatorname{Hom}}_{\mathbb{C}}(T\Sigma, E)),$$

so there exists a complex-antilinear bundle isomorphism

$$B : E \to \overline{\operatorname{Hom}}_{\mathbb{C}}(T\Sigma, E).$$

Choosing a Hermitian bundle metric $\langle\ ,\ \rangle_E$ on E, we can also arrange by Proposition 2.4 that B satisfies the symmetry condition

$$\operatorname{Re}\langle \xi, B\eta(X) \rangle_E = \operatorname{Re}\langle B\xi(X), \eta \rangle_E \quad \text{for all } (X, \xi, \eta) \in T\Sigma \oplus E \oplus E. \tag{6.1}$$

This gives rise to a 1-parameter family of real-linear Cauchy-Riemann type operators on $\widetilde{E}$, defined by

$$\widetilde{\mathbf{D}}_\tau = \varphi^*(\mathbf{D} + \tau B) = \widetilde{\mathbf{D}} + \tau \widetilde{B}$$

for $\tau \in \mathbb{R}$, where we abbreviate $\widetilde{B} := \varphi^* B$ with B regarded as an $\overline{\operatorname{End}}_{\mathbb{C}}(E, J)$-valued $(0,1)$-form. Let $Z(d\varphi) \geq 0$ denote the algebraic count of branch points of φ, which is $-\chi(\widetilde{\Sigma}) + d\chi(\Sigma)$ by the Riemann-Hurwitz formula. Then

$$\begin{aligned} \operatorname{ind}(\widetilde{\mathbf{D}}_\tau) &= m\chi(\widetilde{\Sigma}) + 2c_1(\varphi^* E) = m\left[d\chi(\Sigma) - Z(d\varphi)\right] + 2dc_1(E) \\ &= d \cdot \operatorname{ind}(\mathbf{D}) - mZ(d\varphi) = -mZ(d\varphi) \leq 0. \end{aligned}$$

Theorem 6.1. *The operators $\widetilde{\mathbf{D}}_\tau : \Gamma(\widetilde{E}) \to \Omega^{0,1}(\Sigma, \widetilde{E})$ defined above are injective for all $\tau \in \mathbb{R}$ outside of a discrete subset.*

Remark 6.2. The proof of Theorem 1.3 only requires the special case of Theorem 6.1 for which $\varphi : (\widetilde{\Sigma}, \tilde{\jmath}) \to (\Sigma, j)$ is unbranched, and in this case the proof below becomes somewhat simpler, e.g. it does not require Lemma 5.7. The general case of Theorem 6.1 may nonetheless be useful for proving stronger super-rigidity results.

As in §2.1, we can use analytic perturbation theory to reduce this theorem to a statement for particular values of τ. We first extend $\widetilde{\mathbf{D}}_\tau$ to a Fredholm operator between Hilbert spaces H^1 and L^2, each regarded as *real* vector spaces (since $\widetilde{\mathbf{D}}_\tau$ itself is real and not complex linear), then complexify and consider the family of complex-linear Fredholm operators

$$\widetilde{\mathbf{D}}_\tau : H^1(\widetilde{E}) \otimes \mathbb{C} \to L^2(\overline{\operatorname{Hom}}_{\mathbb{C}}(T\widetilde{\Sigma}, \widetilde{E})) \otimes \mathbb{C}$$

for $\tau \in \mathbb{C}$. This family depends holomorphically on τ. Note that for $\tau \in \mathbb{R}$, the underlying operator $\widetilde{\mathbf{D}}_\tau$ is injective whenever its complexification is injective. Thus by Proposition A.1 in the appendix, in order to prove Theorem 6.1, it suffices to establish the following:

Lemma 6.3. *The operator $\widetilde{\mathbf{D}}_\tau$ is injective for all sufficiently large $\tau > 0$.*

Proof. Choose a Hermitian bundle metric on $T\widetilde{\Sigma}$ that matches the standard Hermitian inner product in some choice of local holomorphic coordinates near each of the branch points of φ. This gives rise to a family of formal adjoint operators $\widetilde{\mathbf{D}}_\tau^*$ with $\widetilde{\mathbf{D}}_0^* =: \widetilde{\mathbf{D}}^*$ such that by Proposition 5.1,

$$\widetilde{\mathbf{D}}_\tau^* \widetilde{\mathbf{D}}_\tau \eta = \widetilde{\mathbf{D}}^* \widetilde{\mathbf{D}} \eta + \tau^2 \widetilde{B}^* \widetilde{B} \eta - \tau (\tilde{\partial}_H \widetilde{B}) \eta,$$

and Lemma 5.7 also implies

$$\left| \tilde{\partial}_H \widetilde{B} \right| \le c_1 |d\varphi|^2$$

for some $c_1 > 0$. Since B is a bundle isomorphism, we can find another constant $c_2 > 0$, such that $|B\eta| \ge c_2 |\eta|$ and thus

$$\left| \widetilde{B} \eta \right| \ge c_2 |d\varphi| \cdot |\eta|.$$

We then find for every $\eta \in \Gamma(\widetilde{E})$,

$$\begin{aligned} \|\widetilde{\mathbf{D}}_\tau \eta\|_{L^2}^2 &= \left\langle \eta, \widetilde{\mathbf{D}}_\tau^* \widetilde{\mathbf{D}}_\tau \eta \right\rangle_{L^2} = \left\langle \eta, \widetilde{\mathbf{D}}^* \widetilde{\mathbf{D}} \eta + \tau^2 \widetilde{B}^* \widetilde{B} \eta - \tau (\tilde{\partial}_H \widetilde{B}) \eta \right\rangle_{L^2} \\ &= \|\widetilde{\mathbf{D}} \eta\|_{L^2}^2 + \tau^2 \|\widetilde{B} \eta\|_{L^2}^2 - \tau \left\langle \eta, (\tilde{\partial}_H \widetilde{B}) \eta \right\rangle_{L^2} \ge \left(\tau^2 c_2^2 - \tau c_1 \right) \left\| |d\varphi| \cdot \eta \right\|_{L^2}^2, \end{aligned}$$

where the constants $c_1, c_2 > 0$ are independent of η. Since $|d\varphi| > 0$ almost everywhere, we conclude that $\widetilde{\mathbf{D}}_\tau$ is injective whenever $\tau^2 c_2^2 - \tau c_1 > 0$. □

Appendix A. Some analytic perturbation theory

The linear perturbation argument of §6 requires a basic ingredient from analytic perturbation theory in the spirit of [Kat95]. Since we were not able to find a reference for the precise result we need, we have included a proof of it in this appendix for the sake of completeness.

Given complex Banach spaces X and Y, denote by $\mathcal{L}(X,Y)$ the Banach space of bounded complex-linear operators $X \to Y$, abbreviate $\mathcal{L}(X) := \mathcal{L}(X,X)$, and let $\mathrm{Fred}(X,Y) \subset \mathcal{L}(X,Y)$ denote the open subset consisting of Fredholm operators. Since $\mathrm{Fred}(X,Y)$ carries a natural complex structure as a subset of $\mathcal{L}(X,Y)$, it makes sense to speak of holomorphic maps into $\mathrm{Fred}(X,Y)$, i.e. maps which are Fréchet differentiable with complex-linear derivative.

Proposition A.1. *Suppose $\mathcal{U} \subset \mathbb{C}$ is a connected open subset and $\mathcal{U} \to \mathrm{Fred}(X,Y) : \tau \mapsto \mathbf{T}_\tau$ is a holomorphic map, and let*

$$Z = \{ \tau \in \mathcal{U} \mid \mathbf{T}_\tau \text{ is not injective} \}.$$

Then either Z is a discrete subset of $\mathcal{U}$, or $Z = \mathcal{U}$.

Proof. Given any $\mathbf{T}_0 \in \mathrm{Fred}(X,Y)$, there exist splittings into closed linear subspaces

$$X = V \oplus \ker \mathbf{T}_0, \qquad Y = W \oplus \operatorname{coker} \mathbf{T}_0$$

such that $\mathbf{T}_0|_V$ is an isomorphism $V \to W$. Using this splitting, we can write any other $\mathbf{T} \in \mathrm{Fred}(X,Y)$ in block form as

$$\mathbf{T} = \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{pmatrix},$$

and define $\mathcal{O} \subset \text{Fred}(X,Y)$ to be the open neighborhood of $\mathbf{T}_0$ for which the block $\mathbf{A}$ is invertible. We can then define a holomorphic map

$$\Phi : \mathcal{O} \to \mathcal{L}(\ker \mathbf{T}_0, \operatorname{coker} \mathbf{T}_0) : \mathbf{T} \mapsto \mathbf{D} - \mathbf{C}\mathbf{A}^{-1}\mathbf{B}.$$

We claim that for all $\mathbf{T} \in \mathcal{O}$, $\ker \mathbf{T} \cong \ker \Phi(\mathbf{T})$. To see this, associate to $\mathbf{T}$ the isomorphism

$$\Psi = \begin{pmatrix} \mathbb{1} & -\mathbf{A}^{-1}\mathbf{B} \\ 0 & \mathbb{1} \end{pmatrix} \in \mathcal{L}(V \oplus \ker \mathbf{T}_0) = \mathcal{L}(X).$$

Then $\mathbf{T}\Psi = \begin{pmatrix} \mathbf{A} & 0 \\ \mathbf{C} & \Phi(\mathbf{T}) \end{pmatrix}$, and since $\mathbf{A}$ is invertible, $\ker \mathbf{T}\Psi = \{0\} \oplus \ker \Phi(\mathbf{T})$, from which the claim follows.

Now if $\mathcal{U} \to \text{Fred}(X,Y) : \tau \to \mathbf{T}_\tau$ is a family of operators depending holomorphically on τ, then fixing any $\tau_0 \in \mathcal{U}$ and placing $\mathbf{T}_{\tau_0}$ in the role of $\mathbf{T}_0$ above, one can define Φ on a neighborhood of $\mathbf{T}_{\tau_0}$ so that

$$\tau \mapsto \Phi(\mathbf{T}_\tau)$$

defines a holomorphic curve mapping into the finite-dimensional complex vector space $\mathcal{L}(\ker \mathbf{T}_{\tau_0}, \operatorname{coker} \mathbf{T}_{\tau_0})$ for τ in a neighborhood of τ_0. The set of all τ near τ_0 for which $\mathbf{T}_\tau$ is not injective then corresponds to the intersections of this holomorphic curve with the stratified complex subvariety of noninjective maps in $\mathcal{L}(\ker \mathbf{T}_{\tau_0}, \operatorname{coker} \mathbf{T}_{\tau_0})$, which has positive codimension. The proposition thus follows from the standard results on intersections of holomorphic curves with complex submanifolds. $\square$

References

[BEH+03] F. Bourgeois, Y. Eliashberg, H. Hofer, K. Wysocki, and E. Zehnder, *Compactness results in symplectic field theory*, Geom. Topol. **7** (2003), 799–888.

[BP01] J. Bryan and R. Pandharipande, *BPS states of curves in Calabi-Yau 3-folds*, Geom. Topol. **5** (2001), 287–318 (electronic).

[BP08] ———, *The local Gromov-Witten theory of curves*, J. Amer. Math. Soc. **21** (2008), no. 1, 101–136. With an appendix by Bryan, C. Faber, A. Okounkov and Pandharipande.

[CM07] K. Cieliebak and K. Mohnke, *Symplectic hypersurfaces and transversality in Gromov-Witten theory*, J. Symplectic Geom. **5** (2007), no. 3, 281–356.

[Eft16] E. Eftekhary, *On finiteness and rigidity of J-holomorphic curves in symplectic three-folds*, Adv. Math. **289** (2016), 1082–1105.

[EGH00] Y. Eliashberg, A. Givental, and H. Hofer, *Introduction to symplectic field theory*, Geom. Funct. Anal., Special Volume (2000), 560–673.

[Fab13] O. Fabert, *Local symplectic field theory*, Internat. J. Math. **24** (2013), no. 5, 1350041, 16.

[FO99] K. Fukaya and K. Ono, *Arnold conjecture and Gromov-Witten invariant*, Topology **38** (1999), no. 5, 933–1048.

[GV] R. Gopakumar and C. Vafa, *M-theory and topological strings–II.* Preprint arXiv:hep-th/9812127.

[HWZ] H. Hofer, K. Wysocki, and E. Zehnder, *Applications of polyfold theory I: the polyfolds of Gromov-Witten theory.* Preprint arXiv:1107.2097.

[Hut14] M. Hutchings, *Lecture Notes on Embedded Contact Homology*, Contact and Symplectic Topology, Bolyai Society Mathematical Studies, vol. 26, Springer, 2014, pp. 389–484.

[Hut] ———, *A magic trick for defining obstruction bundles.* Blog post, available at `http://floerhomology.wordpress.com/2013/09/04/`.

[IPa] E.-N. Ionel and T. H. Parker, *A natural Gromov-Witten virtual fundamental class.* Preprint arXiv:1302.3472.

[IPb] ———, *The Gopakumar-Vafa formula for symplectic manifolds.* Preprint arXiv:1306.1516.

[Kat95] T. Kato, *Perturbation theory for linear operators*, Classics in Mathematics, Springer-Verlag, Berlin, 1995. Reprint of the 1980 edition.

[LP07] J. Lee and T. H. Parker, *A structure theorem for the Gromov-Witten invariants of Kähler surfaces*, J. Differential Geom. **77** (2007), no. 3, 483–513.

[LP12] ———, *An obstruction bundle relating Gromov-Witten invariants of curves and Kähler surfaces*, Amer. J. Math. **134** (2012), no. 2, 453–506.

[LT98] J. Li and G. Tian, *Virtual moduli cycles and Gromov-Witten invariants of general symplectic manifolds*, Topics in symplectic 4-manifolds (Irvine, CA, 1996), First Int. Press Lect. Ser., I, Int. Press, Cambridge, MA, 1998, pp. 47–83.

[MS04] D. McDuff and D. Salamon, *J-holomorphic curves and symplectic topology*, American Mathematical Society, Providence, RI, 2004.

[OZ09] Y.-G. Oh and K. Zhu, *Embedding property of J-holomorphic curves in Calabi-Yau manifolds for generic J*, Asian J. Math. **13** (2009), no. 3, 323–340.

[Rau04] D. Rauch, *Perturbations of the D-bar operator*, Ph.D. Thesis, Harvard University, 2004.

[Rua99] Y. Ruan, *Virtual neighborhoods and pseudo-holomorphic curves*, Proceedings of 6th Gökova Geometry-Topology Conference, 1999, pp. 161–231.

[RT97] Y. Ruan and G. Tian, *Higher genus symplectic invariants and sigma models coupled with gravity*, Invent. Math. **130** (1997), 455–516.

[Sie] B. Siebert, *Gromov-Witten invariants of general symplectic manifolds.* Preprint arXiv:9608005.

[Ste51] N. Steenrod, *The Topology of Fibre Bundles*, Princeton University Press, Princeton, N. J., 1951.

[Tau96a] C. H. Taubes, *Counting pseudo-holomorphic submanifolds in dimension* 4, J. Differential Geom. **44** (1996), no. 4, 818–893.

[Tau96b] C. H. Taubes, SW $\Rightarrow$ Gr*: from the Seiberg-Witten equations to pseudo-holomorphic curves*, J. Amer. Math. Soc. **9** (1996), no. 3, 845–918.

[Wen10] C. Wendl, *Automatic transversality and orbifolds of punctured holomorphic curves in dimension four*, Comment. Math. Helv. **85** (2010), no. 2, 347–407.

[Wena] ———, *Lectures on holomorphic curves in symplectic and contact geometry.* Preprint arXiv:1011.1690v2.

[Wenb] ———, *Transversality and super-rigidity for multiply covered holomorphic curves.* Preprint arXiv:1609.09867.

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